

Application of Homotopy Analysis Method for Periodic Heat Transfer in Convective Straight Fins with Temperature-Dependent Thermal Conductivity

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Abstract: In this paper, the homotopy analysis method is applied to analyze the heat transfer of the oscillating base temperature processes occurring in a convective rectangular fin with variable thermal conductivity. This method is a powerful and easy-to-use tool for non-linear problems and it provides us with a simple way to adjust and control the convergence region of solution series. Without the need of iteration, the obtained solution is in the form of an infinite power series and the results indicated that the series has high accuracy by comparing it with those generated by the complex combination method.

Keywords: homotopy analysis method, fin, periodic heat transfer

Nomenclature

A_c	cross-sectional area of the fin m^2
B	dimensionless frequency of oscillation $\omega b^2/\alpha$
b	fin length m
C	specific heat $\text{J/kg} \cdot \text{K}$
C_1	integral constant
C_2	integral constant
G	dimensionless fin parameter $[Phb^2/k_a A_c]^{1/2}$
h	heat transfer coefficient $\text{W/m}^2 \cdot \text{K}$
k	thermal conductivity $\text{W/m} \cdot \text{K}$
L	auxiliary linear operator
N	non-linear operator
P	fin perimeter m
Q	dimensionless temperature gradient

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S	dimensionless amplitude of oscillation
T	temperature K
T_{bm}	average temperature K
t	time s
w	fin thickness m
x	axial distance measured from fin tip
X	dimensionless distance x/b

Greek symbols

α	thermal diffusivity m^2/s
β	slop of the thermal conductivity-temperature divided by the intercept k_a
ε	thermal conductivity parameter $(k_b - k_a)/k_a$
η	fin efficiency
\hbar	non-zero auxiliary parameter
θ	dimensionless temperature $(T - T_a)/(T_{bm} - T_a)$
ρ	density kg/m^3
τ	dimensionless time $\alpha t/b^2$
ω	frequency $1/\text{s}$

Subscripts

a	ambient
b	base of the fin
m	number of terms in the series

1 Introduction

In the real world, almost every natural phenomenon arises in a non-linear system. Still, we are far less understanding of non-linear systems than linear ones. Non-linear problems are complicated and unpredictable, making them difficult to solve precisely. Even if an exact solution is obtained, the necessary calculations may be too complicated to be practical. Therefore, researchers make endless efforts try to find ways to solve them or to decrease the error of the solution, such as the perturbation method [Cole (1986); Nayfeh (1981, 1985)], Lyapunov's artificial small parameter method [Lyapunov (1992)], the δ -expansion method [Karmishin, Zhukov, and Kolosov (1990)], Adomian's decomposition method [Adomian (1976,

1984, 1994)] and homotopy analysis method [Liao (2003)] etc. Thanks to these proposed methods, we can solve lots of non-linear problems efficiently.

Among scientific issues, heat transfer problems are often very important because they are closely related to our lives. Extended surfaces (fins) have been especially widely used in applications to enhance the heat transfer between a solid surface and its surroundings. [Hung and Appl (1967)] presented the performance of a straight fin with temperature-dependent conductivity and internal heat generation. [Yang (1972)] solved the heat transfer in straight fins with the method of complex combination and obtained a closed form solution. This was further extended to an annular fin and temperature-dependent conductivity by [Aziz (1975)]. Later, [Aziz and Enamul Hug (1975)] and [Krane (1976)] used the regular perturbation method and the numerical solution method to analyze a straight convecting fin with temperature-dependent thermal conductivity. Recently, [Chiu and Chen (2002)] applied Adomian's decomposition method (ADM) to determine the performance of a longitudinal fin with a constant heat transfer coefficient and variable thermal conductivity. [Abbasbandy (2006)] investigated non-linear equations arising in heat transfer based on homotopy analysis method (HAM) and showed the validity and great potential of the HAM for non-linear problems in science and engineering. In addition, [Mueller Jr. and Abu-Mulaweh (2006)] predicted the temperature in a long horizontal fin rod with natural convection and radiation conditions. The results show that the heat loss due to radiation is typically 15-20% of the total. While [Coşkun and Atay (2008)] investigated the fin efficiency with variational iteration method (VIM). The authors observed that the value of thermo-geometric fin parameter is another factor affecting the behavior of the solution. Both [Inc (2008)] and [Domairry and Fazeli (2009)] studied the fin efficiency of convective straight fins with temperature-dependent thermal conductivity and showed that the obtained solution agreed well with both the exact solution and ADM's. Furthermore, fins are also used for some engineering applications which often operate under periodic thermal conditions, such as electronic components, solar collectors and internal combustion engines etc. [Yang, Chien, and Chen (2008)] researched the periodic base temperature in convective longitudinal fins with the double decomposition method. Their results showed that the double decomposition solution has more advantages than the Adomian decomposition solution.

