 CMES, vol.85, no.2, pp.157-175, 2012

Computation of the Time-Dependent Green's Function for the Longitudinal Vibration of Multi-Step Rod

V.G.Yakhno[1] **and D. Ozdek**[2]

Abstract: The present paper describes computation of the time-dependent Green's function for the equations of longitudinal vibration of a multi-step rod with a piecewise constant varying cross-section. This computation is based on generalization of the Fourier series expansion method. The time-dependent Green's function is derived in the form of the Fourier series. The basic functions of this series are eigenfunctions of an ordinary differential equation with boundary and matching conditions. Constructing the eigenvalues and eigenfunctions of this differential equation and then derivation of the Fourier coefficients of the Green's function are main steps of the method. Computational experiments confirm the robustness of the method.

Keywords: Green's function, longitudinal vibration, multi-step rod, analytical method, simulation.

1 Introduction

Green's functions can be considered as a useful tool for different methods in presentation of acoustic, electromagnetic, elastic and other fields, in particular, for the method of moments and boundary element method [Chew (1990)]; [Tewary (1995)]; [Tewary (2004)]; [Ting (2000)]; [Rashed (2004)]; [Ting (2005)]; [Nakamura and Tanuma (1997)]; [Pan and Yuan (2000)]; [Yang and Tewary (2008)]; [Gu; Young and Fan (2009)]; [Chen; Ke and Fan (2009)]. When the Green's function can be constructed it leads to significant simplification of modelling waves and allows engineers to overcome calculational difficulties [Tewary; Bartolo and Powell (2002)]. Green's functions have been used to solve problems of the wave propagation in composite elastic materials. For example, [Inceoğlu and M. Gürgöze (2000)] have used the Green's function to derive an exact solu-

[1] Electrical and Electronics Engineering Department, Dokuz Eylul University, Izmir, Turkey.
[2] Department of Mathematics, The Graduate School of Natural and Applied Sciences, Dokuz Eylul University, Izmir, Turkey.

tion for the problem of longitudinal vibration on a mechanical system including two-clamped-free rods with mass. [Matsuda et al (1995)] have studied free vibrations of a non-uniform circular bar using Green's function and transforming the differential equation into a boundary integral equation. More recently, using Green's function, [Mehri; Davar; and Rahmani (2009)] have investigated the linear dynamic response of a uniform Euler Bernouli beam under different boundary conditions excited by a moving load.

The methods of the Green's functions construction for dynamic equations of linear anisotropic elasticity have been developed by [Wang and Achenbach (1995)], [Tewary (1995)], [Wang et al. (2007)], [Khojasteh et al. (2008)], [Rangelov et al. (2005)], [Rangelov (2003)], [Vavrycuk (2001)], [Vavrycuk (2002)], [Garg et al. (2004)], [Yakhno and Yaslan (2011a)], [Yakhno and Yaslan (2011b)] and other authors.

In composite elastic materials, the dynamic equations defining the Green's function are partial differential equations with piecewise constant coefficients. Inhomogeneous terms in these equations contain the generalized Dirac delta function. Methods of the construction of the Green's functions for the time-dependent partial differential equations with piecewise constant coefficients have not been developed so far because of additional singularities in the behavior of the Green's functions.

The purpose of the present paper is to compute the time-dependent Green's function of the equations of longitudinal vibration of a multi-step rod with a piecewise constant varying cross section. For the computation of the Green's function we suggest a new analytical method. This method is based on the generalization of the Fourier series expansion method and consists of the following. Firstly, the eigenvalues and eigenfunctions of an ordinary differential equation with boundary and matching conditions, corresponding to the partial differential equation of the longitudinal vibration, are determined. These eigenfunctions form an orthonormal set of basic functions which are used in the Fourier series expansions. The Green's function is constructed in the form of the formal Fourier series related to this set of eigenfunctions. An approximation of the Green's function is obtained in the form of the Fourier series with a finite number of terms. Using this approximation the simulation of the longitudinal vibration of two-step rod has been made.

The paper is organized as follows. The equations of the longitudinal vibration of a multi-step rod are listed in Section 2. Section 3 describes the steps of the computation of the time-dependent Green's function. These steps include solving the eigenvalue-eigenfunction problem for the ordinary differential equation with the piecewise constant coefficient; the regularization of the Dirac delta function, appearing in the initial data, by a partial sum of the Fourier series with eigenfunctions as basic functions; the construction of the regularized Green's function by solving an initial boundary value problem. Computational experiments are given in Section 4. Conclusion and appendices with technical details are at the end of the paper.

2 The time-dependent Green's function of the longitudinal vibration of a composite multi-step rod

Let $\rho(x)$, $E(x)$ and $A(x)$ be functions of the form

$$\rho(x) = \{\ \rho_i, \quad x \in (\ell_{i-1}, \ell_i),\ i = 1,2,\ldots,N\ \} \tag{1}$$

$$E(x) = \{\ E_i, \quad x \in (\ell_{i-1}, \ell_i),\ i = 1,2,\ldots,N\ \} \tag{2}$$

$$A(x) = \{\ A_i, \quad x \in (\ell_{i-1}, \ell_i),\ i = 1,2,\ldots,N\ \} \tag{3}$$

where ρ_i, E_i, A_i, ℓ_i, $i = 1,2,\ldots,N$ are given real numbers such that $\rho_i > 0$, $E_i > 0$, $A_i > 0$, $0 = \ell_0 < \ell_1 < \ell_2 < \ldots < \ell_N$. Here, $\rho(x), E(x), A(x)$ are the density, Young's module and the cross-sectional area of the composite multi-step rod, respectively.

The Green's function of the longitudinal vibration of this rod is defined as a generalized function $G(x,t;x^0)$ satisfying the following partial differential equation with piecewise constant coefficients

$$\rho(x)A(x)\frac{\partial^2 G}{\partial t^2} = \frac{\partial}{\partial x}(E(x)A(x)\frac{\partial G}{\partial x}) + \delta(x - x^0)\delta(t), \tag{4}$$

where $x \in (0,\ell_1) \cup \ldots \cup (\ell_{N-1}, \ell_N)$, $t \in \mathbf{R}$, the initial data, boundary and matching conditions

$$G(x,t;x^0)\Big|_{t<0} = 0, \tag{5}$$

$$G(+0,t;x^0) = 0, \quad G(\ell_N - 0,t;x^0) = 0, \tag{6}$$

$$G(\ell_i - 0,t;x^0) = G(\ell_i + 0,t;x^0), \quad E_i A_i \frac{\partial G}{\partial x}\Big|_{x=\ell_i - 0} = E_{i+1}A_{i+1}\frac{\partial G}{\partial x}\Big|_{x=\ell_i + 0}, \tag{7}$$

where $i = 1,2,\ldots,N-1$; $x^0 \in (0,\ell_1) \cup \ldots \cup (\ell_{N-1}, \ell_N)$ is a fixed point and $\delta(x - x^0)\delta(t)$ are the Dirac delta functions concentrated at $x = x^0$ and $t = 0$, respectively.

