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A Direct Integral Equation Method for a Cauchy Problem for the Laplace Equation in 3-Dimensional Semi-Infinite Domains

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Abstract: We consider a Cauchy problem for the Laplace equation in a 3-dimensional semi-infinite domain that contains a bounded inclusion. The canonical situation is the upper half-space in $\mathbb{R}^3$ containing a bounded smooth domain. The function value of the solution is specified throughout the plane bounding the upper half-space, and the normal derivative is given only on a finite portion of this plane. The aim is to reconstruct the solution on the surface of the bounded inclusion. This is a generalisation of the situation in Chapko and Johansson (2008) to three-dimensions and with Cauchy data only partially given. We represent the solution in terms of a sum of a layer potential over the surface over the inclusion with an unknown density and a layer potential involving a Green's function and a known density (the given data on the plane). The Cauchy problem is then reduced to identifying the unknown density. To construct it, we match up the data on the finite portion of the plane, where both function values and the normal derivative are specified, and this gives rise to a integral equation of the first kind over the (bounded) surface of the inclusion having a smooth kernel. We show that this boundary integral equation is uniquely solvable for a certain class of data in the usual Sobolev and Hölder type spaces. To numerically solve this equation, we employ Weinert's method [Wienert (1990)]. This involves rewriting the integral equation over the unit sphere under the assumption that the surface of the inclusion can be mapped one-to-one to the unit sphere. The density is then represented in terms of a linear combination of spherical harmonics, and this generates a linear system to solve for the coefficients in this representation. Due to the ill-posedness of the Cauchy problem, Tikhonov regularization is incorporated. Numerical results are given as well, showing that accurate reconstructions of the solution and its normal derivative can be obtained on the surface of the inclusion with small computational effort. We also investigate the case when the normal derivative is given throughout the plane and

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the function value is only specified at a finite portion, and compare the accuracy of the reconstructions.

Keywords: Laplace equation; Cauchy problem; Semi-infinite 3D domain; Single- and double layer potentials; Green's function; Integral equation of the first kind; Tikhonov regularization; Discrete projection method.

1 Introduction

The Cauchy problem for the Laplace equation, i.e. the problem of reconstructing a harmonic function from knowledge of the function and its normal derivative on an arc (in $\mathbb{R}^2$) or surface (in $\mathbb{R}^3$) goes back to Hadamard (1923). Hadamard showed that the Cauchy problem is ill-posed, i.e. the solution does not depend continuously on the data; in fact, for perturbed data, there might not even be a solution. Criteria on the data to guarantee the existence of a solution in $\mathbb{R}^3$ are given in Johnson (1935). Uniqueness is clear from Carleman (1939); Calderón (1958). Formulas for the solution, i.e. extension of an analytic function from Cauchy data given on an arc were presented in Carleman (1926). However, these formulas are unstable and appear to be not suitable for numerical computations. In Lavrent'ev (1956); Mergelyan (1956); Pucci (1955), investigations and stable formulas were presented for the Cauchy problem in two and three dimensions. Since then, a large number of publications for the Cauchy problem for the Laplace equation have been presented, in particular on the issue of stable numerical calculation of the solution, see for example, Berntsson and Eldén (2001); Cao, Klibanov and Pereverzev (2009); Dinh Nho Hào (1998); Ingham, Yuan and Han (1991); Karageorghis, Lesnic and Marin (2011); Kubo (1994); Lavrent'ev (1967); Li and Syngellakis (1995); Payne (1975); Reinhardt, Dinh Nho Hào and Han (1999); Tarchanov (1999) and the references therein. Recently, iterative methods based on the ideas in Kozlov and Maz'ya (1989) have been popular, some of these works are listed in Helsing and Johansson (2010). Also, non-linear problems have been investigated Avdonin, Kozlov, Maxwell and Truffer (2009); Baravdish and Svensson (2011); Egger and Leitão (2009) (for more on the development of the theory of ill-posed problems and their stable solution, see for example Engl, Hanke, and Neubauer (1996); Isakov (1998); Kirsch (1996); Tikhonov and Arsenin (1977); Vainikko and Veretennikov (1986)).

The large number of publications related to the Cauchy problem for the Laplace equation is partly due to the importance of this problem in engineering applications. It appears, for example, in cardiology Colli, Franzone, and Magenes (1979), corrosion detection Cakoni, Kress and Schuft (2010), electrostatics van Berkel and Lionheart (2007), geophysics Glasko (1971), leak identification Escriva, Baranger and Tlatli (2007), non-destructive testing Alessandrini (1993) and plasma physics Gorenflo (1965).

However, from the numerical side, it is mainly results in two-dimensional bounded domains that have been presented. Note though that engineering problems posed in unbounded domains are common and important, and include the irrotational flow of an incompressible fluid exterior to a body Hess and Smith (1967), heat flow in the oceans Shih (1971) and the distribution of stresses in an infinite medium with holes or inclusions Chen and Wu (2007). The surrounding media can then be treated as infinite, thus leading to unbounded domain problems. From a theoretical point of view, partial differential equations in unbounded domains are challenging in itself and non-standard function spaces are needed to prove properties of solutions, for an overview, see the introduction in Seager and Carey (1990).

Using standard domain discretisation methods such as the Finite Element Method (FEM) in unbounded domains, truncation of the solution domain is usually needed, leading to additional difficulties for the Cauchy problem. In Chapko and Johansson (2008), the authors presented numerical results for a Cauchy problem posed in an unbounded planar domain containing a bounded inclusion. The method was based on an integral equation technique in combination with a generalization of the alternating method Kozlov and Maz'ya (1989). The problem was reformulated as an integral equation over the bounded boundary of the inclusion, making it computationally efficient. In particular, no artificial boundary was needed. Integral equation methods somewhat into the above spirit have been used earlier in the literature for the Cauchy problem in bounded domains, see Cheng, Hon, Wei and Yamamoto (2001); Helsing and Johansson (2010); Hon and Li (2003); Zeb, Elliott, Ingham, and Lesnic (1997) (and in Marin and Lesnic (2005) for Helmholtz-type equations).

As mentioned in Chapko and Johansson (2008), the similar approach could potentially be generalized to unbounded domains in $\mathbb{R}^3$. In this paper, we shall undertake the challenging task to present and implement an integral equation approach and produce numerical results for a Cauchy problem in an unbounded domain in $\mathbb{R}^3$, and this is the main novelty the work.

To formulate the Cauchy problem to be studied, let $D_1 \subset \mathbb{R}^3$ be a semi-infinite region with boundary Λ and let D_2 be a simply connected bounded domain in $\mathbb{R}^3$ with boundary surface Γ, such that $\bar{D_2} \subset D_1$. We let $D = D_1 \setminus \bar{D}_2$.

Assume that we have stationary heat conduction and let the temperature function $u \in C^2(D) \cap C^1(\bar{D})$ satisfy the Laplace equation

$$\Delta u = 0 \quad \text{in } D \tag{1}$$

and the regularity condition

$$u(x) = o(|x|^{-1}), \quad x \in D, \quad |x| \to \infty. \tag{2}$$

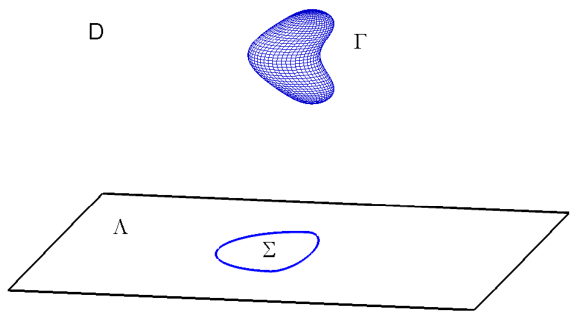


Figure 1: A semi-infinite solution domain D bounded by the plane Λ (with finite portion Σ) and the inclusion with boundary Γ.

Compared with Chapko and Johansson (2008), we not only generalize that work to higher dimensions but we shall also consider the more realistic case when Cauchy data is only partially known; the corresponding two linear inverse problems that we consider are:

A) *Neumann data measurement*. Let the function (temperature) f_Λ be given on the exterior boundary Λ together with knowledge of the heat flux g_Σ on the finite portion Σ of Λ, that is (1) is supplied with the boundary conditions

$$u = f_\Lambda \quad \text{on } \Lambda \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = g_\Sigma \quad \text{on } \Sigma. \tag{3}$$

The inverse problem we are concerned with is: Find the corresponding Cauchy data u and $\frac{\partial u}{\partial \nu}$ on the boundary Γ of the inclusion.