Regarding the view mentioned above, it's beneficial for us to discuss practical situations. In this paper, we are going to analyze the fin's heat transfer of the oscillating base temperature by using a powerful method (HAM). The basic idea of the HAM will be first introduced in the next section, and also the results of why HAM shows performance that equals the complex combination method.

2 Basic idea of HAM

In this section, we apply the homotopy analysis method to the discussed problem. To illustrate the basic ideas of this method, we consider the following general non-linear differential equation

$$N[u(x, t)] = 0 \quad (1)$$

where N is a non-linear operator, x and t denote independent variables, and $u(x, t)$ is an unknown function, respectively. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM, we first construct the so-called zero-order deformation equation

$$(1 - p)L[\phi(x, t; p) - u_0(x, t)] = p\hbar H(x, t)N[\phi(x, t; p)], \quad (2)$$

where $p \in [0, 1]$ is the embedding parameter, \hbar is a non-zero auxiliary parameter, $H(x, t)$ is a non-zero auxiliary function, L is an auxiliary linear operator, $u_0(x, t)$ is an initial guess of $u(x, t)$ and $\phi(x, t; p)$ is an unknown function, respectively. Obviously, when $p = 0$ and $p = 1$ it holds

$$\phi(x, t; 0) = u_0(x, t), \quad (3)$$

$$\phi(x, t; 1) = u(x, t), \quad (4)$$

respectively. The solution $\phi(x, t; p)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$ as p increases from 0 to 1. [Liao (2003)] expanded $\phi(x, t; p)$ in Taylor series with respect to p , one has

$$\phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t) p^m, \quad (5)$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; p)}{\partial p^m}. \quad (6)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter \hbar and the auxiliary function $H(x, t)$ are properly chosen, the series (5) converges at $p = 1$, as

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t). \quad (7)$$

Define the vector

$$\vec{u}_m = \{u_0(x, t), u_1(x, t), \dots, u_m(x, t)\}$$

Differentiating Eq.(2) m times with respect to the embedding parameter p and then dividing them by $m!$ and finally setting $p = 0$, we have the so-called m th-order deformation equation

$$L[u_m(x, t) - \chi(m)u_{m-1}(x, t)] = \hbar H(x, t) \mathcal{R}(\vec{u}_{m-1}), \quad (8)$$

where

$$\mathcal{R}(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x, t; p)]}{\partial p^{m-1}} \right|_{p=0}, \quad (9)$$

$$\text{and } \chi(m) = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases}.$$

It should be emphasized that $u_m(x, t)$ for $m \geq 1$ is governed by the linear equation (8) with the boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3 The governing equation and boundary condition

As Fig. 1 shows, a rectangular fin with length, b , and thickness, w , is considered. The fin surface exposed to a convective environment at temperature T_a and heat transfer coefficient, h , is assumed to be uniform. The thermal conductivity of the fin material, k , is assumed to vary as a linear function of the temperature, i.e.

$$k(T) = k_a [1 + \beta (T - T_a)],$$

where k_a is the thermal conductivity at ambient temperature and β is the slope of the thermal conductivity temperature curve divided by the intercept, k_a .

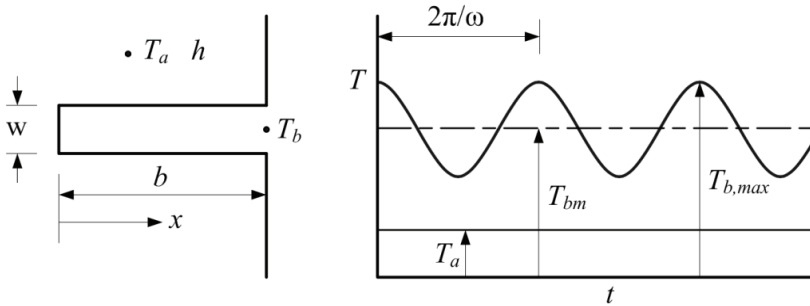


Figure 1: Fin geometry and base temperature oscillation

The fin tip is assumed to be adiabatic while the base temperature T_b is allowed to vary periodically around T_{bm} with frequency ω . The axial distance x is measured from the fin tip in figure 1. According to [Lau and Tan (1973)], one can neglect the effect of heat conduction in y -direction if the fin's Biot number, hw/k , is less than 0.1. In the one-dimensional system, the energy equation and the boundary conditions are