Using the technique of generalized functions (see, for example [Vladimirov (1971)]) the following remark holds.

Remark 1. Let $x^0 \in (0,\ell_1) \cup \ldots \cup (\ell_{N-1}, \ell_N)$ be a fixed point, $\Theta(t)$ be the Heaviside function ($\Theta(t) = 1$ for $t \geq 0$ and $\Theta(t) = 0$ for $t < 0$) and $g(x,t;x^0)$ be a generalized function satisfying

$$\rho(x)A(x)\frac{\partial^2 g}{\partial t^2} = \frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial g}{\partial x}\right),\ x \in (0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N),\ t>0 \tag{8}$$

$$g(x,0;x^0)=0,\ \frac{\partial g}{\partial t}(x,0;x^0)=\frac{1}{\rho(x^0)A(x^0)}\delta(x-x^0), \tag{9}$$

$$g(0,t;x^0)=0, \quad g(\ell_N,t;x^0)=0, \tag{10}$$

$$g(\ell_i-0,t;x^0)=g(\ell_i-0,t;x^0), \tag{11}$$

$$E_iA_i\frac{\partial g}{\partial x}(\ell_i-0,t;x^0)=E_{i+1}A_{i+1}\frac{\partial g}{\partial x}(\ell_i+0,t;x^0), \tag{12}$$

for $i=1,2,\ldots,N-1$. Then $G(x,t;x^0)=\Theta(t)g(x,t;x^0)$ is the generalized function satisfying $(4)-(7)$.

Therefore the construction of the time-dependent Green's function $G(x,t;x^0)$ can be based on the determination of the generalized function $g(x,t;x^0)$ satisfying $(8)-(12)$.

3 Computation of the approximate Green's function

3.1 Step 1: Eigenvalue-Eigenfunction problem solving

Let $d(x)$ be defined as a piecewise constant function of the form

$$d(x)=\left\{\ d_i=\rho_i/E_i,\quad x\in(\ell_{i-1},\ell_i),\ i=1,2,\ldots,N\ \right\}.$$

Let us consider the following ordinary differential equation

$$-y''(x)=\lambda d(x)y(x),\quad x\in(0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N) \tag{13}$$

subjects to the following conditions

$$y(0)=0,\ \ y(\ell_N)=0, \tag{14}$$

$$y(\ell_i-0)=y(\ell_i+0), \tag{15}$$

$$y'(\ell_i-0)=\beta_i y'(\ell_i+0),\quad \beta_i=\frac{E_{i+1}A_{i+1}}{E_iA_i}, \tag{16}$$

where λ is a parameter, $i=1,2\ldots,N-1$.

We note that a number λ for which there exists a nonzero function $y(x)$ satisfying $(13)-(16)$ is called an eigenvalue, and nonzero solution $y(x)$ of $(13)-(16)$ for this eigenvalue is called an eigenfunction corresponding to λ. The problem $(13)-(16)$ is called the eigenvalue-eigenfunction problem (EEP).

Remark 2. The eigenvalue-eigenfunction problems which are similar to $(13)-(16)$ have been considered in works [Bulavin and Kashcheev (1965)], [Faydaoğlu and Guseinov (2003)], [Mulholland and Cobble (1972)], [Ozışık (1980)], [Tittle (1965)].

Remark 3. The differential operator $-\frac{1}{d(x)}\frac{d^2}{dx^2}$, defined in the class of twice differentiable functions over $(0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N)$ satisfying $(14)-(16)$, is positive definite (see, Appendix A). Using the general theory of positive definite operators (see, for example, [Vladimirov (1971)]) we find that eigenvalues of EEP $(13)-(16)$ are positive.

The procedure of finding eigenvalues and corresponding to them eigenfunctions consists of the following. The general solution of the equation (13) can be written in each interval (ℓ_{i-1},ℓ_i) in the form

$$y_i(x) = A_i\cos\sqrt{\lambda d_i}(x) + B_i\sin\sqrt{\lambda d_i}(x), \tag{17}$$

where A_i, B_i are arbitrary constants. We find $A_1 = 0$ from (14) and take for further consideration $B_1 = 1$. Using (17), relations $(14)-(16)$ can be written in the form

$$\mathbf{Q}(\lambda)\mathbf{S} = \mathbf{0}, \tag{18}$$

where $\mathbf{0}$ is the zero column vector, $\mathbf{S}$ is the column vector with components 1, A_2, B_2, $\ldots$, A_N, B_N and Q is a block matrix of the form

$$\mathbf{Q}(\lambda)=\begin{bmatrix} \mathbf{P}_1(\lambda) & \mathbf{R}_1(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P_2}(\lambda) & \mathbf{R}_2(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{P_3}(\lambda) & \mathbf{R}_3(\lambda) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \mathbf{P}_{N-2}(\lambda) & \mathbf{R}_{N-2}(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{P}_{N-1}(\lambda) & \mathbf{R}_{N-1}(\lambda) \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{P}_N(\lambda) \end{bmatrix}, \tag{19}$$

where $\mathbf{P}_k(\lambda)$, $\mathbf{R}_j(\lambda)$ are the submatrices defined by

$$\mathbf{P}_1(\lambda) = \begin{bmatrix} \sin(\sqrt{\lambda d_1}\ell_1) \\ \sqrt{\lambda d_1}\cos(\sqrt{\lambda d_1}\ell_1) \end{bmatrix}_{2\times 1};$$

$$\mathbf{P}_k(\lambda) = \begin{bmatrix} \cos(\sqrt{\lambda d_k}\ell_k) & \sin(\sqrt{\lambda d_k}\ell_k) \\ -\sqrt{\lambda d_k}\sin(\sqrt{\lambda d_k}\ell_k) & \sqrt{\lambda d_k}\cos(\sqrt{\lambda d_k}\ell_k) \end{bmatrix}_{2\times 2}, k = 2,\ldots,N-1;$$

$$\mathbf{P}_N(\lambda) = \begin{bmatrix} \cos(\sqrt{\lambda d_N}\ell_N) & \sin(\sqrt{\lambda d_N}\ell_N) \end{bmatrix}_{1\times 2};$$

$$\mathbf{R}_j(\lambda) = \begin{bmatrix} -\cos(\sqrt{\lambda d_{j+1}}\ell_j) & -\sin(\sqrt{\lambda d_{j+1}}\ell_j) \\ \beta_j\sqrt{\lambda d_{j+1}}\sin(\sqrt{\lambda d_{j+1}}\ell_j) & -\beta_j\sqrt{\lambda d_{j+1}}\cos(\sqrt{\lambda d_{j+1}}\ell_j) \end{bmatrix}_{2\times 2},$$

for $j=1,\ldots,N-1$.