B) *Dirichlet data measurement*. Let the function (heat flux) g_Λ be given on the exterior boundary Λ together with knowledge of the temperature f_Σ on the finite portion Σ, that is (1) is supplied with the boundary conditions

$$\frac{\partial u}{\partial \nu} = g_\Lambda \quad \text{on } \Lambda \qquad \text{and} \qquad u = f_\Sigma \quad \text{on } \Sigma. \tag{4}$$

The inverse problem is also in this case: Find the Cauchy data u and $\frac{\partial u}{\partial \nu}$ on the boundary Γ.

For an illustration of the configuration and solution domain, see Figure 1. For simplicity, we assume that this is the configuration, i.e. D_1 is the half-space $x_3 > 0$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Note that other semi-infinite domains are possible as well, for example, an octant.

We shall represent the solution to each of these inverse problems as a sum of a layer potential over the surface of the inclusion with an unknown density and a layer potential involving a Green's function with a known density (the given data on Λ). The Cauchy problem is then reduced to identifying the unknown density. For the construction of it, we match up the additional data on the finite portion Σ. This generates a boundary integral equation of the first kind over the bounded surface Γ of the inclusion having a smooth kernel. This equation is uniquely solvable for a certain class of data in the usual Sobolev and Hölder type spaces.

To numerically solve the obtained boundary integral equation, we employ Weinert's method Wienert (1990). This is a timely approach since this method has attracted much attention recently, see Chapko, Johansson and Protsyuk (2011); Ganesh and Graham (2004); Ganesh, Graham and Sivaloganathan (1998); Graham and Sloan (2002). The discretisation first involves rewriting the boundary integral equation over the unit sphere under the assumption that the surface of the inclusion can be mapped one-to-one to the unit sphere. The density is then represented in terms of a linear combination of spherical harmonics, and this generates a linear system to solve for the coefficients in this representation. Due to the ill-posedness of the Cauchy problem, Tikhonov regularization is incorporated when solving this linear system.

Clearly, the assumption that the surface of the inclusion can be mapped one-to-one to unit sphere is a restriction of our approach. There is however a sufficiently large class of such inclusions useful in applications, that have this property. Moreover, one can generalize and use differential geometry and assume that the inclusion is parametrized by surface patches of the unit sphere or one can even only numerically construct an approximation to such a map. This is though deferred to future work.

For the outline of this work, in Chapter 2, we introduce some notation and function spaces, and discuss for which data there exist a solution to the Cauchy problem. In Chapter 3, we show how to reformulate the above two Cauchy problems in terms of boundary integral equations, see Theorem 3.2 and 3.5, and discuss the solvability of these equations. Moreover, we rewrite these equations over the unit sphere, see Theorem 3.3 and 3.6. In Chapter 4, we discretise these equations with Wienert's method [Wienert (1990)]. In Chapter 5, numerical results are presented, showing that accurate reconstructions of the solution and its normal derivative can be obtained on the boundary surface of the inclusion with small computational effort. We also compare the accuracy of the reconstructions for the two inverse problems described above.

2 Notation, function spaces and smoothness of the data

Let $L^2(D_0)$ be the standard L^2-space with the standard norm for a smooth domain D_0 in $\mathbb{R}^3$. Then $H^1(D_0)$ is the Sobolev space of real-valued functions in D_0 with finite norm given by $\|u\|^2_{H^1(D_0)} = \|u\|^2_{L^2(D_0)} + \|\nabla u\|^2_{L^2(D_0)}$, where $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. For trace spaces, we recall that the space of traces of functions from $H^1(D_0)$ on ∂D_0 is $H^{1/2}(\partial D_0)$. By $C^k(D_0)$, where k is a non-negative integer, we mean the set of functions differentiable up to order k and bounded in the usual sup-norm.

We then briefly discuss what we require in terms of smoothness of the given data for the above inverse problems. We assume that the data are smooth such that there exists a classical solution to each of the inverse problems. For compatibility conditions on the data to guarantee the existence of such a solution, see Johnson (1935). However, our proposed method will work for more general classes of data. Let us therefore indicate what one such class of data can be.

For unbounded domains, it is known that for given L^2-data, the Dirichlet or Neumann problem for the Laplace equation in the half-space $\mathbb{R}^3_+$ can have a solution which is not in $L^2(\mathbb{R}^3_+)$ although its gradient is. A natural framework is then to use weighted spaces. Let $L^{2,1}(\mathbb{R}^3_+)$ be the weighted L^2-space with weight $(1+|x|^2)^{-1/2}$. The space $W^{1,0}(\mathbb{R}^3_+)$ consists of functions with generalized derivatives of order ≤ 1, such that

$$\|u\|^2_{W^{1,0}(\mathbb{R}^3_+)} = \|u\|^2_{L^{2,1}(\mathbb{R}^3_+)} + \|\nabla u\|^2_{L^2(\mathbb{R}^3_+)}$$

is finite. Functions in $W^{1,0}(\mathbb{R}^3_+)$ have a well-defined trace on $\mathbb{R}^2$ in the trace space $W^{1/2,0}(\mathbb{R}^2)$ and such elements have in particular a finite $L^{2,1}(\mathbb{R}^2)$ norm. Moreover, the Dirichlet problem in the half-space $\mathbb{R}^3_+$ for the Laplace equation, with data in the trace space $W^{1/2,0}(\mathbb{R}^2)$ has a unique solution in $W^{1,0}(\mathbb{R}^3_+)$, see Theorem 6 in Boulmezaoud (2003). Similarly, the Neumann problem for the Laplace equation in $\mathbb{R}^3_+$ has a unique solution in $W^{1,0}(\mathbb{R}^3_+)$ for data in the dual space $W^{-1/2,0}(\mathbb{R}^2)$, see Theorem 9 in Boulmezaoud (2003).

Let us then show that the above inverse problem with Dirichlet data measurement on the finite portion Σ is solvable for a large class of data. Let u be the solution to the Neumann problem in $\mathbb{R}^3_+$ for a given g_Λ in $W^{-1/2,0}(\mathbb{R}^2)$ and let f_Σ be the restriction of u to Σ. Varying g_Λ over $W^{-1/2,0}(\mathbb{R}^2)$, we claim that we get a dense set of elements f_Σ in $L^2(\Sigma)$. Assume not, then there exists an element h in $L^2(\Sigma)$ with $(f_\Sigma, h) = 0$ (with $(\cdot,\cdot)$ being the inner product in $L^2(\Sigma)$), for every f_Σ constructed in the above manner. Since smooth functions are dense in L^2, we adjust the situation such that we can extend h by 0 to an element $\tilde{h}$ in $W^{-1/2,0}(\mathbb{R}^2)$. Let v be the solution to the Neumann problem for the Laplace equation with the extension of h as data. From the definition of a solution and since Green's formula is valid, see

Theorem 1 in Boulmezaoud (2003), we find that

$$\int_{\mathbb{R}^2} v \frac{\partial u}{\partial \nu}\, ds = \int_{\mathbb{R}^2} u \frac{\partial v}{\partial \nu}\, ds,$$

from which it follows that

$$\int_{\mathbb{R}^2} v g_\Lambda \, ds = \int_{\Sigma} f_\Sigma h \, ds = 0 \tag{5}$$

since $\tilde{h}$ is zero outside Σ and also due to the assumption $(f_\Sigma, h) = 0$. Clearly, this is a contradiction, since we are free to vary g_Λ and can make sure that the first integral in (5) is not zero. This in turns implies that the Cauchy problem is solvable for a dense set of data with g_Λ taken from the space $W^{-1/2,0}(\mathbb{R}^2)$ and $f_\Sigma \in L^2(\Sigma)$. Similar considerations holds for the other inverse problem.

Thus, it is justifiable to make the following assumptions. For the Neumann data measurement case, we assume that the data are given (with $g_\Sigma \in L^2(\Sigma)$) and compatible such that there exists a solution to the inverse problem (interpreted in the classical or weak solution sense). Similarly, for the Dirichlet data measurement case, we assume that the data are given (with $f_\Sigma \in L^2(\Sigma)$) and compatible such that there exists a solution to the inverse problem (interpreted in the classical or weak solution sense). We shall only use f_Λ and g_Λ that are continuous functions possibly with a growth condition imposed, although, as shown above, more general functions from the relevant trace spaces can be used.

3 Integral equation formulations

In this section, we shall reformulate the inverse problems in terms of boundary integral equations.