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) - \frac{hP}{A_c} (T - T_a) = \rho C \frac{\partial T}{\partial t}, \quad (10)$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0, \quad (11)$$

$$T(b, t) = T_b = T_{bm} + (T_{bm} - T_a) S \cos \omega t. \quad (12)$$

By using the following dimensionless variables,

$$\theta = \frac{T - T_a}{T_{bm} - T_a}, \quad X = \frac{x}{b}, \quad \varepsilon = \frac{k_b - k_a}{k_a} = \beta (T_{bm} - T_a),$$

$$G^2 = \frac{Phb^2}{k_a A_c}, \quad \tau = \frac{\alpha t}{b^2}, \quad B = \frac{\omega b^2}{\alpha}.$$

We have

$$(1 + \varepsilon \theta) \frac{\partial^2 \theta}{\partial X^2} + \varepsilon \left(\frac{\partial \theta}{\partial X} \right)^2 - G^2 = \frac{\partial \theta}{\partial \tau}, \quad (13)$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{X=0} = 0, \quad (14)$$

$$\theta(1, \tau) = 1 + S \cos B\tau, \quad (15)$$

where S is the dimensionless amplitude of oscillation.

4 Analysis

Since the governing equation (13) is a partial differential equation, it will be difficult for us to choose a proper base function, which includes two independent variables to express the solution. So, here we temporarily take the variable τ as a constant and choose $\theta_0(X, \tau) = 1 + S \cos B\tau$ to be our initial guess. In this way, we can express the solution by a set of base functions

$$\{ a_m X^{2m} \mid m = 0, 1, 2, 3, \dots \}, \quad (16)$$

which still obey the rule of solution express [Liao (2003)]. According to (13), we choose the linear operator

$$L[\phi(X, \tau; p)] = \frac{\partial^2 \phi(X, \tau; p)}{\partial X^2}, \quad (17)$$

with the property

$$L[C_1 X + C_2] = 0,$$

where the integral constant C_1 and C_2 are determined by the boundary conditions. From Eq. (13), we define a non-linear operator

$$N[\phi(X, \tau; p)] = [1 + \varepsilon \phi(X, \tau; p)] \frac{\partial^2 \phi(X, \tau; p)}{\partial X^2} + \varepsilon \left(\frac{\partial \phi(X, \tau; p)}{\partial X} \right)^2 - G^2 - \frac{\partial \phi(X, \tau; p)}{\partial \tau}. \quad (18)$$

Then, from Eq. (9), we have

$$\mathcal{R}_m(\theta_{m-1}) = \frac{\partial^2 \theta_{m-1}}{\partial X^2} - G^2 \theta_{m-1} + \varepsilon \frac{\partial}{\partial X} \left(\sum_{j=0}^{m-1} \frac{\partial}{\partial X} \theta_{m-1-j} \cdot \theta_j \right) - \frac{\partial \theta_{m-1}}{\partial \tau}.$$

To obey both the rule of solution expression and the rule of the coefficient ergodicity, the corresponding auxiliary function should be determined uniquely $H(X, \tau) = 1$. Now, the solution of the m th-order deformation equation (8) for $m \geq 1$ becomes

$$\theta_m = \chi(m) \theta_{m-1} + \hbar \int_0^X \int_0^X H(X, \tau) \cdot \mathcal{R}_m(\theta_{m-1}) dX d\tau + C_1 X + C_2. \quad (19)$$

Therefore, the first three iterates are expressed as

$$\theta_0(X, \tau) = 1 + S \cos B\tau$$

$$\theta_1(X, \tau) = -\frac{1}{2} \hbar [X^2 G^2 (1 + S \cos B\tau) - X^2 S B \sin B\tau - G^2 (1 + S \cos B\tau) + S B \sin B\tau]$$

$$\begin{aligned} \theta_2(X, \tau) = & -\frac{1}{24} \hbar^2 X^4 [S B^2 \cos B\tau - S G^4 \cos B\tau + 2 S B G^2 \sin B\tau - G^4] \\ & - \frac{1}{24} \hbar X^4 [-12 S B \hbar G^2 \sin B\tau + 24 \varepsilon S \hbar G^2 \cos B\tau + \dots] + \dots \end{aligned}$$