Hence finding the nonzero functions $y(x)$ satisfying $(13)-(16)$ is reduced to determine the numbers $A_2,B_2,\ldots,A_N,B_N$ satisfying (18). The equation (18) is the homogeneous linear algebraic system which is written in the matrix form. This system is consistent if and only if the determinant of the matrix $\mathbf{Q}(\lambda)$ is equal to zero. Moreover, the roots of $\det(\mathbf{Q}(\lambda))=0$ are eigenvalues of $(13)-(16)$. The roots of $\det(\mathbf{Q}(\lambda))=0$ can be computed by MATLAB tools (see, Section 4, Computational experiments).

Let us assume that roots λ_m, $m=1,2,\ldots$ of $\det(\mathbf{Q}(\lambda))=0$ have been derived. Substituting $\lambda=\lambda_m$ into (18), we find the following relations:

$$\mathbf{R}_j(\lambda_m)\begin{bmatrix} A^m_{j+1} \\ B^m_{j+1} \end{bmatrix} = -\mathbf{P}_j(\lambda_m)\begin{bmatrix} A^m_j \\ B^m_j \end{bmatrix}, \quad \mathbf{R}_1(\lambda_m)\begin{bmatrix} A^m_2 \\ B^m_2 \end{bmatrix} = -\mathbf{P}_1(\lambda_m). \tag{20}$$

Since $A^m_1=0$, $B^m_1=1$ and $\det\mathbf{R}_j(\lambda_m)=\beta_j\sqrt{\lambda_m d_{j+1}}\neq 0$ then the values of A^m_2, $B^m_2,\ldots A^m_N$, B^m_N are found by the following recurrence relations:

$$\begin{bmatrix} A^m_{j+1} \\ B^m_{j+1} \end{bmatrix} = -\mathbf{R}_j^{-1}(\lambda_m)\mathbf{P}_j(\lambda_m)\begin{bmatrix} A^m_j \\ B^m_j \end{bmatrix}, \quad \mathbf{R}_1(\lambda_m)\begin{bmatrix} A^m_2 \\ B^m_2 \end{bmatrix} = -\mathbf{P}_1(\lambda_m). \tag{21}$$

Substituting the coefficients $A^m_2,B^m_2,\ldots,A^m_N,B^m_N$ into (17), we find the explicit formula for the eigenfunction $y_m(x)$ corresponding to λ_m :

$$y_m(x)=A^m_i\cos(\sqrt{\lambda_m d_i}x)+B^m_i\sin(\sqrt{\lambda_m d_i}x),\, x\in(\ell_{i-1},\ell_i)\, i=1,2,\ldots,N,$$

where $m=1,2,\ldots$.

3.2 Properties of eigenvalues and eigenfunctions

Property 1. Let $\rho(x), A(x)$ be functions defined by $(1),(3)$ and $y_m(x)$, $m=1,2,\ldots$ be eigenfunctions of $(13)-(16)$;

$$X_m(x)=y_m(x)/\sqrt{\alpha_m}, \quad x\in(0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N),\ m=1,2,\ldots, \tag{22}$$

where $\alpha_m=\int_0^{\ell_N}\rho(x)A(x)y^2_m(x)dx$. Then the set of functions $X_m(x)$, $m=1,2,\ldots$ and $X_{m'}(x)$, $m'=1,2,\ldots$ is orthonormal, i.e.

$$\int_0^{\ell_N}\rho(x)A(x)X_m(x)X_{m'}(x)dx=\begin{cases} 0, & \text{if } m\neq m', \\ 1, & \text{if } m=m'. \end{cases} \tag{23}$$

The validity of Property 1 is given in Appendix B.

Property 2. There exists a countable set of eigenvalues of the eigenvalue-eigenfunction problem $(13)-(16)$ and positive real numbers m^0 and C such that

$$\lambda_m \leq C \cdot m^2, \quad \text{for any } m > m^0. \tag{24}$$

Here C does not depend on m.

Proof. The proof is similar to the proof of Theorem 5.2 of the paper [Faydaoğlu and Guseinov (2003)]. □

Property 3. Let $h(x)$ be a four times differentiable function over $[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]$ such that $h(x)$ satisfies the following conditions:

a) $\int_0^{\ell_N} |h^j(x)|dx < \infty,\ j = 1,2,3,4;$

b) $h(0) = h(\ell_N) = 0,\ h(\ell_i - 0) = h(\ell_i + 0),\ i = 1,2,\ldots,N-1;$

c) $E_iA_ih'(\ell_i - 0) = E_{i+1}A_{i+1}h'(\ell_i + 0)\ i = 1,2,\ldots,N-1.$

Then the function $h(x)$ can be expanded in the following Fourier series

$$h(x) = \sum_{m=1}^{\infty} h_m X_m(x),\ \ x \in [0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N], \tag{25}$$

where the Fourier coefficients are defined by

$$h_m = \int_0^{\ell_1} \rho_1 A_1 h(x) X_m(x) dx + \ldots + \int_{\ell_{(N-1)}}^{\ell_N} \rho_N A_N h(x) X_m(x) dx, \quad m = 1,2,\ldots. \tag{26}$$

The series in (25) is uniformly convergent to $h(x)$ over $[0,\ell_1)\cup\ldots\cup(\ell_{(N-1)},\ell_N]$ and the series obtained by differentiation up to second order are uniformly convergent over $[0,\ell_1)\cup\ldots\cup(\ell_{(N-1)},\ell_N]$. Moreover, there exists a positive real number m_0 and a constant D_1 such that

$$|h_m| \leq \frac{D_1}{m^4}, \quad \text{for } \ m > m_0. \tag{27}$$

where D_1 does not depend on m.

The validity of Property 3 is given in Appendix C.

Property 4. Let $x^0 \in (0,\ell_1)\cup\ldots\cup(\ell_{(N-1)},\ell_N)$ be a fixed point and M be a fixed natural number; $X_m(x)$, $m = 1,2,\ldots$ be the orthonormal eigenfunctions defined by (22). Then

$$\delta_M(x,x^0) = \rho(x^0)A(x^0)\sum_{m=1}^{M} X_m(x^0)X_m(x) \tag{28}$$

is an approximation (regularization) of $\delta(x-x^0)$, i.e.

$$\lim_{M\to\infty}\int_0^{\ell_N}\rho(x)A(x)\delta_M(x,x^0)h(x)dx=\int_0^{\ell_N}\rho(x)A(x)\delta(x-x^0)h(x)dx$$

for any $h(x)$ satisfying the conditions $a)$, $b)$, $c)$ of Property 3.