3.1 Neumann data measurement

By G we denote the Green's function for the equation $\Delta u = 0$ in D_1 in the case of a Dirichlet boundary condition on Λ, that is, G is defined for all $x \neq y$ in $\bar{D}_1$ and of the form

$$G(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|} - \tilde{G}(x,y),$$

where, for fixed $y \in D_1$, the function $\tilde{G}$ satisfies the Laplace equation in D_1 with respect to x and $G(\cdot\,,y) = 0$ on Λ. The solution w satisfying $\Delta w = 0$ in D_1 and the Dirichlet condition $w = f_\Lambda$ on Λ can be represented in the form

$$w(x) = -\int_\Lambda \frac{\partial G(x,y)}{\partial \nu(y)}\, f_\Lambda(y)\, ds(y), \quad x \in D_1. \tag{1}$$

This is a well-studied representation and if f_Λ is continuous and bounded, then w is a classical solution in D_1. One can also consider other types of spaces such as L^p-spaces, see further Chapter 3 Section 2 in Stein (1970), Finkelstein and Scheinberg (1975), Gardiner (1981) and Chapter 4 in Axler, Bourdon and Ramey (1992).

Seeking the unique solution of (1)–(3) in the form

$$u(x) = \int_\Gamma G(x,y)\phi(y)\,ds(y) + w(x), \quad x \in D \tag{2}$$

and taking into account the given normal derivative on Σ, leads to the following integral equation of the first kind with a smooth kernel

$$\int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)}\phi(y)\,ds(y) = g_\Sigma(x) - \frac{\partial w}{\partial \nu}(x), \quad x \in \Sigma, \tag{3}$$

to be solved for the unknown density ϕ. Since (1) is a classical solution provided that $f_\Lambda(y)$ is bounded and continuous, the normal derivative is well-defined in $L^2(\Sigma)$ according to the trace theorem. More generally, we let $X(\Lambda)$ be a function space for which the representation (1) makes sense and for which the normal derivative on Σ is well-defined in $L^2(\Sigma)$.

We introduce the operator

$$(K\phi)(x) := \int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)}\,\phi(y)\,ds(y), \quad x \in \Sigma, \tag{4}$$

which is the normal derivative on Σ of the single-layer potential (with density on Γ).

Analogously to Chapko and Johansson (2012) we can prove the following theorem, which is of importance in connection with the regularization of the ill-posed integral equation (3) of the first kind. For the sake of completeness and since the proof in Chapko and Johansson (2012) was given in $\mathbb{R}^2$, we include a proof of the result here.

Theorem 3.1 *The operator $K : L^2(\Gamma) \to L^2(\Sigma)$ given by* (4) *is injective and has dense range.*

Proof. We first show that the operator is injective. Let $\phi \in L^2(\Gamma)$ satisfy $K\phi = 0$. Then for the function $u(x) = \int_\Gamma G(x,y)\phi(y)\,ds(y)$, where $x \in D$, we have $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on Σ. Therefore, by Holmgren's theorem, $u = 0$ in $\bar{D}$. Then, from the properties of the single-layer potential, we have that $u = 0$ also in the interior of Γ. Since $\phi = \frac{\partial u^-}{\partial \nu}|_\Gamma - \frac{\partial u^+}{\partial \nu}|_\Gamma$, we obtain that $\phi = 0$, that is K is injective.

To establish that the operator K has dense range, it suffices to show that the adjoint operator $K^* : L^2(\Sigma) \to L^2(\Gamma)$ given by

$$(K^* f)(x) = \int_\Sigma \frac{\partial G(x,y)}{\partial \nu(y)} f(y)\, ds(y), \quad x \in \Gamma$$

is injective. Let f satisfy $K^* f = 0$. Define the function $v(x) = \int_\Sigma \frac{\partial G(x,y)}{\partial \nu(y)} f(y)\, ds(y)$, for $x \in \mathbb{R}^3 \setminus \Sigma$. Clearly, we have $\Delta v = 0$ in $\mathbb{R}^3 \setminus \Sigma$, $v(x) = o(|x|^{-1})$, when $|x| \to \infty$, and $v = 0$ on Γ. The exterior Dirichlet problem in the exterior of Γ (with the condition imposed at infinity) for harmonic functions has a unique solution. In particular, zero data on Γ gives the trivial solution. Thus, since $v = 0$ on Γ and due to the condition at infinity, it follows that v coincides with the trivial solution in $\mathbb{R}^3 \setminus \Sigma$ (in the exterior of Γ), i.e. $v = 0$ in $\mathbb{R}^3 \setminus \Sigma$. Therefore, since $f = v^+|_\Sigma - v^-|_\Sigma$, we conclude that $f = 0$. We have then shown that K has dense range in $L^2(\Sigma)$. $\square$

Note that the denseness of the range for the operator K is in line with the observation in Section 2 that the Cauchy problems under study are solvable for a dense set of data in $L^2(\Sigma)$.

From properties of single-layer potentials we have the following integral representation for Cauchy data on the surface Γ.

Theorem 3.2 (Neumann measurement) *Let $f_\Lambda \in X(\Lambda)$ and $g_\Sigma \in L^2(\Sigma)$ be given. The value of the solution to* (1)–(3) *on the surface Γ of the inclusion is given by*

$$u(x) = \int_\Gamma G(x,y)\phi(y)\, ds(y) + w(x), \quad x \in \Gamma \tag{5}$$

and the normal derivative is given by

$$\frac{\partial u}{\partial \nu}(x) = -\frac{1}{2}\phi(x) + \int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)} \phi(y)\, ds(y) + \frac{\partial w}{\partial \nu}(x), \quad x \in \Gamma, \tag{6}$$

where w is given by (1) *and the density ϕ is constructed from equation* (3).

3.2 Rewriting the Neumann measurement case integral equation over the unit sphere

Assume that the boundary surface Γ of the inclusion can be bijectively mapped onto the unit sphere $\mathbb{S}^2$, i.e. there exists a one-to-one mapping $q : \mathbb{S}^2 \to \Gamma$ having a smooth Jacobian J_q. We can then rewrite the integral equations from the previous section over the unit sphere.

Taking into account the parametric representation of Γ, we reduce the integral equation (3), corresponding to the Neumann data measurement, to the following equation

$$\int_{\mathbb{S}^2} Q(x,\hat{y})\psi(\hat{y})J_q(\hat{y})\,ds(\hat{y}) = g(x), \quad x \in \Sigma. \tag{7}$$

Here, we used the following notation $\hat{y} = p(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$, $\psi(\hat{y}) = \phi(q(\hat{y}))$, $g(x) = g_\Sigma(x) - \frac{\partial w}{\partial \nu}(x)$ and $Q(x,\hat{y}) = \frac{\partial G}{\partial \nu(x)}(x,\hat{y})$. We can then rewrite the representations (5) and (6) for the Cauchy data over the unit sphere, and we have the corresponding parametric forms

$$u(\hat{x}) = \int_{\mathbb{S}^2} \left[\frac{R(\hat{x},\hat{y})}{|\hat{x}-\hat{y}|} - \tilde{Q}(\hat{x},\hat{y})\right] \psi(\hat{y})J_q(\hat{y})\,ds(\hat{y}) + w_1(\hat{x}), \quad \hat{x} \in \mathbb{S}^2 \tag{8}$$

and

$$\frac{\partial u}{\partial \nu}(\hat{x}) = -\frac{1}{2}\psi(\hat{x}) + \int_{\mathbb{S}^2} \left[\frac{R_1(\hat{x},\hat{y})}{|\hat{x}-\hat{y}|} - \tilde{Q}_1(\hat{x},\hat{y})\right] \psi(\hat{y})J_q(\hat{y})\,ds(\hat{y}) + w_2(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{9}$$

where

$$w_1(\hat{x}) = w(q(\hat{x})), \quad w_2(\hat{x}) = \frac{\partial w}{\partial \nu}(q(\hat{x})), \quad \tilde{Q}(\hat{x},\hat{y}) = \tilde{G}(q(\hat{x}),q(\hat{y})),$$

$$R(\hat{x},\hat{y}) = \frac{1}{4\pi}\frac{|\hat{x}-\hat{y}|}{|q(\hat{x})-q(\hat{y})|}, \quad R_1(\hat{x},\hat{y}) = \nu(q(\hat{x}))\cdot(q(\hat{x})-q(\hat{y}))R(\hat{x},\hat{y})$$

and $\tilde{Q}_1(\hat{x},\hat{y}) = \frac{\partial \tilde{G}}{\partial \nu(x)}(q(\hat{x}),q(\hat{y}))$.
To further simply the discretisation of these equations, we shall move the weak singularity in the corresponding integrals to the north pole $\hat{n} = (0,0,1)$. To do this, we consider the orthogonal transformation $T_{\hat{x}}$ such that $T_{\hat{x}}\hat{x} = \hat{n}$ for all $\hat{x} \in \mathbb{S}^2$, see Ganesh and Graham (2004); Ivanyshyn and Kress (2010); Wienert (1990). We also introduce an induced transformation $\mathscr{T}_{\hat{x}}$ on $C(\mathbb{S}^2)$ as

$$\mathscr{T}_{\hat{x}}\psi(\hat{y}) = \psi(T_{\hat{x}}^{-1}\hat{y}), \quad \hat{y} \in \mathbb{S}^2, \ \psi \in C(\mathbb{S}^2)$$

and its bivariate analogue

$$\mathscr{T}_{\hat{x}}\psi(\hat{y}_1,\hat{y}_2) = \psi(T_{\hat{x}}^{-1}\hat{y}_1, T_{\hat{x}}^{-1}\hat{y}_2), \quad \psi \in C(\mathbb{S}^2 \times \mathbb{S}^2).$$