Summing these terms, it is observed that

$$\psi_m(X, \tau) = \sum_{n=0}^{m-1} \theta_n = \theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{m-1}. \quad (20)$$

Thus, components of θ are determined and written as a m -terms approximation. With a proper chosen of auxiliary parameter \hbar , it will converge to θ as $m \rightarrow \infty$. Here, we take m to be seven. The temperature distribution along the fin axial and fin performance can be easily performed by using Eq.(20).

5 Results and discussion

While \hbar is a convergence-controller parameter, we first plot the so-called \hbar -curve of $\psi_7''(0,0)$ in figure 2. From the figure, it's easy to discover the valid domain of \hbar ($\hbar \in [-1.1, -0.3]$) for the present problem. Therefore, we choose $\hbar = -0.6$ and consider the same conditions in [Yang (1972)] that is defined as the thermal conductivity parameter $\varepsilon = 0$, amplitude parameter $S = 0.1$ and frequency parameter $B = 1.0$. Fig. 3 shows the axial temperature distribution at $G = 1.0$. A perfect match between the HAM results and [Yang (1972)] is observed, which confirms the validity of the homotopy analysis method. Obviously, the present method gives quick and accurate results instead of complicated numerical integration and iteration procedures.

Consequently, we further discuss the effects on the fin for different parameters with $G = 0.3, 0.5, 1.0$ and $\varepsilon = 0, \pm 0.2, \pm 0.4$. Figures 4a-4c show the non-dimensional temperature distribution along with the fin surface for different values of $G = 0.3, G = 0.5, G = 1.0$ with ε varying from -0.4 to 0.4 , respectively. The mean temperature increases with the increasing ε of the fin. This is because an improvement in thermal conductivity is obtained for a raise of ε and a greater heat transfer will be more easily transferred to the fin tip. In addition, it could be found that under a fixed value of ε , the temperature distribution at fin tip decreases with fin parameter G increasing. This is due to the larger fin parameter, the stronger convection heat transfer around the fin surface.

The temperature distributions at the fin tip ($X = 0$) with the variable parameters ε and G are shown in Fig. 5a-5c. At a fixed value of G , the variation of thermal conductivity ε affects both the amplitude and phase angle of temperature variation. The decreasing ε decreases the amplitude but increases the phase angle. And these become obvious for the lower fin parameter.

Both the fin efficiency and the total energy transferred from the fin base are of great interest in engineering. The fin efficiency can be defined as the ratio of the actual heat transfer to heat transfer from the fin surface at base temperature $\eta =$

