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A Domain Decomposition Method Based on Natural BEM and Mixed FEM for Stationary Stokes Equations on Unbounded Domains

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Abstract: In this paper, a new domain decomposition method is suggested for the stationary Stokes equations on unbounded domain and its convergence is proved. We draw an artificial boundary to make the domain into two parts: one is bounded, in which we use the mixed finite element method; the other is unbounded, in which we apply the natural boundary reduction. Then we change the sub-problem on the unbounded domain onto a one in a bounded domain and we use the Dirichlet to Neumann(DtN) alternating algorithm to solve the resulting mixed system. The theoretical results as well as the numerical examples show that this method is very effective especially for problems over unbounded domains.

Keywords: domain decomposition, nature boundary element, mixed finite element method, Stokes equations, convergence.

1. Introduction

Let Ω be a simply connected domain in $\mathbb{R}^2$ with a Lipschitz-continuous boundary Γ_0, and denote Ω^c the complement of $\Omega\bigcup\Gamma_0$. We consider the following exterior boundary value Stokes equation

$$\begin{cases} -\eta\Delta\vec{u}+\nabla p &= \vec{f} \quad \text{in } \Omega^c \\ \operatorname{div}\vec{u} &= 0 \quad \text{in } \Omega^c \\ \vec{u} &= 0 \quad \text{on } \Gamma_0 \end{cases} \tag{1.1}$$

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where $\vec{u} = (u_1, u_2)^T$ is the velocity vector of the fluid which we assume it is limited at infinity, p is the kinematic static pressure, $\eta > 0$ is kinematic viscosity, $\vec{f} = (f_1, f_2)^T$ is a given density of outer volume force. Here we assume $\vec{f}$ has compact support, i.e. $supp\vec{f} \subseteq \Omega_0$, Ω_0 is a disk with radius R ($R > 0$ is a constant).

Many physicists and mathematicians has been attracted by this problem because of its wide range of practical applications. There are many kinds of methods to solve the Stokes problems. Girault and Raviart (1986) discussed the finite element method of the Stokes problems and Temam (1984); Verfurth (1984) introduced a detail mathematical theory and numerical analysis for this problem. For the problems over unbounded domains, the standard techniques such as FEM(finite element method), which is effective for most problems over bounded domains, will meet some difficulties and the corresponding computing cost will be very high. As a alternative, the boundary element method is considered and developed for this kind of problems(see, e. g. Han and Wu (1985b); Bao (2000); Meddahi and Sayas (2000); Reidinger and Steinbach (2003)).

The mixed finite element method(refer to Brezzi (1974); Raviart and Thomas (1977)), which is a general technique for the solution of partial differential equations, which arise in many fields of applications. In recent years, there has been a rapidly growing interest in developing the combination of mixed finite element method with boundary integral method(see, e.g. Han and Wu (1985b), Brink, Carstensen, and Stein (1996), Carstensen and Stefan (2000), Gatica and Wendland (1997)). One feather of our paper is that we'll use the mixed finite element method to solve this problem over unbounded domains. Another feather we want to mention in this paper is that natural boundary reduction method is used. Natural boundary reduction method, which is also known as the exact artificial boundary condition method, was suggested and developed by K. Feng, D. Yu and H. Han in the early 1980s(refer to Feng and Yu (1982); Han and Wu (1985a); Yu (1993, 2002)). Compared with many other approaches of reduction, natural boundary reduction method has its own advantages(see. e.g.Feng and Yu (1982); Yu (1993, 2002); Yang, Hu, and Yu (2005); Yu and Huang (2008); Liu and Yu (2008); Yang and Yu (2011)). Liu and Yu (2008) has solved this problem by the coupling of natural BEM and FEM. But here we'll consider the domain decomposition method based on natural boundary reduction and mixed finite element method and provide a DtN alternate algorithm to approximate the discrete system. The theoretical and numerical result will show that this algorithm is convergent and very effective for the stationary Stokes equations in 2-dimensional unbounded plane.

The rest of our paper is organized as following. In section 2, we apply the natural boundary reduction and derive the mixed variational formulation for the Stokes exterior problem. We also prove the existence and uniqueness of the resulting system.

In section 3, we propose a Dirichlet-Neumann(DtN) alternating method for solving mixed formulation of the problem in the interior domain and prove the convergence of this iterative method. Finally, the results of numerical experiments show that our theoretical results and algorithm are very effective for unbounded domains.