The validation of Property 4 is given in Appendix E.

3.3 Step 2: Regularization of initial data

Let us consider a problem $(8)-(12)$, where $\delta(x-x^0)$ is replaced by $\delta_M(x,x^0)$, i.e. the problem of finding $g_M(x,t;x^0)$ satisfying

$$\rho(x)A(x)\frac{\partial^2 g_M}{\partial t^2}=\frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial g_M}{\partial x}\right),\quad x\in(0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N), \tag{29}$$

$$g_M(x,+0;x^0)=0,\quad \frac{\partial g_M}{\partial t}(x,+0;x^0)=\frac{1}{\rho(x^0)A(x^0)}\delta_M(x,x^0), \tag{30}$$

$$g_M(0,t;x^0)=0,\quad g_M(\ell_N,t;x^0)=0, \tag{31}$$

$$g_M(\ell_i-0,t;x^0)=g_M(\ell_i+0,t;x^0), \tag{32}$$

$$E_iA_i\frac{\partial g_M}{\partial x}\Big|_{x=\ell_i-0}=E_{i+1}A_{i+1}\frac{\partial g_M}{\partial x}\Big|_{x=\ell_i+0}, \tag{33}$$

where $i=1,2,\ldots,N-1$ and $x^0\in(0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N)$.

3.4 Step 3: Computation of the solution $g_M(x,t;x^0)$ of $(29)-(33)$

A solution of $(29)-(33)$ is found in the form of the Fourier series

$$g_M(x,t;x^0)=\sum_{m=1}^{M}T_m(t;x^0)X_m(x),\quad x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N],\ t>0, \tag{34}$$

where $T_m(t;x^0)$ are unknown functions and $X_m(x)$ are defined in (22). Substituting (34) into $(29)-(30)$ and using the orthogonality property (23) we get the following ordinary differential equation and initial data

$$T''_m(t;x^0)+\lambda_m T_m(t;x^0)=0, \tag{35}$$

$$T_m(0;x^0)=0,\quad T'_m(0;x^0)=X_m(x^0) \tag{36}$$

for each $m=1,2,\ldots,M$. Solutions of $(35)-(36)$ are $T_m(t;x^0)=\frac{X_m(x^0)}{\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}t)$, for $m=1,2,\ldots,M$. Hence, a solution of $(29)-(33)$ is found by

$$g_M(x,t;x^0)=\sum_{m=1}^{M}\frac{X_m(x^0)}{\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}t)X_m(x). \tag{37}$$

3.5 *A regularized solution of* (4) – (7)

Let M be a natural number; $x^0 \in (0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N)$ and $t \geq 0$ be fixed; $g_M(x,t;x^0)$ be a function defined by (37) and $h(x)$ be a function satisfying the conditions of Property 3. Then using (26), we find

$$\int_0^{\ell_N} \rho(x)A(x)h(x)g_M(x,t;x^0)dx = \sum_{m=1}^{M} \frac{h_m}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}t)X_m(x^0).$$

Applying Properties 1, 2, 3 we find that the series $\sum_{m=1}^{\infty} \frac{h_m}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}t)X_m(x^0)$ is convergent for any fixed $t \geq 0$ and $x^0 \in (0,\ell_1)\cup\ldots(\ell_{N-1},\ell_N)$. We define the linear functional $g(x,t;x^0)$ by

$$< g,h >= \lim_{M\to\infty} \int_0^{\ell_N} \rho(x)A(x)h(x)g_M(x,t;x^0)dx = \sum_{m=1}^{\infty} \frac{h_m}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}t)X_m(x^0)$$

for any function $h(x)$ satisfying the conditions of Property 3.
We note that the function $g_M(x,t;x^0)$ defines the linear functional by

$$< g_M,h >= \int_0^{\ell_N} \rho(x)A(x)h(x)g_M(x,t;x^0)dx = \sum_{m=1}^{M} \frac{h_m}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m}t)X_m(x^0)$$

for any function $h(x)$ satisfying the conditions of Property 3.
We say that the sequence $g_M(x,t;x^0)$ converges to $g(x,t;x^0)$ as M tends to ∞ and denote $g_M(x,t;x^0) \to g(x,t;x^0)$ if $< g,h > - < g_M,h >$ approaches zero for any function $h(x)$ satisfying the conditions of Property 3.

Using Properties 2,4, we can show that $g_M(x,t;x^0) \to g(x,t;x^0)$, $\frac{\partial^k}{\partial t^k} g_M(x,t;x^0) \to \frac{\partial^k}{\partial t^k} g(x,t;x^0)$, $\frac{\partial^k}{\partial x^k} g_M(x,t;x^0) \to \frac{\partial^k}{\partial x^k} g(x,t;x^0)$, $k = 1,2$ as M tends to ∞.
Moreover, using Property 3, we find that $\delta_M(x,x^0) \to \delta(x-x^0)$ as M tends to ∞.
From above mentioned reasonings we find that $g(x,t;x^0)$ is a generalized solution of $(8)-(12)$ and $g_M(x,t;x^0)$ is an approximate (regularized) solution of $(8)-(12)$.
Moreover, using Remark 1 we find that $G(x,t;x^0) = \Theta(t)g(x,t;x^0)$ is the Green's function of the equation of the longitudinal vibration and $G_M(x,t;x^0) = \Theta(t)g_M(x,t;x^0)$ is an approximation (regularization) of $G(x,t;x^0)$.

4 Computational Experiments

For computational experiment, the two-step rod with the following properties has been taken. The density, the Young's module and the sectional area of the first material are

$$\rho_1 = 1.4\times 10^3 kg/m^3,\ \ E_1 = 4.34\times 10^9 N/m^2,\ \ A_1 = 3m^2.$$

The density, the Young's module and the sectional area of the second material are

$\rho_2 = 1.19 \times 10^3 kg/m^3, \;\; E_2 = 3.3 \times 10^9 N/m^2, \;\; A_2 = 2m^2,$

and we take $x^0 = 3$, $\ell_1 = 5\ m$, $\ell_2 = 8\ m$.

The main goal of the experiment is to compute the values of the approximated Green's function $G_M(x,t;x^0)$ and its derivative $\frac{\partial}{\partial t} G_M(x,t;x^0)$.