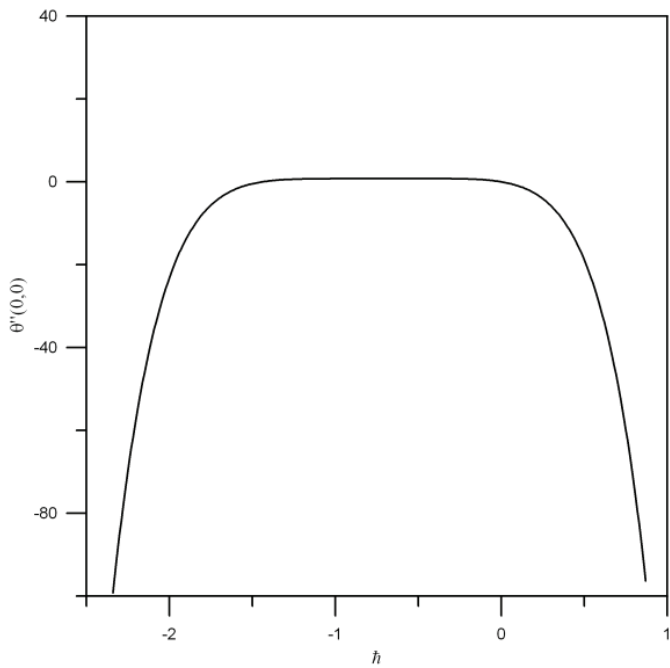


Figure 2: The h -curve for 7th-order approximation of $\theta''(0,0)$

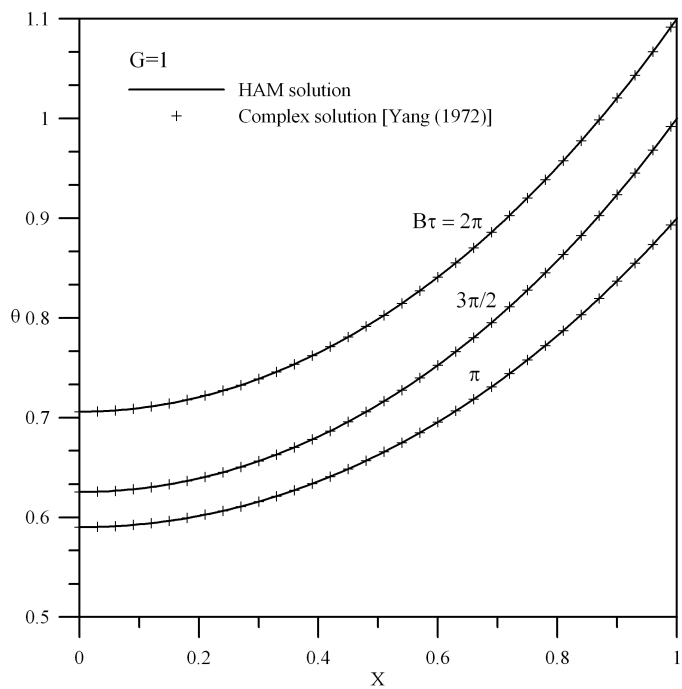


Figure 3: The variation relationships of θ and X for several assigned values of $B\tau$