2. The mixed variational formulation

Define the Sobolev space

$$W^1(\Omega^c) = \left\{ w \left| \frac{w}{\sqrt{1+r^2}\ln(2+r^2)} , \frac{\partial w}{\partial x_i} \in L^2(\Omega^c),\ i=1,2,\ r=\sqrt{x_1^2+x_2^2} \right.\right\}, \tag{2.2}$$

$$L_0^2(\Omega^c) = \left\{ q \in L^2(\Omega^c);\ \int_{\Omega^c} q dx = 0 \right\} \tag{2.3}$$

and let

$$W_0^1(\Omega^c) = \left\{ \vec{v} \in W_0^1(\Omega^c)^2 | \vec{v} = 0, \text{on } \ \Gamma_0 \right\}, \tag{2.4}$$

Define $H := W_0^1$ and $M = L_0^2(\Omega^c)$ with norm

$$||\vec{v}||_H^2 := ||\vec{v}||_{1,\Omega_1}^2 + ||\vec{v}||_{\frac{1}{2},\Gamma_1}^2. \tag{2.5}$$

Then the exterior Stokes problem (1.1) is equivalent to the following variational form: Find $(\vec{u}, p) \in H \times M$, such that

$$\begin{cases} D(\vec{u},\vec{v}) - \displaystyle\int_{\Omega^c} p \text{div} \vec{v} dx_1 dx_2 = F(\vec{v}),\ \ \forall\, \vec{v} \in H, \\ \displaystyle\int_{\Omega_c} q \text{div} \vec{u} dx_1 dx_2 = 0,\ \ \forall\, q \in M. \end{cases} \tag{2.6}$$

where

$$D(\vec{u},\vec{v}) = 2\eta \sum_{i,j=1}^{2} \int\int_{\Omega^c} \varepsilon_{ij}(\vec{u}) \varepsilon_{ij}(\vec{v}) dx_1 dx_2, \tag{2.7}$$

$$\varepsilon_{ij}(\vec{v}) = \frac{1}{2} \left\{ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\},\ i,j=1,2, \tag{2.8}$$

$$F(\vec{v}) = \int_{\Omega^c} \vec{f} \vec{v} dx_1 dx_2 \tag{2.9}$$

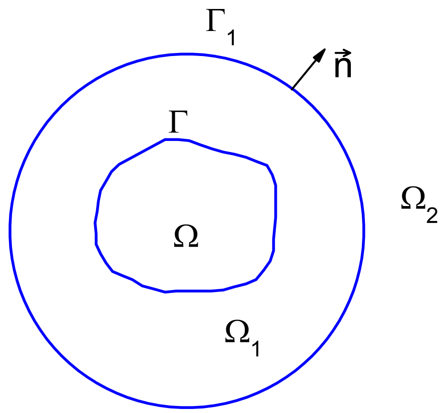

Figure 1: $\Omega^c = \Omega_1 \cup \Omega_2$, and Γ_1 is an auxiliary circle

Draw an artificial boundary Γ_1 with radius R dividing Ω^c into an unbounded part Ω_2 and a bounded part Ω_1 containing the support of $\vec{f}$(see Fig1.). Apply the natural boundary reduction to Ω_2 and let $\mathscr{K}$ be the natural integral operator of Stoke problem with respect to Ω_2, $\mathscr{K} : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$. Let

$$\widehat{D}_2(\vec{u}_0, \vec{v}_0) = \int_{\Gamma_1} \vec{v}_0 \cdot \mathscr{K} \vec{u}_0 ds. \tag{2.10}$$

We have the explicit expression of the natural integral equation on Γ_1(refer to Yu (1993)-Yu (2002))

$$\mathscr{K}\vec{u}|_{\Gamma_1} = \frac{2\eta}{R} \begin{bmatrix} -\dfrac{1}{4\pi\sin^2\frac{\theta}{2}} & 0 \\ 0 & -\dfrac{1}{4\pi\sin^2\frac{\theta}{2}} \end{bmatrix} * \begin{bmatrix} u_1(R,\theta) \\ u_2(R,\theta) \end{bmatrix} \tag{2.11}$$

where $*$ denotes the convolution with respect to θ. By the energy invariance we obtain

$$D_2(\vec{u}, \vec{v}) = \widehat{D}_2(\vec{u}_0, \vec{v}_0) + \int_{\Omega_2} p \mathrm{div}\vec{v} dx_1 dx_2. \tag{2.12}$$