In the first step, we construct the eigenvalues and corresponding to them eigenfunctions. Notice that in this example $N = 2$ and $d(x) = d_1$ for $x \in [0, \ell_1)$; $d(x) = d_2$ for $x \in (\ell_1, \ell_2]$. Using technique of Section 3 we have found

$$y_m(x) = \begin{cases} y_{1m}(x), & 0 \le x < \ell_1, \\ \\ y_{2m}(x), & \ell_1 < x \le \ell_2. \end{cases} \tag{38}$$

where $y_{1m}(x) = \sin(\sqrt{d_1 \lambda_m} x)$, and

$$y_{2m}(x) = \frac{1}{2\beta_1 \sqrt{d_2}} \Big[(\beta_1 \sqrt{d_2} + \sqrt{d_1}) \sin(\sqrt{\lambda_m}((\sqrt{d_1} - \sqrt{d_2})\ell_1 + \sqrt{d_2} x)) + (\beta_1 \sqrt{d_2} - \sqrt{d_1}) \sin(\sqrt{\lambda_m}((\sqrt{d_1} + \sqrt{d_2})\ell_1 - \sqrt{d_2} x)) \Big].$$

Here the values of d_1, d_2, β_1 are defined by $d_i = \rho_i / E_i$, $i = 1, 2$; $\beta_1 = \frac{E_2 A_2}{E_1 A_1}$.

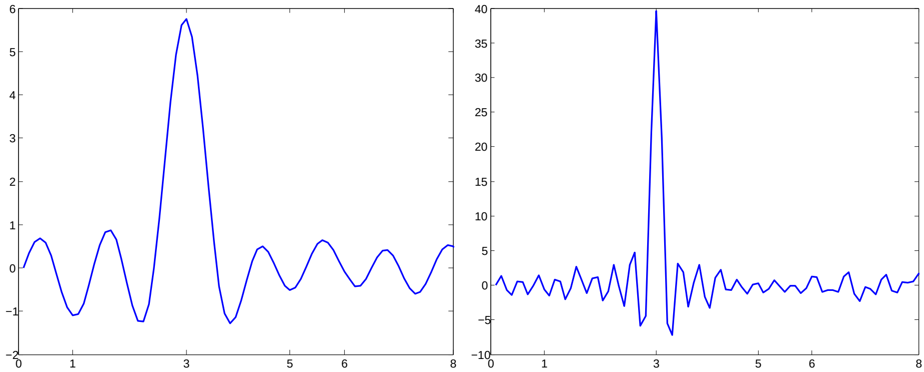
In the second step, the parameter M of the regularization (approximation) has been chosen by the following empirical observation and natural logic. Using the formula (28), the regularization (approximation) $\delta_M(x, x^0)$ of the Dirac delta function $\delta(x - x^0)$ has been made for the different values of M. We note that the values of the Dirac delta function have been interpreted as zero for all points except $x = x^0$. The value at $x = x^0$ is $+\infty$. We have computed and drawn the graphs of $\delta_M(x, x^0)$ for $x^0 = 3$, $M = 5, 10, \ldots, 200, 210, \ldots$ The results of the computation are presented in Figs. 1 and 2 for $M = 30$ and $M = 200$, respectively.

Using reasoning described above, we have chosen number $M = 200$ as an optimal value of the parameter of approximation.

In the third step, the problem $(29) - (33)$ has been solved for $M = 200$, $x^0 = 3$, $\ell_1 = 5$, $\ell_2 = 8$, $N = 2$. The solution $g_M(x,t;x^0)$ of $(29) - (33)$ has been computed by formula (37). The values of the function $g_M(x,t;x^0)$ $(M = 200)$ have been excepted as values of an approximate solution of $(8) - (12)$. This means

$$G_M(x,t;x^0) = \Theta(t) \sum_{m=1}^{M} \frac{X_m(x^0)}{\sqrt{\lambda_m}} \sin(\sqrt{\lambda_m} t) X_m(x),$$

$$\frac{\partial G_M}{\partial t}(x,t;x^0) = \Theta(t) \sum_{m=1}^{M} X_m(x^0) \cos(\sqrt{\lambda_m} t) X_m(x)$$



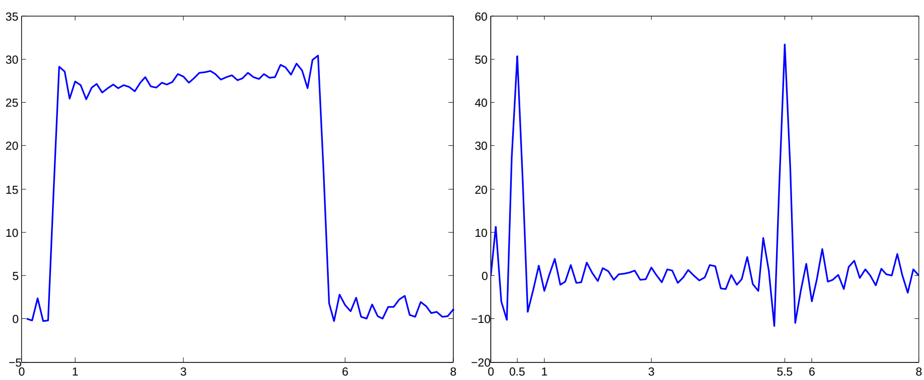
(a) $\delta_M(x,3)$ for $M = 30$ (b) $\delta_M(x,3)$ for $M = 200$

Figure 1: Screenshots for the regularization of the Dirac delta function

are the approximate Green's function and its approximate derivative for the equations of the longitudinal vibration $(4)-(7)$, respectively.

Using these formulae, the visualization of Green's function and its derivative has been made. The results are presented in Figures 2-6. The vertical axis are magnitude of G_M and $\frac{\partial G_M}{\partial t}$ for the fixed time and the horizontal axis is x-axis.

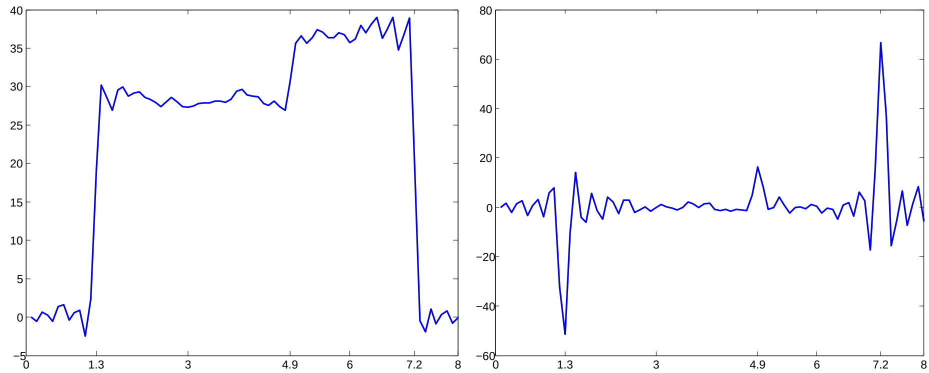
At the time $t = 0.1s$, the values of G_M and $\frac{\partial G_M}{\partial t}$ are just slightly different from the values of the initial excitation at $t = 0$. In the graph of Figure 2(b) there are splashes