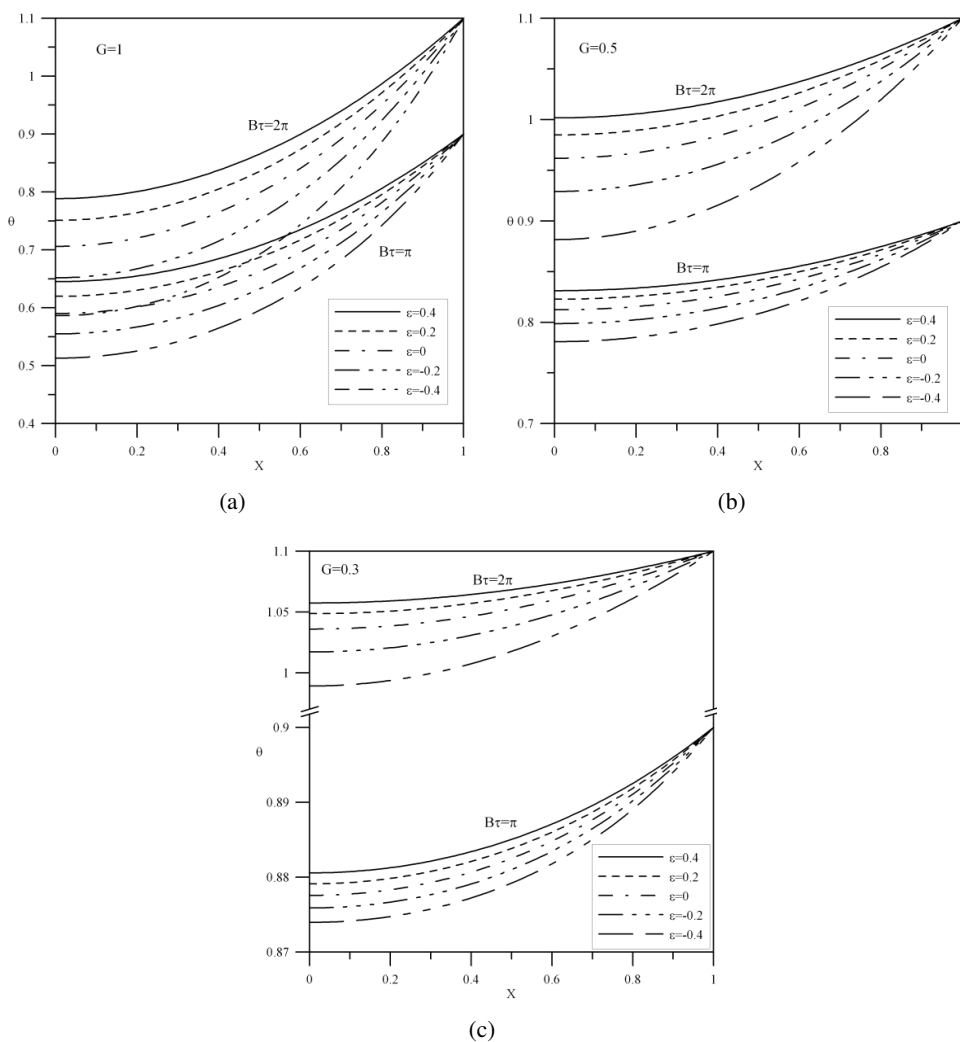


Figure 4: (a) Effect of parameter ε and $B\tau$ on the axial temperature distribution for $G = 1.0$. (b) Effect of parameter ε and $B\tau$ on the axial temperature distribution for $G = 0.5$. (c) Effect of parameter ε and $B\tau$ on the axial temperature distribution for $G = 0.3$

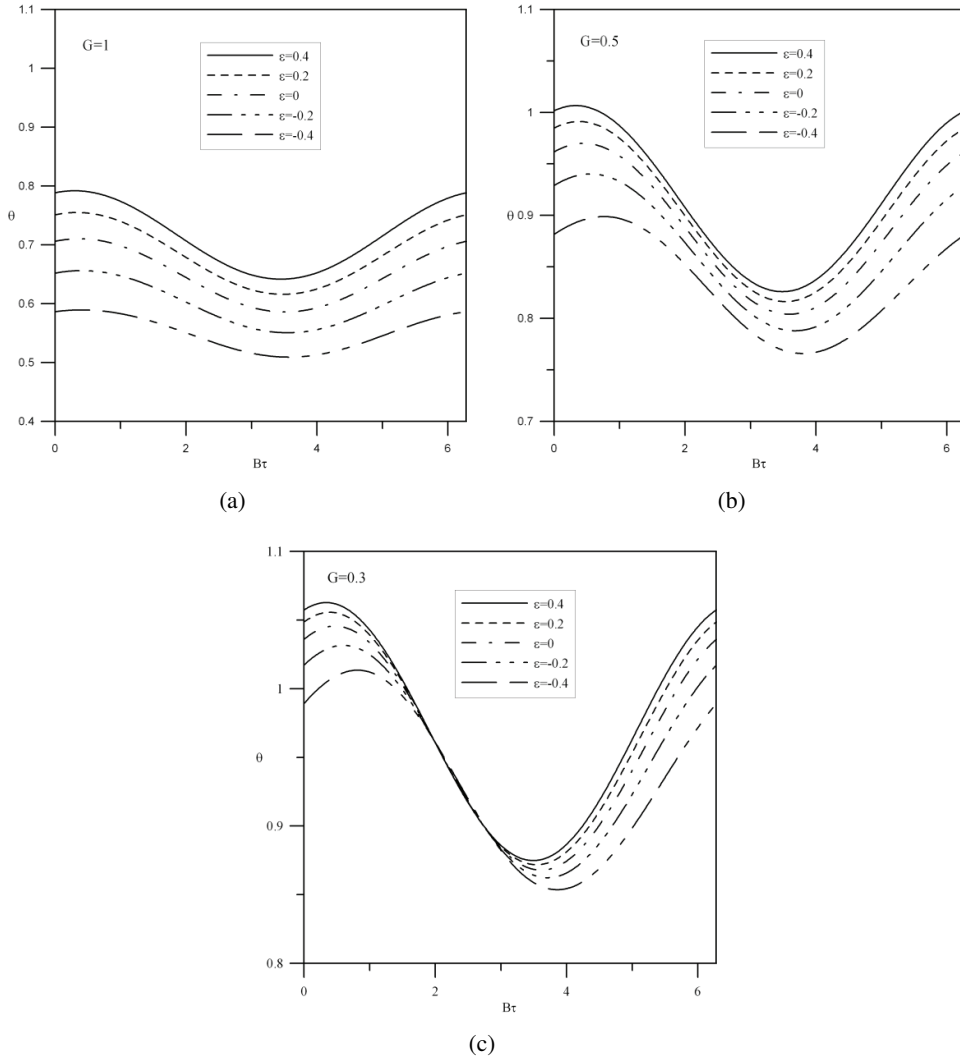


Figure 5: (a) Effect of parameter ε on the temperature oscillation at fin tip ($X=0$) for $G=1.0$. (b) Effect of parameter ε on the temperature oscillation at fin tip ($X=0$) for $G=0.5$. (c) Effect of parameter ε on the temperature oscillation at fin tip ($X=0$) for $G=0.3$.