We denote by $\tilde{\mathscr{R}}(\hat{x},\hat{y}) := R(\hat{x},\hat{y})J_q(\hat{y})$ and $\tilde{\mathscr{R}}_1(\hat{x},\hat{y}) := R_1(\hat{x},\hat{y})J_q(\hat{y})$. Since $|\hat{x}-\hat{y}| = |T_{\hat{x}}^{-1}(\hat{n}-\hat{\eta})| = |\hat{n}-\hat{\eta}|$ with $\hat{\eta} = T_{\hat{x}}\hat{y}$, the representations with Neumann data measurement (8) and (9) can be transformed into the following expressions.

Theorem 3.3 (Neumann data measurement) *The representation* (5) *for the value of the solution to* (1)–(3) *on the surface of the inclusion* Γ *can be rewritten over the unit sphere* $\mathbb{S}^2$*, using the above notation, as*

$$u(\hat{x}) = \int_{\mathbb{S}^2} \frac{\mathscr{T}_{\hat{x}}\tilde{\mathscr{R}}(\hat{n},\hat{\eta})}{|\hat{n}-\hat{\eta}|}\mathscr{T}_{\hat{x}}\psi(\hat{\eta})\,ds(\hat{\eta}) - \int_{\mathbb{S}^2}\tilde{Q}(\hat{x},\hat{y})\psi(\hat{y})J_q(\hat{y})\,ds(\hat{y}) + w_1(\hat{x}), \qquad (10)$$

and similarly the equation (6) *for the normal derivative can be written over* $\mathbb{S}^2$ *as*

$$\frac{\partial u}{\partial \nu}(\hat{x}) = -\frac{1}{2}\psi(\hat{x}) + \int_{\mathbb{S}^2} \frac{\mathscr{T}_{\hat{x}}\tilde{\mathscr{R}}_1(\hat{n},\hat{\eta})}{|\hat{n}-\hat{\eta}|}\mathscr{T}_{\hat{x}}\psi(\hat{\eta})\,ds(\hat{\eta}) - \int_{\mathbb{S}^2}\tilde{Q}_1(\hat{x},\hat{y})\psi(\hat{y})J_q(\hat{y})\,ds(\hat{y}) + w_2(\hat{x}). \qquad (11)$$

3.3 Dirichlet measurement

Let N be the Green's function for the Laplace equation in D_1 in the case of a Neumann boundary condition on Λ, that is, N is defined for all $x \neq y$ in $\bar{D}_1$ and of the form

$$N(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|} + \tilde{N}(x,y),$$

where, for fixed $y \in D_1$, the function $\tilde{N}$ satisfies the Laplace equation in D_1 with respect to x and $\frac{\partial N(\cdot,y)}{\partial \nu(y)} = 0$ on Λ.

The solution $\mathbb{S}^2$ to the Neumann problem in D_1 with boundary data $\partial\omega/\partial\nu = g_\Lambda$ on Λ can be represented in the form

$$\omega(x) = \int_\Lambda N(x,y)g_\Lambda(y)\,ds(y), \quad x \in D_1. \qquad (12)$$

This is also a well-studied representation and if $g_\Lambda(y)$ is continuous and satisfying a growth condition, then ω is a classical solution in D_1. Furthermore, if $g_\Lambda \in L^{2,1}(\mathbb{R}^2)$, then ω is a weak solution in $W^{1,0}(\mathbb{R}^3_+)$. One can also consider other types of spaces such as L^p-spaces, see further Gardiner (1981); Shu, Tanaka and Yanagishita (2011). We let $Y(\Lambda)$ be a function space for which the representation (12) makes sense and has a well-defined trace in $L^2(\Sigma)$.

We then search for the unique solution of the Cauchy problem (1),(2),(4) in the form

$$u(x) = \int_\Gamma N(x,y)\upsilon(y)\,ds(y) + \omega(x), \quad x \in D, \qquad (13)$$

where the unknown density υ satisfies the integral equation of the first kind with a smooth kernel,

$$\int_\Gamma N(x,y)\upsilon(y)\,ds(y) = f_\Sigma(x) - \omega(x), \quad x \in \Sigma. \tag{14}$$

Due to the assumption on the Cauchy data in particular since $g_\Lambda \in Y(\Lambda)$, the left-hand side is in $L^2(\Sigma)$.

For the operator

$$(S\upsilon)(x) := \int_\Gamma N(x,y)\upsilon(y)\,ds(y), \quad x \in \Sigma, \tag{15}$$

we have the following properties, which can be proved analogously to Theorem 3.1.

Theorem 3.4 *The operator $S : L^2(\Gamma) \to L^2(\Sigma)$ defined by* (15) *is injective and has dense range.*

Again, this is as expected since it is shown in Section 2 that the Neumann measurement case has a solution for a dense set of data in $L^2(\Sigma)$.

The Cauchy data on Γ can by calculated as follows.

Theorem 3.5 (Dirichlet measurement) *Let $g_\Lambda \in Y(\Lambda)$ and $f_\Sigma \in L^2(\Sigma)$ be given. The value of the solution to* (1),(2),(4) *on the surface Γ of the inclusion is given by*

$$u(x) = \int_\Gamma N(x,y)\upsilon(y)\,ds(y) + \omega(x), \quad x \in \Gamma \tag{16}$$

and the normal derivative is given by

$$\frac{\partial u}{\partial \nu}(x) = -\frac{1}{2}\upsilon(x) + \int_\Gamma \frac{\partial N(x,y)}{\partial \nu(x)}\upsilon(y)\,ds(y) + \frac{\partial \omega}{\partial \nu}(x), \quad x \in \Gamma, \tag{17}$$

where ω is given by (12) *and the density υ is constructed from equation* (14).

3.4 Rewriting the Dirichlet measurement case integral equation over the unit sphere

Using the above notation, in the case of a Dirichlet data measurement, we obtain the parametrized integral equation over the unit sphere $\mathbb{S}^2$,

$$\int_{\mathbb{S}^2} L(x,\hat{y})\vartheta(\hat{y})J_q(\hat{y})\,ds(\hat{y}) = f(x), \quad x \in \Sigma, \tag{18}$$

where $L(x,\hat{y}) = N(x,\hat{y})$, $\vartheta(\hat{y}) = \upsilon(q(\hat{y}))$ and $f(x) = f_\Sigma(x) - \omega(x)$.

The Cauchy data on Γ have in this case the following parametrization

$$u(\hat{x}) = \int_{\mathbb{S}^2} \left[\frac{R(\hat{x},\hat{y})}{|\hat{x}-\hat{y}|} + \tilde{L}(\hat{x},\hat{y}) \right] \vartheta(\hat{y}) J_q(\hat{y})\, ds(\hat{y}) + \omega_1(\hat{x}), \quad \hat{x} \in \mathbb{S}^2$$

and

$$\frac{\partial u}{\partial \nu}(\hat{x}) = -\frac{1}{2}\vartheta(\hat{x}) + \int_{\mathbb{S}^2} \left[\frac{R_1(\hat{x},\hat{y})}{|\hat{x}-\hat{y}|} + \tilde{L}_1(\hat{x},\hat{y}) \right] \vartheta(\hat{y}) J_q(\hat{y})\, ds(\hat{y}) + \omega_2(\hat{x}), \quad \hat{x} \in \mathbb{S}^2,$$

where we used the notation

$$\omega_1(\hat{x}) = \omega(q(\hat{x})), \quad \omega_2(\hat{x}) = \frac{\partial \omega}{\partial \nu}(q(\hat{x})),$$

$$\tilde{L}(\hat{x},\hat{y}) = \tilde{N}(q(\hat{x}),q(\hat{y})), \quad \tilde{L}_1(\hat{x},\hat{y}) = \frac{\partial \tilde{N}}{\partial \nu(x)}(q(\hat{x}),q(\hat{y})).$$

We can again move the singularity in each of these equations to the north pole.