(a) G_M for $t = 9s$ (b) $\frac{\partial G_M}{\partial t}$ for $t = 9s$

Figure 2: Screenshots of the Green's function and its derivative for $t = 9s$

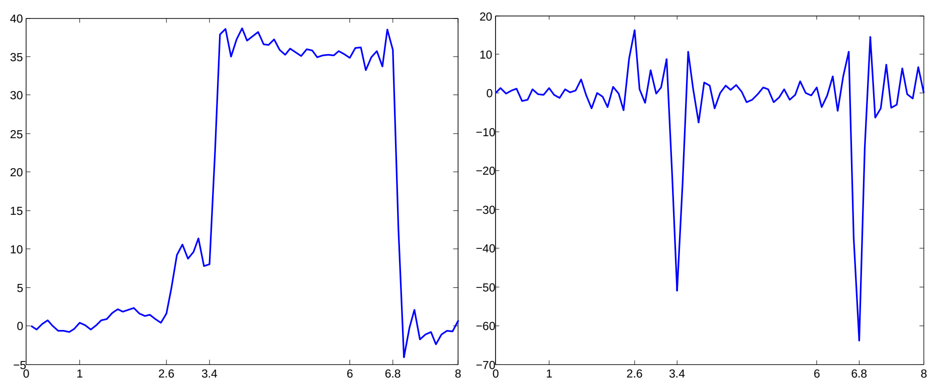
of the magnitude. Both of them have maximum of magnitude. These two splashes correspond to the waves moving in two opposite directions. One of them goes to the origin of the coordinates and another one goes to $+\infty$. In the graph of Figure



(a) G_M for $t = 15s$ (b) $\frac{\partial G_M}{\partial t}$ for $t = 15s$

Figure 3: Screenshots of the Green's function and its derivative for $t = 15s$

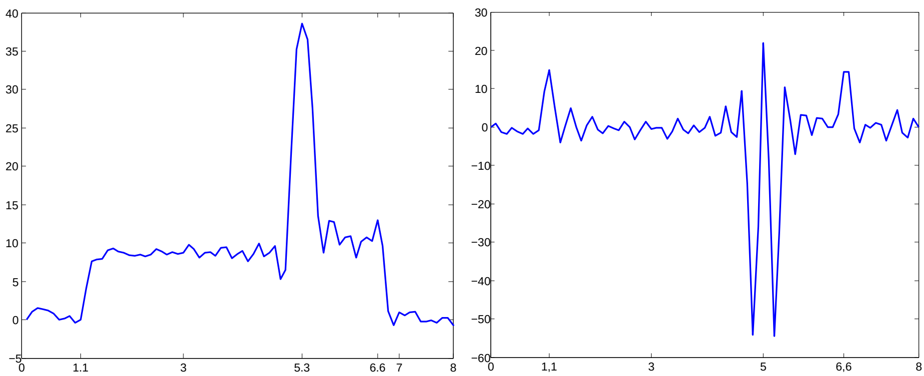
3(b), there are three splashes of the displacement magnitude for $t = 15s$. The first splash at $x = 1.3$ corresponds to the wave which is reflected from the origin and is moving in the direction of $x-$axis, the second splash at $x = 4.9$ belongs to the reflected wave from the interface of two materials and the third splash at $x = 7.2$ is related to the transmitted wave from one material to another. In the graph of



(a) G_M for $t = 23s$ (b) $\frac{\partial G_M}{\partial t}$ for $t = 23s$

Figure 4: Screenshots of the Green's function and its derivative for $t = 23s$

Figure 4(b) there are three splashes of the displacement magnitude for $t = 23s$. The first small splash at $x = 2.6$ corresponds to the reflected wave from the interface of two different materials, the second splash at $x = 3.4$ corresponds to the wave which continues to move in the direction of $x-$axis, the third one at $x = 6.8$ is related to the reflected wave from the boundary $x = 8$ (end of the rod). The splash of the reflected wave has a local maximum of the magnitude at $x = 2.6$, the splashes at $x = 3.4$ and $x = 6.8$ have a local minimum of the magnitude. In the graph of Figure



(a) G_M for $t = 28.5s$ (b) $\frac{\partial G_M}{\partial t}$ for $t = 28.5s$

Figure 5: Screenshots of the Green's function and its derivative for $t = 28.5s$

5(b) there are four splashes of the displacement magnitude for $t = 28.5s$. The first small splash is related to the reflected wave which is moving into the origin of coordinates. This splash has a local maximum at $x = 1.1$. Near $x = 5$ there exist two splashes. The splash, on the left hand side of $x = 5$, belongs to the wave which is getting closer to the interface of two different materials. The splash, on the right hand side of $x = 5$, is related to the transmitted wave form the interface of two different materials. These splashes have a local minimum of the magnitude. The last splash is related to the reflected wave from the interface of two different materials. This small splash has a local maximum at $x = 6.6$.

5 Conclusion

The method of computing the time-dependent Green's function for the equation of the longitudinal vibration on a multi-step rod is suggested. This method is based on a non-classical eigenvalue-eigenfunction problem with piecewise constant coefficients, generalization of the Fourier series expansion method and regularization of the Dirac delta function. Computational experiments confirm the robustness of our method for the approximate computation of the Green's function and its derivative.

The suggested method can be generalized for the computation of the Green's function of the longitudinal vibrations on multi-step rod with the boundary conditions of the form

$$\frac{\partial G}{\partial x}(+0,t;x^0)=0, \quad G(\ell_N-0,t;x^0)=0,$$

or

$$G(+0,t;x^0)=0, \quad \frac{\partial G}{\partial x}(\ell_N-0,t;x^0)=0,$$

or

$$\frac{\partial G}{\partial x}(+0,t;x^0)=0, \quad \frac{\partial G}{\partial x}(\ell_N-0,t;x^0)=0.$$

Appendix A:

Let us show that $-\frac{1}{d(x)}\frac{d^2}{dx^2}$ is positive in the class of twice differentiable functions over $[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]$ satisfying $(14)-(16)$. For this we need to show that

$$\int_0^{\ell_N}\rho(x)A(x)\left(-\frac{1}{d(x)}\frac{d^2}{dx^2}y(x)\right)y(x)dx>0, \tag{39}$$

for any twice differentiable function $y(x)$ for $x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]$ satisfying $(14)-(16)$. The left hand side of (39) can be written as

$$\int_0^{\ell_N}E(x)A(x)[y'(x)]^2dx.$$

Since $E(x)$ and $A(x)$ have positive values we find that the last integral is positive and this means that operator $-\frac{1}{d(x)}\frac{d^2}{dx^2}$ is positive-definite.

Appendix B:

Let λ_n, λ_m be the eigenvalues of **EEP** and y_n, y_m be corresponding to them eigenfunctions.