Therefore problem (2.6) is equivalent to the problem in bounded subdomain Ω_1: Find $(\vec{u}, p) \in H \times M$ such that

$$\begin{cases} A(\vec{u}, \vec{v}) - B(\vec{v}, p) = F(\vec{v}), \ \ \forall\, \vec{v} \in H, \\ B(\vec{u}, q) = 0, \ \ \forall\, q \in M. \end{cases} \tag{2.13}$$

where

$$A(\vec{u},\vec{v}) = D_1(\vec{u},\vec{v}) + \widehat{D}_2(\gamma\vec{u},\gamma\vec{v}), \tag{2.14}$$

$$B(\vec{v},q) = \int_{\Omega_1} q\mathrm{div}\vec{v}dx_1dx_2 \tag{2.15}$$

and γ is the Dirichlet boundary value operator with respect to Γ_1.

The following lemma is proved in Yu (2002), which will be used in the following theorem.

Lemma 2.1 *The bilinear form $\widehat{D}_2(\vec{u}_0,\vec{v}_0)$ is symmetric, continuous and V-elliptic in $H^{\frac{1}{2}}(\Gamma_1)/\mathbb{R} \times H^{\frac{1}{2}}(\Gamma_1)/\mathbb{R}$, that's means the natural boundary integral operator $\mathscr{K}$ is $H^{\frac{1}{2}}(\Gamma_1)/\mathbb{R}$-elliptic, i.e., there exists $C > 0$ such that*

$$\langle \mathscr{K}\vec{v}_0,\vec{v}_0\rangle \geq C||\vec{v}_0||^2_{[H^{1/2}(\Gamma_1)/\mathbb{R}]^2}, \quad \forall \vec{v}_0 \in [H^{\frac{1}{2}}(\Gamma_1)/\mathbb{R}]^2. \tag{2.16}$$

Consider the closed subspace of H given by

$$H^0 = \{\vec{v} \in H | \mathrm{div}\vec{v} = 0, \text{ in } \Omega_1\} \tag{2.17}$$

Theorem 2.1 *Assume that the following hypotheses hold:*

(a) There exists a constant $\alpha > 0$ such that

$$A(\vec{v},\vec{v}) \geq C||\vec{v}||^2_H \quad \forall \vec{v} \in H^0 \tag{2.18}$$

(b) There exist a constant $\beta > 0$ such that

$$\sup_{\vec{v}\in H} \frac{B(\vec{v},q)}{||\vec{v}||_H} \geq \beta||q||_M \quad \forall q \in M. \tag{2.19}$$

Then the mixed variational problem (2.13) has a unique solution.

Proof: Condition (a) is straightforward to verigy. From Poincar*é*-Friedrichs inequality and Lemma 2.1, we derive

$$A(\vec{v},\vec{v}) = 2\eta|\vec{v}|^2_{1,\Omega_1} + \langle \mathscr{K}(\gamma\vec{v}),\gamma\vec{v}\rangle|_{\Gamma_1} \geq \alpha(||\vec{v}||^2_{1,\Omega_1} + ||\vec{v}||^2_{\frac{1}{2},\Gamma_1}) \geq \alpha||\vec{v}||^2_H. \tag{2.20}$$

For the proof of the so-called inf-sup condition (b), see Girault and Raviart (1986). The proof is complete.

3. A D-N Alternating algorithm

In this section we design a Dirichlet-Neumann(DtN) alternating algorithm for solving the resulting mixed system (2.13). For simplicity of exposition, we here discuss only the D-N alternating algorithm of continuous problems.

Algorithm 3.1

(1) Choose initial value $\vec{\lambda}^0 \in H^{\frac{1}{2}}(\Gamma_1)$*, and set* $n := 0, k := 0, p_0^1 \in L_0^2(\Omega_1)$.

(2) Solve the natural integral equation on Γ_1

$$\vec{t}^{2n} = \mathscr{K}\vec{\lambda}^n \tag{3.21}$$

(3) Solve the mixed boundary value problem in the annular subdomain Ω_1*:*

$$\begin{cases} A(\vec{v}^{2n+1}, p^{2n+1}) & = \vec{f} & \text{in } \Omega_1, \\ \mathrm{div}\vec{u}^{2n+1} & = 0 & \text{in } \Omega_1, \\ \vec{u}^{2n+1} & = 0 & \text{on } \Gamma_0, \\ \vec{t}^{2n+1} & = -\vec{t}^{2n} & \text{on } \Gamma_1. \end{cases} \tag{3.22}$$

where δ_{ij} *is the Kronecker Delta,* $\vec{n} = (n_1, n_2)$ *is the outward normal direction to* Ω_2 *and* $\vec{t} = (t_1, t_2)^T$

$$t_i = \sum_{i,j=1}^{2} \sigma_{ij}(\vec{u}, p) n_j, i = 1, 2, \tag{3.23}$$

$$\sigma_{ij}(\vec{u}, p) = -\delta_{ij} p + 2\eta \varepsilon_{ij}(\vec{u}), i, j = 1, 2, \tag{3.24}$$