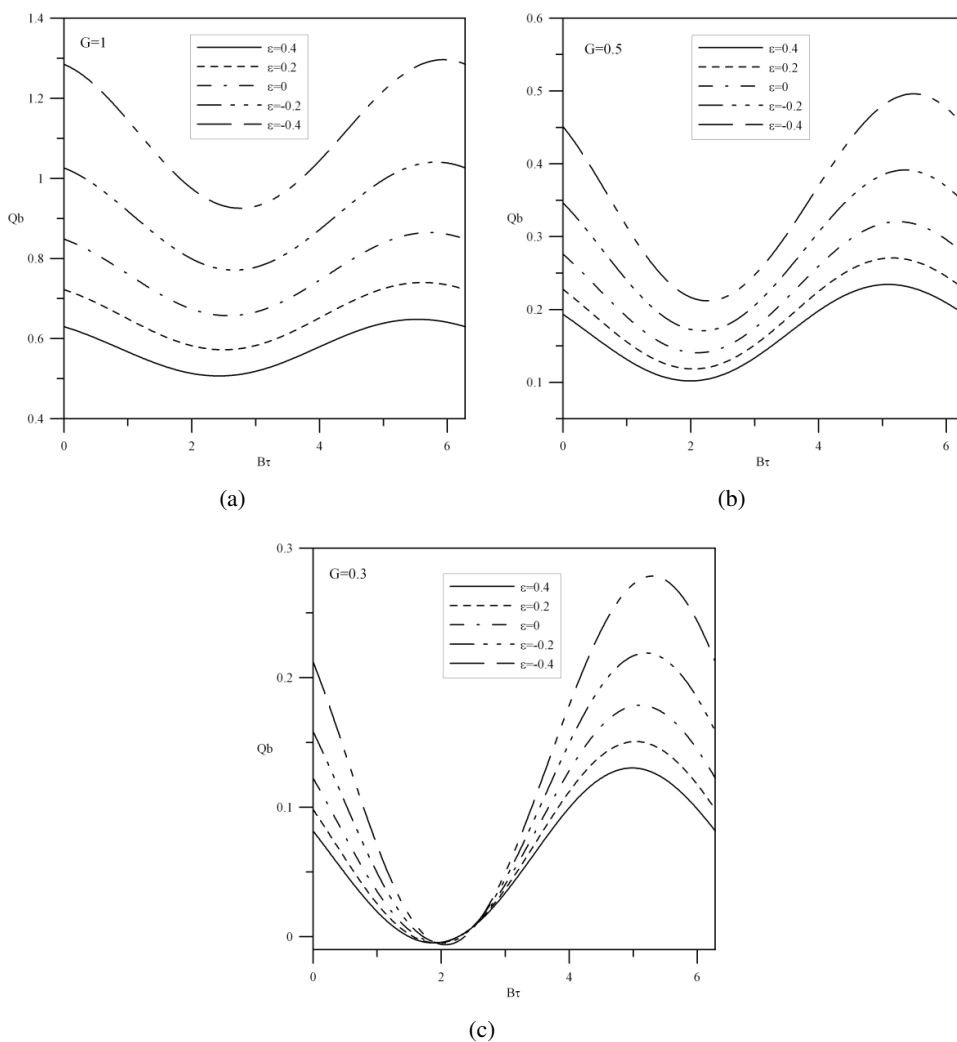


Figure 6: (a) The variation of the base temperature gradient Q_b with parameter ϵ at $G = 1.0$. (b) The variation of the base temperature gradient Q_b with parameter ϵ at $G = 0.5$. (c) The variation of the base temperature gradient Q_b with parameter ϵ at $G = 0.3$