Multiplying the ordinary differential equations (13) by $y_m(x)$ and $y_n(x)$ respectively, we find

$$y_{im}(x)y''_{in}(x)+\lambda_n d_i(x)y_{im}(x)y_{in}(x)=0, \quad x\in(\ell_{i-1},\ell_i),$$
$$y_{in}(x)y''_{im}(x)+\lambda_m d_i(x)y_{in}(x)y_{im}(x)=0, \quad x\in(\ell_{i-1},\ell_i).$$

Subtracting gives

$$\frac{d}{dx}\left[y_{im}(x)y'_{in}(x)-y_{in}(x)y'_{im}(x)\right]+(\lambda_n-\lambda_m)d_i(x)y_{im}(x)y_{in}(x)=0.$$

Multiplying the last equality by $E(x)A(x)$ and then integrating the obtained relation from ℓ_1 to ℓ_N and using boundary and matching conditions $(14)-(16)$ we have

$$(\lambda_n-\lambda_m)\left[\int_0^{\ell_1}\rho_1 A_1 y_{1m}(x)y_{1n}(x)dx+\ldots+\int_{\ell_{(N-1)}}^{\ell_N}\rho_N A_N y_{Nm}(x)y_{Nn}(x)dx\right]=0.$$

Defining α_m as

$$\alpha_m=\int_0^{\ell_1}\rho_1 A_1 y_{1m}^2(x)dx+\ldots+\int_{\ell_{(N-1)}}^{\ell_N}\rho_N A_N y_{Nm}^2(x)dx$$

we find

$$\int_0^{\ell_N}\rho(x)A(x)y_n(x)y_m(x)dx=\begin{cases}0, \text{ if } m\neq n\\ \alpha_m, \text{ if } m=n\end{cases}.$$

Appendix C:

Let $h(x)$ be a four times differentiable function on $[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]$ such that $h(x)$ satisfies the conditions of the Property 3 and let λ_m, $m=1,2,\ldots$ be the eigenvalues and $X_m(x)$, $m=1,2,\ldots$ be the corresponding to them eigenfunctions of $(13)-(16)$. Using (13) we find

$$-X_m^{(4)}(x)=\lambda_m^2 d^2(x)X_m(x),\quad x\in(0,\ell_1)\cup\ldots\cup(\ell_1,\ell_2).$$

From (26) and the last relation we have

$$h_m=\int_0^{\ell_1}[-\frac{\rho_1 A_1}{\lambda_m^2 d_1^2}]h(x)X_m^{(4)}(x)dx+\ldots+\int_{\ell_{N-1}}^{\ell_N}[-\frac{\rho_N A_N}{\lambda_m^2 d_N^2}]h(x)X_m^{(4)}(x)dx,\ m=1,2,\ldots$$

Applying integration by parts four times, we get

$$h_m=-\frac{\rho_1 A_1}{\lambda_m^2 d_1^2}\int_0^{\ell_1}h^{(4)}(x)X_m(x)dx-\ldots-\frac{\rho_N A_N}{\lambda_m^2 d_N^2}\int_{\ell_{N-1}}^{\ell_N}g^{(4)}(x)X_m(x)dx,\ m=1,2,\ldots.$$

Using (24) we find a constant D_1 such that

$$|h_m|\leq\frac{D_1}{m^4},\quad \text{as } m\to+\infty.$$

Then the uniform convergence of the series $\sum_{m=1}^{\infty}h_m X_m(x)$ in (25) to $h(x)$ follows in a similar way from the Theorem 6.4 of the paper [Faydaoğlu and Guseinov (2003)]. To show that the series obtained by differentiation up to second order are uniformly convergent over $[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]$ we differentiate the right hand series in (25) term by term

$$\sum_{m=1}^{\infty} h_m X'_m(x) \text{ and } \sum_{m=1}^{\infty} h_m X''_m(x),$$

where $X_m(x),\ m=1,2,\ldots$ are orthonormal eigenfunctions, defined by (22).
It follows from (13) that

$$|X''(x)| \le |\lambda_m||d(x)||X_m(x)|.$$

Denoting $C_5=\max_{x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]}|d(x)|$ and using (24), (22) we have

$$|X''(x)| < C_5\cdot C\cdot m^2,\ x\in[0,\ell_1-\varepsilon]\cup\ldots\cup[\ell_{N-1}+\varepsilon,\ell_N].$$

Using (27), from the last inequality we find

$$|h_m X''_m(x)| \le \frac{D_3}{m^2},\ \ x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N],$$

where $D_3=\max\{C_5,D_1\}$ for any $\varepsilon>0$.
Integrating (13) from ℓ_{i-1} to x we have

$$X'_m(x)=X'_m(\ell_{i-1})-\lambda_m d(x)\int_{\ell_{i-1}}^{x} X_m(s)ds.$$

We find

$$|X'_m(x)| < C_3+C_4\cdot C\cdot m^2,\ \ x\in[\ell_{i-1}+\varepsilon,\ell_i],\ i=1,2,\ldots,N,$$

where $C_3=|X'_m(\ell_{i-1})|$, $C_4=\max_{x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]}|d(x)||\int_{\ell_{i-1}}^{x} ds|$.
Using (27), from the last inequality we get

$$|h_m X'_m(x)| \le \frac{D_2}{m^2},\ \ x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N],$$

where $D_2=\max\{C_1,C_2,C_3,C_4,D_1\}$ for any $\varepsilon>0$. Then by Weierstrass M-Test [Rudin (1967)], there exists functions $h'(x)$ and $h''(x)$ such that

$$h'(x)=\sum_{m=1}^{\infty} H_m X'_m(x),\ \ x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N]$$

$$h''(x)=\sum_{m=1}^{\infty} h_m X''_m(x),\ \ x\in[0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N].$$

Moreover these series uniformly converge to $h'(x)$ and $h''(x)$, respectively as $\varepsilon\to 0$.

Appendix D:

Let M be a natural number; $x^0 \in (0,\ell_1)\cup\ldots\cup(\ell_{N-1},\ell_N)$ be fixed; $X_m(x)$ be eigenfunctions for $m=1,2,\ldots$; $\delta_M(x,x^0)$ be a function defined by (28) and $h(x)$ be any four times differentiable function satisfying the conditions a), b), c) of the Property 3, then

$$\lim_{M\to\infty}\int_0^{\ell_N}\rho(x)A(x)\delta_M(x,x^0)h(x)dx=\rho(x^0)A(x^0)\sum_{n=1}^{\infty}X_n(x^0)h_n,$$

and according to the Theorem 6.4 in [Faydaoğlu and Guseinov (2003)], the series $\sum_{m=1}^{\infty}h_mX_m(x^0)$ converges to $h(x^0)$, so

$$\lim_{M\to\infty}\int_0^{\ell_N}\rho(x)A(x)\delta_M(x,x^0)h(x)dx=\rho(x^0)A(x^0)h(x^0)$$
$$=\int_0^{\ell_N}\rho(x)A(x)\delta(x-x^0)h(x)dx.$$

Hence, we can say that $\rho(x^0)A(x^0)\sum_{m=1}^{\infty}X_m(x^0)X_m(x)$ is the regularization of the Dirac delta function $\delta(x-x^0)$.