Theorem 3.6 (Dirichlet data measurement) *The integral representation* (16) *for the value of the solution to* (1),(2),(4) *on the surface of the inclusion* Γ *can be rewritten over the unit sphere* $\mathbb{S}^2$*, using the above notation, as*

$$u(\hat{x}) = \int_{\mathbb{S}^2} \frac{\mathscr{T}_{\hat{x}} \tilde{\mathscr{R}}(\hat{n},\hat{\eta})}{|\hat{n}-\hat{\eta}|} \mathscr{T}_{\hat{x}} \vartheta(\hat{\eta})\, ds(\hat{\eta}) + \int_{\mathbb{S}^2} \tilde{L}(\hat{x},\hat{y}) \vartheta(\hat{y}) J_q(\hat{y})\, ds(\hat{y}) + \omega_1(\hat{x}), \qquad (19)$$

and the formula (17) *for the normal derivative is given by*

$$\frac{\partial u}{\partial \nu}(\hat{x}) =$$
$$-\frac{1}{2}\vartheta(\hat{x}) + \int_{\mathbb{S}^2} \frac{\mathscr{T}_{\hat{x}} \tilde{\mathscr{R}}_1(\hat{n},\hat{\eta})}{|\hat{n}-\hat{\eta}|} \mathscr{T}_{\hat{x}} \vartheta(\hat{\eta})\, ds(\hat{\eta}) + \int_{\mathbb{S}^2} \tilde{L}_1(\hat{x},\hat{y}) \vartheta(\hat{y}) J_q(\hat{y})\, ds(\hat{y}) + \omega_2(\hat{x}). \qquad (20)$$

4 Discretisation and Tikhonov regularization

We shall describe how to discretise the equations (10)–(11) and (19)–(20), and how to solve the obtained linear systems in a stable way.

4.1 Quadrature rules

The following quadrature is used for continuous integrands

$$\int_{\mathbb{S}^2} f(\hat{y})\, ds(\hat{y}) \approx \sum_{\rho'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'} \tilde{a}_{s'} f(p(\theta_{s'}, \varphi_{\rho'})), \qquad (1)$$

where $\varphi_{\rho'} = \rho'\pi/(n'+1)$, $\theta_{s'} = \arccos z_{s'}$ with $z_{s'}$ being the zeros of the Legendre polynomials $P_{n'+1}$, $\tilde{a}_{s'} = 2(1-z_{s'}^2)/((n'+1)P_{n'}(z_{s'}))^2$ and $\tilde{\mu}_{\rho'} = \pi/(n'+1)$.

For the case of a weak singularity, we have the quadrature rule

$$\int_{\mathbb{S}^2} \frac{f(\hat{y})}{|\hat{n}-\hat{y}|}\, ds(\hat{y}) \approx \sum_{\rho'=0}^{2n'+1}\sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'}\tilde{b}_{s'} f(p(\theta_{s'},\varphi_{\rho'})) \tag{2}$$

with weights

$$\tilde{b}_{s'} = \frac{\pi\tilde{a}_{s'}}{n'+1}\sum_{i=0}^{n'} P_i(z_{s'}).$$

Both quadratures are obtained by approximation of the regular part of the integrand via spherical harmonics and then employing exact integration. Note that according to results in Ganesh and Graham (2004); Wienert (1990) these quadrature rules have super-algebraic convergence order.

We also have several integrals over the plane ($\mathbb{R}^2$). For a continuous integrand we suggest the sinc-quadrature Stenger (1993)

$$\int_{\mathbb{R}^2} f(y)\, ds(y) \approx h_\infty^2 \sum_{i,j=-M_1}^{M_1} f(ih_\infty, jh_\infty). \tag{3}$$

If the function f is from the standard Hardy space and has the following asymptotic behaviour $|f(x_1,x_2)| \leq \hat{C}e^{-\sigma_1|x_1|}e^{-\sigma_2|x_2|}$ with $\hat{C} > 0$, $\sigma_1 > 0$ and $\sigma_2 > 0$, then the approximation (3) converges exponentially.

Let $\Sigma = \{x = (x_1,x_2), a \leq x_1 \leq b, c \leq x_2 \leq d\}$ and $\Sigma_{n_2}^{n_1} = \{\check{x}_{ij} = (\tilde{x}_{1i},\tilde{x}_{2j}), \tilde{x}_{1i} = a + h_1 i, h_1 = (b-a)/\tilde{n}_1, i = 0,\ldots,\tilde{n}_1, \tilde{x}_{2j} = c + h_2 j, h_2 = (d-c)/\tilde{n}_2, j = 0,\ldots,\tilde{n}_2\}$.

For the integral over the plane having a weak singularity, we make the following transformation

$$\int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\, ds(y) = \int_{\mathbb{R}^2\setminus\Sigma} \frac{f(y)}{|x-y|}\, ds(y) + \int_{\Sigma} \frac{f(y)}{|x-y|}\, ds(y), \quad x \in \Sigma.$$

Then for the first integral that does not have a singularity we can do the corresponding substitutions and reduce it to the case of (3). Using piecewise constant approximation for the smooth function f in the second integral on subdividing $\Sigma_{n_2}^{n_1}$ and after exact integration, we obtain the following quadrature rule

$$\int_{\Sigma} \frac{f(y)}{|x-y|}\, ds(y) \approx \sum_{i=1}^{\tilde{n}_1}\sum_{j=1}^{\tilde{n}_2} f(\check{x}_{ij})R_{ij}(x), \quad x \in \Sigma,$$

where

$$R_{ij}(x) = \int_{\tilde{x}_{1,i-1}}^{\tilde{x}_{1i}} \int_{\tilde{x}_{2,j-1}}^{\tilde{x}_{2j}} \frac{dy_1 dy_2}{|x-y|} = (\tilde{x}_{2j} - x_2) \ln \frac{\tilde{x}_{1i} - x_1 + |\check{x}_{ij} - x|}{\tilde{x}_{1,i-1} - x_1 + |\check{x}_{i-1,j} - x|} +$$

$$(\tilde{x}_{1i} - x_1) \ln \frac{\tilde{x}_{2j} - x_2 + |\check{x}_{ij} - x|}{\tilde{x}_{2,j-1} - x_2 + |\check{x}_{i,j-1} - x|} + (x_2 - \tilde{x}_{2,j-1}) \ln \frac{\tilde{x}_{1i} - x_1 + |\check{x}_{i,j-1} - x|}{\tilde{x}_{1,i-1} - x_1 + |\check{x}_{i-1,j-1} - x|} +$$

$$(x_1 - \tilde{x}_{1,i-1}) \ln \frac{\tilde{x}_{2j} - x_2 + |\check{x}_{i-1,j} - x|}{\tilde{x}_{2,j-1} - x_2 + |\check{x}_{i-1,j-1} - x|}.$$

An analogous approach is employed for hypersingular integrals over the plane. The quadrature for integrals over the domain Σ has in this case the following form

$$\int_{\Sigma} \frac{f(y)}{|x-y|^3} \, ds(y) \approx \sum_{i=1}^{\tilde{n}_1} \sum_{j=1}^{\tilde{n}_2} f(\check{x}_{ij}) F_{ij}(x), \quad x \in \Sigma, \tag{4}$$

where (see Nazarchuk and Kulynych (2009))

$$F_{ij}(x) = \int_{\tilde{x}_{1,i-1}}^{\tilde{x}_{1i}} \int_{\tilde{x}_{2,j-1}}^{\tilde{x}_{2j}} \frac{dy_1 dy_2}{|x-y|^3} = \Delta_x \int_{\tilde{x}_{1,i-1}}^{\tilde{x}_{1i}} \int_{\tilde{x}_{2,j-1}}^{\tilde{x}_{2j}} \frac{dy_1 dy_2}{|x-y|} = \Delta R_{ij}(x), \quad x \notin \Sigma_{n_2}^{n_1}.$$

4.2 Linear system and regularization

Now, we return to the ill-posed integral equation (7). The use of quadrature rules (1), (3) and (4) lead to an approximating equation, which we reduce by collocation to the linear system

$$\sum_{\rho'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'} \tilde{a}_{s'} Q(x_{ij}, \hat{y}_{s',\rho'}) \tilde{\psi}(\hat{y}_{s',\rho'}) J_q(\hat{y}_{s',\rho'}) = \tilde{g}(x_{ij}), \quad x_{ij} \in \Sigma,$$

where $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$, $n_1 n_2 > 2(n'+1)^2$ and the right-hand side has the form

$$\tilde{g}(x) \;=\; g_{\Sigma}(x) - \frac{1}{2\pi} \sum_{i=1}^{\tilde{n}_1} \sum_{j=1}^{\tilde{n}_2} f_{\Lambda}(\check{x}_{ij}) F_{ij}(x) + \frac{h_\infty^2}{4\pi} \sum_{i,j=-M}^{M} f_{\Lambda}(ih_\infty, jh_\infty) \hat{G}(x, ih_\infty, jh_\infty) +$$

$$\frac{h_\infty^2}{4\pi} \sum_{\ell=1}^{4} \sum_{i,j=-M}^{M} f_{\Lambda}(y(\tilde{\psi}_{1\ell}(ih_\infty), \tilde{\psi}_{2\ell}(jh_\infty))) \frac{\tilde{\psi}'_{1\ell}(ih_\infty) \tilde{\psi}'_{2\ell}(jh_\infty)}{|x - y(\tilde{\psi}_{1\ell}(ih_\infty), \tilde{\psi}_{2\ell}(jh_\infty))|^3},$$

$$x \in \Sigma,$$

with the continuous kernel

$$\hat{G}(x,y) = \frac{\partial^2}{\partial x_3 \partial y_3}\left[\tilde{G}(x,y) - \frac{1}{4\pi}\frac{1}{|x-y^*|}\right].$$

Since the system is ill-posed we incorporate for its numerical solution Tikhonov regularization with regularization parameter $\lambda > 0$.