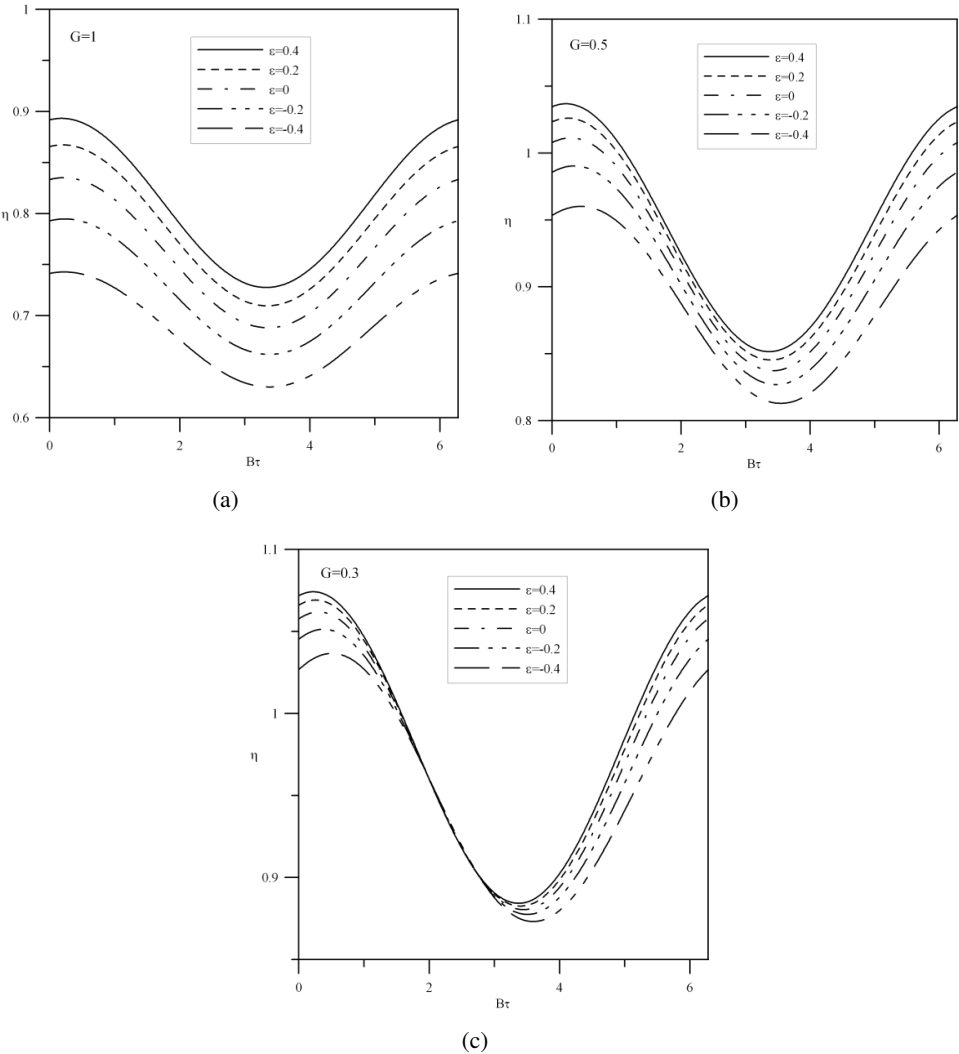


Figure 7: (a) Effect of parameter ε on instantaneous fin efficiency at $G = 1.0$. (b) Effect of parameter ε on instantaneous fin efficiency at $G = 0.5$. (c) Effect of parameter ε on instantaneous fin efficiency at $G = 0.3$

$\int_0^1 \theta(X, \tau) dX$ and the total energy transferred from the fin base can be indicated by the dimensionless temperature gradient Q_b with the definition $Q_b = \left. \frac{\partial \theta}{\partial X} \right|_{X=1}$. Fig. 6a-6c illustrate the effects of parameter ε with $G = 1.0, 0.5, 0.3$ on the total energy transferred. The amplitude of oscillation and phase angle are affected by the changing ε and G . Under a fixed parameter G , the parameter ε decreases, causing Q_b increase. Note that the phase angle changes significantly as ε varies from -0.4 to 0.4 . Fig. 7a-7c show the time-dependent fin efficiency under different values of ε and G . The amplitude of the oscillation increases significantly as the parameter G is reduced. In addition, for higher value of parameter ε lead to a better fin efficiency.

6 Conclusions

In this work, the HAM is applied to a non-linear, convective, rectangular fin with variable thermal conductivity and the oscillation base temperature problem. The results show that the HAM is effective and reliable. Different from all other analytic methods, the obtained solution offers many advantages over other methods. It provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter \hbar . Such a powerful method might be applied to solve other strongly non-linear or linear problems in science and engineering.

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