References

Bulavin, P. E.; Kashcheev, V. M. (1965): Solution of the non-homogeneous heat conduction equation for multilayered bodies. *International Chemical Engineering*, vol. 5, pp. 112-115.

Chen, J. T.; Ke, J. N.; Liao, H. Z. (2009): Construction of Green's function using null-field integral approach for Laplace problems with circular boundaries. *CMC: Computers, Materials and Continua*, vol. 9, pp. 93-109.

Chew, W. C. (1990): *Waves and Fields in Inhomogeneous Media*. Van Nostrand Reinhold, New York.

Faydaoglu, S.; Guseinov, G. Sh. (2003): Eigenfunction expansion for a Sturm-Liouville boundary value problem with impulse. *International Journal of Pure and Applied Mathematics*, vol. 8, pp. 137-170.

Garg, N.R.; Goel, A.; Miglani, A.; Kumar, R. (2004): Elastodynamic response of an anisotropic medium due to a line-load. *Earth Planets Space*, vol. 56, pp. 407-417.

Gu, M. H.; Young, D. L.; Fan, C. M. (2009): the method of fundamental solutions for one-dimensional wave eqautions. *CMC: Computers, Materials and Continua*, vol. 11, pp. 185-208.

İnceoğlu, S.; Gürgöze, M. (2000): Longitudinal vibrations of rods coupled by several spring-mass system. *Journal of Sound and Vibration* vol. 234(5), pp. 895-905.

Khojasteh, A.; Rahimian, M.; Pak, R.Y.S. (2008): Three-dimensional dynamic Green's functions in transversely isotropic bi-materials. *Int. J. Solids Struct.*, vol. 45, pp. 4952-4972.

Matsuda, H.; Sakiyama, T.; Morita, C.; and Kawakami, M. (1995): Longitudinal impulsive response analysis of variable cross-section bars. *Journal of Sound and Vibration* vol. 181, pp. 541-551.

Mehri, B.; Davar, A.; and Rahmani,O. (2009): Dynamic Green function solution of beams under a moving load with different boundary conditions. *Scientia Iranica* vol. 16(3), pp.273-279.

Mulholland G. P.; Cobble M. H. (1972): Diffusion through composite media. *International Journal of Mass Transfer*, vol. 15, pp. 147-160.

Nakamura, G.; Tanuma, K. (1997): A formula for the fundamental solution of anisotropic elasticity. *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 50, pp. 179-194.

Ozışık M. N. (1980): *Heat Conduction*, John Wiley and Sons, New York.

Pan, E.; Yuan, F. G. (2000): Three-dimensional Green's functions in anisotropic bimaterials. *International Journal of Solids and Structures*, vol. 37, pp. 5329-5351.

Rashed, Y. F. (2005): Green's first identity method for boundary-only solution of self-weight in BEM formulation for thick slabs. *CMC: Computers, Materials and Continua*, vol. 1, pp. 319-326.

Rangelov, T.V.; Manolis, G.D.; Dineva, P.S. (2005): Elastodynamic fundamental solutions for certain families of 2D inhomogeneous anisotropic domains: basic derivations. *Eur. J. Mech. A/Solids*, vol. 24, pp. 820-836.

Rangelov, T.V. (2003): Scattering from cracks in an elasto-anisotropic plane. *J. Theoret. Appl. Mech.*, vol. 33, pp. 55-72.

Rudin W. (1967): *Functional Analysis*, McGraw-Hill, New York.

Tewary, V. K. (1995): Computationally efficient representation for elastodynamic and elastostatic Green's functions for anisotropic solids. *Physical Review B*, vol. 51, pp. 15695–15702.

Tewary, V. K. (2004): Elastostatic Green's function for advanced materials subject to surface loading. *Journal of Engineering Mathematics*, vol. 49, pp. 289–304.

Tewary, V. K.; Bartolo L. M.; Powell A. C. (2002): Green's functions experts meeting and the Green digital library. NIST Workshop Report. [Online]. *Available: http://www.ctcms.nist.gov/gf/ [April 25, 2002].*

Ting, T. C. T. (2000): Recent developments in anisotropic elasticity. *International Journal of Solids and Structures*, vol. 37, pp. 401-409.

Ting, T. C. T. (2005): Green's functions for a bimaterial consisting of two orthotropic quarter planes subjected to an antiplane force and a secrew dislocation. *Mathematics and Mechanics of Solids*, vol. 10, no. 2, pp. 197 - 211.

Tittle C. W. (1965): Boundary value problems in composite media: quasi-orthogonal functions. *Journal of Applied Physics*, vol. 36, pp. 1486-1488.

Wang, V.Y.; Achenbach, J.D. (1995): Three-dimensional time-harmonic elastodynamic Green's functions for anisotropic solids. *Proc. R. Soc. Lond.*, vol. A449, pp. 441-458.

Wang, X.; Pan, E.; Feng, W.J. (2007): Time-dependent Green's functions for an anisotropic bimaterial with viscous interface. *Eur. J. Mech. A/Solids*, vol. 26, pp. 901-908.

Vavrycuk, V. (2001): Exact elastodynamic Green functions for simple types of anisotropy derived from higher-order ray theory.*Stud. Geophys. Geod.*, vol. 45, pp. 67-84.

Vavrycuk, V. (2002): Asymptotic elastodynamic Green function in the kiss singularity in homogeneous anisotropic solids. *Stud. Geophys. Geod.*, vol. 46, pp. 249-266.

Vladimirov V. S. (1971): *Equations of Mathematical Physics*, Marcel Dekker, New York.

Yakhno, V.; Cerdik Yaslan, H. (2011b): Computation of the time-dependent fundamental solution for equations of elastodynamics in general anisotropic media. *Computers and Structures*, vol. 89, pp. 646 -655.

Yakhno, V.; Cerdik Yaslan, H. (2011b): Computation of the time-dependent Green's function of three dimensional elastodynamics in 3D quasicrystals. *CMES: Computer Modelling in Engineering and Sciences, Tech Science Press*, vol.81, no. 3, pp. 295-310.

Yang, B.; Tewary, V. K. (2008): Green's function for multilayered with interfacial membrane and flexural rigidities. *CMC: Computers, Materials and Continua*, vol. 8, pp. 23-31.