The approximation for the density ψ can be calculated via projection on the subspace of spherical harmonics

$$\tilde{\psi}(\hat{x}) = \sum_{\ell=0}^{n'} \sum_{|j|\le\ell} \sum_{\rho'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'} \tilde{a}_{s'} \tilde{\psi}(\hat{y}_{s',\rho'}) Y_{\ell,j}^R(\hat{y}_{s',\rho'}) Y_{\ell,j}^R(\hat{x}), \quad \hat{x} \in \mathbb{S}^2,$$

where

$$Y_{\ell,k}^R = \begin{cases} \sqrt{2}\mathrm{Im}\, Y_{\ell,|k|}, & 0 < k < \ell, \\ \mathrm{Re}\, Y_{\ell,|k|}, & k = 0, \\ \sqrt{2}\mathrm{Re}\, Y_{\ell,|k|}, & -\ell \le k < 0, \end{cases}$$

with the spherical harmonics $Y_{\ell,k}$ as defined in Abramowitz and Stegun (1972).

We can then find the approximation for the Cauchy data on Γ via the discretisation of (10) and (11). The approximation of the function value on Γ is given by

$$\tilde{u}(\hat{x}) = \sum_{\rho'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'} \left[\tilde{b}_{s'} \tilde{\mathscr{R}}(\hat{x}, T_{\hat{x}}\hat{y}_{s',\rho'}) \tilde{\psi}(T_{\hat{x}}\hat{y}_{s',\rho'}) - \tilde{a}_{s'} \tilde{Q}(\hat{x}, \hat{y}_{s',\rho'}) \tilde{\psi}(\hat{y}_{s',\rho'}) J_q(\hat{y}_{s',\rho'})\right] + \tilde{w}_1(\hat{x})$$

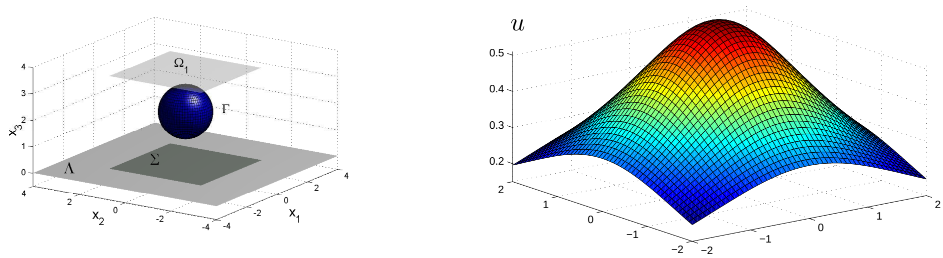
and the approximation of the normal derivative is

$$\frac{\partial \tilde{u}}{\partial \nu}(\hat{x}) = \sum_{\rho'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'} \left[\tilde{b}_{s'} \tilde{\mathscr{R}}_1(\hat{x}, T_{\hat{x}}\hat{y}_{s',\rho'}) \tilde{\psi}(T_{\hat{x}}\hat{y}_{s',\rho'}) - \tilde{a}_{s'} \tilde{Q}_1(\hat{x}, \hat{y}_{s',\rho'}) \tilde{\psi}(\hat{y}_{s',\rho'}) J_q(\hat{y}_{s',\rho'})\right] - \frac{1}{2}\tilde{\psi}(\hat{x}) + \tilde{w}_2(\hat{x}).$$

Here $\tilde{w}_1$ and $\tilde{w}_2$ are calculated using the sinc-quadrature rule (3) for the corresponding integrals (see (1)). The discretisation of the equation (14) and approximation of the Cauchy data (16) and (17) in the case of Dirichlet data measurement can be realized analogously.

5 Numerical examples

We use synthetic data for our numerical experiments, i.e. the Cauchy data on Λ are constructed as follows: the Dirichlet boundary value problem with boundary conditions $u = g$ on Γ and $u = f_\Lambda$ on Λ, for given boundary functions g and f_Λ, is numerically solved by the boundary integral equation approach Chapko, Johansson and Protsyuk (2011). Then we find the trace of the normal derivative of the solution on Σ (to generate the input data g_Σ). For the modelling of noisy input data (and to avoid the "inverse crime") random pointwise errors have been added to the values of the normal derivative g_Σ on Σ with the percentage given in terms of the L^2-norm.



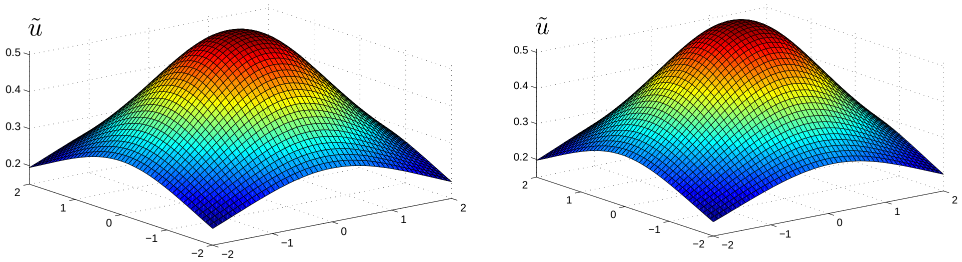
a) Domain and surfaces for Ex.1 b) Exact solution on the plane Ω_1

Figure 2: Input data for Example 1

5.1 Example 1

We assume that the semi-infinite region is a half-space $D_1 = \{x \in \mathbb{R}^3, x_3 > 0\}$ with the boundary $\Lambda = \{x \in \mathbb{R}^3, x_3 = 0\}$ and the inclusion D_2 is a ball with the boundary $\Gamma = \{p(\theta,\varphi) + (0,0,2), \theta \in [0,\pi], \varphi \in [0,2\pi]\}$ (see Fig.2a). The boundary functions are $g = 1$ on Γ, $f_\Lambda = 0$ on Λ and the data g_Σ on $\Sigma = \{x \in \mathbb{R}^3, -2 \le x_1, x_2 \le 2, x_3 = 0\}$ was generated by solving the corresponding direct problem.

We are interested to see whether it is possible to generate an accurate approximation also behind (above) the inclusion and shall therefore calculate the approximation on $\Omega_1 = \{x \in \mathbb{R}^3, -2 \le x_1, x_2 \le 2, x_3 = 3.5\}$. In Fig.2b is the exact solution $u|_{\Omega_1}$, generated by solving the direct problem. In Fig.3a and Fig.3b are the numerical solutions $\tilde{u}|_{\Omega_1}$, calculated via the outlined integral equation approach, for exact and 3% noisy input data g_Σ, respectively. Here, we used the following parameters: $n = 8$, $n_1 = n_2 = 50$ and $\lambda = 10^{-9}$ for the case of exact data and $\lambda = 10^{-6}$ for noisy data. The relative L^2- error on the domain Ω was calculated as $e_2(\Omega) = \|\tilde{u} - u\|_{2,\Omega}/\|u\|_{2,\Omega}$. Note that $e_2(\Gamma) = 0.05$ and $e_2(\Gamma) = 0.06$ for exact and noisy data, respectively.



(a) Exact data ($e_2(\Omega_1) = 0.024$) (b) 3% noisy data ($e_2(\Omega_1) = 0.036$)

Figure 3: The reconstruction of the solution on Ω_1 for Example 1

One can vary the above parameters and check the stability of the approximation. Making the mesh finer does not improve the results much further. The choice of the regularization parameter was made by trial and error, although there are rules on how to choose it that can be incorporated. Calculating the values of the approximation closer to where the Cauchy data is originally given renders, as one expects, an even better reconstruction. Choosing Σ larger also improves the approximation. Moreover, Ω_1 does not have to be parallel to Λ.

5.2 Example 2

We consider now an inclusion having a non-constant curvature and choose a cushion-shaped cavity (see Fig. 4a) with the parametrization

$$\Gamma = 0.8\sqrt{0.8 + 0.5(\cos 2\varphi - 1)(\cos 4\theta - 1)}\,p(\theta, \varphi) + (0, 0, 2), \tag{1}$$

where $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi]$.
The boundary data functions are given by

$$g(x) = x_1^2, \quad x \in \Gamma$$

and

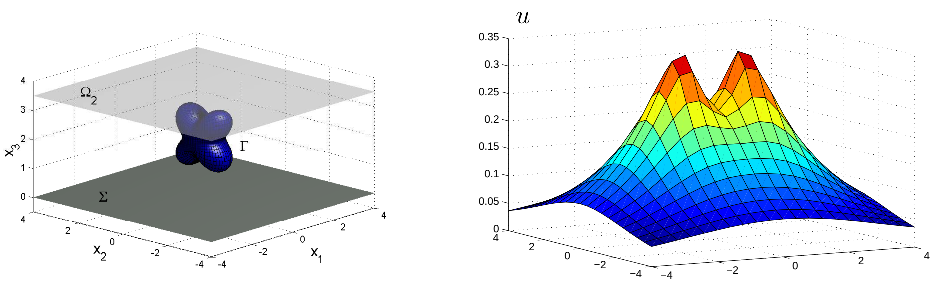
$$f_\Lambda(x) = 10\exp(-2|x|^2),\ x \in \Lambda$$

and the set $\Sigma \subset \Lambda$ is $\Sigma = \{x \in \mathbb{R}^3, -4 \leq x_1, x_2 \leq 4, x_3 = 0\}$.
Again, we shall show that we can generate an accurate approximation also behind (above) the inclusion and choose $\Omega_2 = \{x \in \mathbb{R}^3, -4 \leq x_1, x_2 \leq 4, x_3 = 3.5\}$. We demonstrate in Fig.4b, Fig.5a and Fig.5b the exact solution $u|_{\Omega_2}$, and the numerical solutions $\tilde{u}|_{\Omega_2}$ for exact and 3% noisy input data, calculated via the outlined integral

equation approach. Here, we used the following parameters: $n = 6$, $n_1 = n_2 = 20$, $M_1 = 40$ and $\lambda = 10^{-9}$ for the case of exact data and $\lambda = 10^{-6}$ for noisy data. The error of the approximation is slightly higher compared with the previous example, which is to be expected since the shape of the inclusion is more complicated.

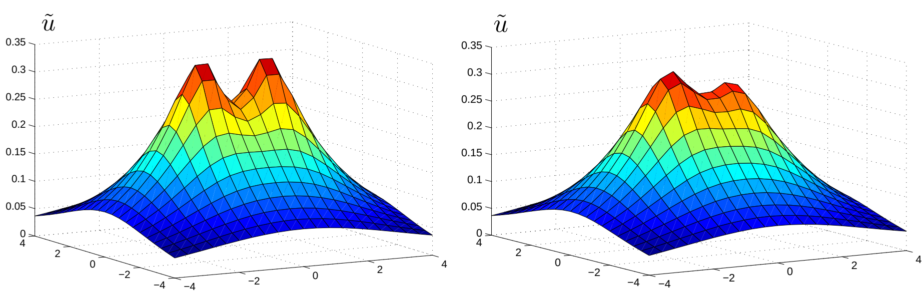
Again, as in the previous example, one can change the parameters and regions to conclude that the method is stable and the reconstructions behaves in the expected way with respect to those. The choice of the regularization parameter was also here made by trial and error.

Finally, we report that the numerical approximation for the other inverse problem, where instead the Dirichlet data is given on the finite Σ, behaves in the similar way for both of these examples.



a) Domain and surfaces for Ex.2 b) Exact solution on the plane Ω_2

Figure 4: Input data for Example 2



a) Exact data ($e_2(\Omega_2) = 0.014$) b) 3% noisy data ($e_2(\Omega_2) = 0.048$)

Figure 5: The reconstruction of the solution on Ω_2 for Example 2

6 Conclusion

We investigated a Cauchy problem for the Laplace equation in a 3-dimensional semi-infinite domain containing a bounded inclusion, where the function values is given on a plane and the normal derivative on a finite portion of this plane. An integral equation approach was presented for this problem, where the solution was represented in terms of a sum of a layer potential over the surface of the inclusion with an unknown density and a layer potential involving a Green's function and a known density. This makes it possible to reduce the Cauchy to a boundary integral equation (over a bounded surface) for identifying the unknown density. We showed that this integral equation is solvable for a dense set in the standard space of square integrable function over the surface of the inclusion. To numerically solve this equation, we employ Weinert's method [Wienert (1990)], and this involved rewriting the integral equation over the unit sphere under the assumption that the surface of the inclusion can be mapped one-to-one to the unit sphere. The unknown density is expressed in terms of a linear combination of spherical harmonics, which generated a linear system to solve for the coefficients in this representation. To solve this system Tikhonov regularization was incorporated, where the regularization parameter was chosen by trial and error. Two examples were numerically investigated, one where the inclusion was a ball and one where the inclusion had a non-constant curvature (cushion-shaped). The numerical results indicated the stability and accuracy of the proposed method, both for the function value and the normal derivative. A similar procedure and results were outlined for the Cauchy problem where instead the normal derivative was given on the plane and the function value on a finite portion of it.

References

Abramowitz, M., Stegun, I. A. (1972): *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Publications, New York.

Alessandrini, G. (1993): Stable determination of a crack from boundary measurements. *Proc. Roy. Soc. Edinburgh A*, vol. 123 no. 3, pp. 497–516.

Avdonin, S., Kozlov, V., Maxwell, D., Truffer, M. (2009): Iterative methods for solving a nonlinear boundary inverse problem in glaciology. *J. Inverse Ill-Posed Probl.* vol. 17 no. 3, pp. 239–258.

Axler, S., Bourdon, P., Ramey, W. (1992): *Harmonic Function Theory*. Springer-Verlag, New York.

Baravdish, G., Svensson, O. (2011): Image reconstruction with $p(x)$-parabolic equations, The Seventh International Conference on Inverse Problems in Engineering, (Eds. A J. Kassab and E. A. Divo), University of Central Florida, Orlando, Florida, USA, pp. 31–36.

Berntsson, F., Eldén, L. (2001): Numerical solution of a Cauchy problem for the Laplace equation. *Inverse Problems* vol. 17 no. 4, pp. 839–853.

Boulmezaoud, T. Z. (2003): On the Laplace operator and on the vector potential problems in the half-space: An approach using weighted spaces. Math. Methods Appl. Sci. vol. 26 no. 8, pp. 633–669.

Cakoni, F., Kress, R., Schuft, C. (2010): Integral equations for shape and impedance reconstruction in corrosion detection. *Inverse Problems* vol. 26 no. 9, 095012.

Calderón, A.-P. (1958): Uniqueness in the Cauchy problem for partial differential equations. *Amer. J. Math.* vol. 80 no. 1, pp. 16–36.

Cao, H., Klibanov, M. V., Pereverzev, S. V. (2009): A Carleman estimate and the balancing principle in the quasi-reversibility method for solving the Cauchy problem for the Laplace equation. *Inverse Problems* vol. 25 no 3, pp. 1–21.

Carleman, T. (1926): *Les Fonctions Quasi Analytiques, Gauthier-Villars*. Paris, (In French).

Carleman, T. (1939): Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes, (French). *Ark. Mat., Astr. Fys.* vol. 26 no. 17, pp. 1–9.

Chapko R., Johansson, B. T. (2008): An alternating boundary integral based method for a Cauchy problem for Laplace equation in semi-infinite domains. *Inverse Probl. Imaging*, vol. 3 no. 2, pp. 317–333.

Chapko, R., Johansson, B. T. (2012): On the numerical solution of a Cauchy problem for the Laplace equation via a direct integral equation approach. *Inverse Probl. Imaging* vol 6 no. 1, pp. 25-38.

Chapko, R., Johansson, B. T., Protsyuk, O. (2011): On an indirect integral equation approach for stationary heat transfer in semi-infinite layered domains in $\mathbf{R}^3$ with cavities. *J. Numer. Appl. Math.* vol. 105 no. 2 (2011), pp. 4–18.

Chen, J. T. and Wu, A. C. (2007): Null-field approach for the multi-inclusion problem under antiplane shears. *J. Appl. Mech.* vol. 74 no. 3, pp. 469–487.

Cheng, J., Hon, Y. C., Wei, T., Yamamoto, M. (2001): Numerical computation of a Cauchy problem for Laplace's Equation. *Z. Angew. Math. Mech.* vol. 81 no. 10, pp. 665–674.

Colli Franzone, P., Magenes, E. (1979): On the inverse potential problem of electrocardiology. *Calcolo* vol. 16 no., pp. 459–538.

Dinh Nho Hào (1998): *Methods for Inverse Heat Conduction Problems*. Peter Lang, Frankfurt/Main.

Egger, H., Leitão, A. (2009): Efficient reconstruction methods for nonlinear elliptic Cauchy problems with piecewise constant solutions. *Adv. Appl. Math. Mech.* vol. 6 no. 1, pp. 729–749.

Engl, H. W., Hanke, M., Neubauer, A. (1996): *Regularization of Inverse Problems*. Kluwer Academic Publishers Group, Dordrecht.

Escriva, X., Baranger, T. N., Tlatli, N. H. (2007): Leak identification in porous media by solving the Cauchy problem. *C. R. Mecanique* vol. 335 no. 7, pp. 401–406.

Finkelstein, M., Scheinberg, S. (1975): Kernels for solving problems of Dirichlet type in a half-plane. *Advances in Math.* vol. 18 no. 1, pp. 108–113.

Ganesh, M., Graham I. G. (2004): A high-order algorithm for obstacle scattering in three dimensions. *J. Comput. Phys.* vol. 198 no. 1, pp. 211–242.

Ganesh, M., Graham, I. G., Sivaloganathan, J. A. (1998): New spectral boundary integral collocation method for three-dimensional potental problems. *SIAM J. Numer. Anal.* vol. 35 no. 2, pp. 778–805.

Gardiner, S. J. (1981): The Dirichlet and Neumann problems for harmonic functions in half-spaces. *J. London Math. Soc.* vol. 24 no. 2, pp. 502–512.

Glasko, V. B. (1971): On the unique determination of the earth's core structure from Rayleigh surface waves. *Zh. Vychisl. Mat. Mat. Fiz.* vol. 11 no. 6, pp. 1498–1509.

Gorenflo, R. (1965): Funktionentheoretische Bestimmung des Aussenfeldes zu einer zweidimensionalen magnetohydrostatischen Konfiguration. Z. Angew. Math. Phys. vol. 16 no. 2, pp. 279–290.

Graham, I. G., Sloan, I. H. (2002): Fully discrete spectral boundary integral methods for Helmholtz problems on smooth closed surfaces in $\mathbf{R}^3$. *Numer. Math.* vol. 92 no. 2, pp. 289–323.

Hadamard, J. (1923) *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Yale University Press, New Haven.

Helsing, J., Johansson, B. T. (2010): Fast reconstruction of harmonic functions from Cauchy data using integral equation techniques. *Inverse Problems Sci. Engn.* vol. 18 no. 3, pp. 381–399.

Hess, J. L. and Smith, A. M. O. (1967): Calculation of potential flow about arbitrary bodies, *Prog. Aeronaut. Sci.* vol. 8 no. 2, pp 1–138.

Hon, Y. C., Li, M. (2003): A meshless scheme for solving inverse problems of Laplace equation, *Recent Development in Theories & Numerics.* World Sci. Publ., River Edge, NJ, pp. 291–300.

Ingham, D. B., Yuan, Y., Han, H. (1991): The boundary element method for an improperly posed problem. *IMA J. Appl. Math.* vol. 47 no. 1, pp. 61–79.

Isakov, V. (1998): *Inverse Problems for Partial Differential Equations*. Springer-Verlag, New York.

Ivanyshyn, O., Kress, R. (2010): Identification of sound-soft 3D obstacles from phaseless data. *Inverse Probl. Imaging* vol. 4 no. 1, pp. 131–149.

Johnson, L. H., Jr. (1935): The Cauchy problem for Laplace's equation in three dimensions. *Amer. Math. Monthly* vol. 42 no. 2, pp. 65–74.

Karageorghis, A., Lesnic, D., Marin, L. (2011): A survey of applications of the MFS to inverse problems. *Inv. Pr. Sci. Eng.* vol. 19 no. 3, pp. 309–336.

Kirsch, A. (1996): *An Introduction to the Mathematical Theory of Inverse Problems*. Springer-Verlag, New-York.

Kozlov, V. A., Maz'ya, V. G. (1989): On iterative procedures for solving ill-posed boundary value problems that preserve differential equations, *Algebra i Analiz*, vol. 5 no. 1, pp. 144–170. English transl.: (1990): *Leningrad Math. J.*, vol. 5 no. 1, pp. 1207–1228.

Kubo, M. (1994): L^2-conditional stability estimate for the Cauchy problem for the Laplace equation. *J. Inv. Ill-Posed Pr.* vol. 2 no. 3, pp. 253–262.

Lavrent'ev, M. M. (1956): On the Cauchy problem for Laplace equation. *Izv. Akad. Nauk SSSR. Ser. Mat.* vol. 20, pp. 819–842 (In Russian).

Lavrent'ev, M. M. (1967): *Some Improperly Posed Problems of Mathematical Physics*. Springer Verlag, Berlin.

Li, B. C., Syngellakis, S. (1995): On improperly posed Cauchy problems and their approximate solution. *IMA J. Appl. Math.* vol. 55, pp. 85–95.

Marin, L., Lesnic, D. (2005): The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations. *Computers and Structures* vol. 83 no. 4–5, pp. 267–278.

Mergelyan, S. N. (1956): Harmonic approximation and approximate solution of the Cauchy problem for the Laplace equation. *Uspehi Mat. Nauk (N.S.)* vol. 71 no 5, pp. 3–26. (In Russian).

Nazarchuk Z. T., Kulynych Ya. P. (2009): The interpolation cubature formula for the calculation of some class of hypersingular integrals, Reports of the National Academy of Sciences of Ukraine, vol. 3, pp. 36–43 (In Ukrainian).

Payne, L. E. (1975): *Improperly Posed Problems in Partial Differential Equations*. Regional Conference Series in Applied Mathematics, No. 22, SIAM, Philadelphia, Pa.

Pucci, C. (1955): Sui problemi di Cauchy non "ben posti". *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* vol. 18 no. 8, pp. 473–477 (In Italian).

Reinhardt, H. J., Dinh Nho Hào and Han, H. (1999): Stability and regularization of a discrete approximation to the Cauchy problem for the Laplace's equation. *SIAM J. Numer. Anal.* vol. 36 no. 3, pp. 890–905.

Seager, M. K., Carey, G. F. (1990): Adaptive domain extension and adaptive grids for unbounded spherical elliptic pdes. *SIAM J. Sci. Stat. Comput.* vol .11 no. 1, pp. 92–11.

Shih, K. -G. (1971): Uniform heat flow in a semi-infinite medium disturbed by a body of different thermal conductivity. *Pure and Appl. Geophys.* vol. 85 no. 1, pp. 298–308.

Shu, F., Tanaka, M., Yanagishita, M. (2011): Neumann problem on a half-space. *Proc. Amer. Math. Soc.* vol. 139 no. 4, pp. 1333–1345.

Stein, E. M. (1970): *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton.

Stenger, F. (1993): *Numerical Methods Based on Sinc and Analytic Functions*. Springer-Verlag, Berlin.

Tarchanov, N. (1999): *The Cauchy Problem for Solutions of Elliptic Equations*. Akad. Verlag, Berlin, 1995.

Tikhonov, A. N., Arsenin, V. Y. (1977): *Solutions of Ill-Posed Problems*. John Wiley & Sons, New York.

Vainikko, G. M., Veretennikov, A. Y (1986): *Iteration Procedures in Ill-Posed Problems*. Nauka Publ., Moscow (in Russian).

van Berkel, C., Lionheart, W. R. B. (2007): Reconstruction of a grounded object in an electrostatic halfspace with an indicator function. *Inv. Pr. Sci. Eng* vol. 15 no. 6, pp. 585–600.

Wienert, L. (1990): *Die Numerische Approximation von Randintegraloperatoren für die Helmholtzgleichung im* $\mathbf{R}^3$. Ph.D. thesis, University of Göttingen.

Zeb, A., Elliott, L., Ingham, D. B., Lesnic, D. (1997): Solution of the Cauchy problem for Laplace equation, In: *First UK Conference on Boundary Integral Methods*, (Eds: L. Elliott, D. B. Ingham and D. Lesnic), Leeds University Press, pp. 297–307.