Uzawa Method of (3.22):

$$\begin{cases} D_1(\vec{u}_{k+1}^{2n+1}, \vec{v}_1) & = F(\vec{v}_1) + (p_k^{2n+1}, \mathrm{div}\vec{v}_1) - \int_{\Gamma_1} \vec{t}^{2n} \vec{v}_1 dx, & \forall \vec{v}_1 \in H, \\ \vec{u}_{k+1}^{2n+1} & = 0, & \text{on } \Gamma_0, \\ p_{k+1}^{2n+1} & = p_k^{2n+1} - \rho \mathrm{div} \vec{u}_{k+1}^{2n+1}, & \end{cases} \tag{3.25}$$

(4) If $||\vec{u}_{k+1}^{2n+1} - \vec{u}_k^{2n+1}|| < \varepsilon_1$ *and* $||p_{k+1}^{2n+1} - p_k^{2n+1}|| < \varepsilon_2$*, then continue; If not, set* $k := k+1$ *and goto step (3).*

(5) *Directly compute the following formula:*

$$\begin{pmatrix} \vec{u}^{2n+2} \\ p^{2n+2} \end{pmatrix} = P\vec{u}_{k+1}^{2n+1}|_{\Gamma_1}$$

(6) *If the solution approximate enough, stop computing; If not, let*

$$\vec{\lambda}^{n+1} = \theta_n \vec{u}_{k+1}^{2n+1} + (1-\theta_n)\vec{\lambda}^n, \quad on\ \Gamma_1, \tag{3.26}$$

then set $n := n+1$ *and goto step (2).*

In step (3) ρ denotes a relaxation factor, $0 < \rho < 2\eta$. In step (5) P is the Poisson integral operator of Stokes equation(refer to Yu (2002)).

Remark 3.1 *Algorithm 3.1 is absolutely different with the usually DtN algorithm. In step (2) we need not solve the Dirichlet boundary value problem on unbounded domain* Ω_2. *We just directly apply the natural boundary integral operator* $\mathscr{K}$, *which is just the DtN operator, i.e. Steklov-Poincaré operator for* Ω_2 *and solve the value of the outward normal derivative* $\vec{t}^{2n}$ *by the given boundary value* $\vec{\lambda}^n$ *on* Γ_1.

Therefore the Algorithm 3.1 can greatly reduce the computational work. We only treat the mixed boundary value problem in the rather small bounded domain where the mixed finite element method will be used. Moreover, it is very important to choose the relaxation factor ρ and θ_n in Algorithm 3.1. The iteration may be not convergent when we choose some bad relaxation factors. Usually we choose $\theta_n = 0.5$ and $\rho = \frac{1}{2}\eta$ or some bigger values.

Next, we analysis the convergence of Algorithm 3.1. The convergence of Uzawa method in step (3) can see Girault and Raviart (1986). Then we have

$$\vec{u}_{k+1}^{2n+1} \to \vec{u}^{2n+1}, \ \ p_{k+1}^{2n+1} \to p^{2n+1}, \ k \to \infty. \tag{3.27}$$

Taking limit as $k \to \infty$ in (3.25), we get

$$\begin{cases} D_1(\vec{u}^{2n+1}, \vec{v}_1) & = F(\vec{v}_1) + (p^{2n+1}, \mathrm{div}\vec{v}_1) - \int_{\Gamma_1} \vec{t}_2^{\,n} \vec{v}_1 dx, & \forall \vec{v}_1 \in H, \\ (q, \mathrm{div}\vec{u}^{2n+1}) & = 0, & \forall q \in M, \\ \vec{u}^{2n+1} & = 0, & \text{on } \Gamma_0, \end{cases} \tag{3.28}$$

Assumed that $\Phi = \left\{\vec{v}|_{\Gamma_1} : \vec{v} \in W^0\right\}$, then $\forall \vec{\varphi} \in \Phi,\ k = 1,2$, find $(\vec{\omega}_k(\vec{\varphi}), \tau_k(\vec{\varphi})) \in W_k^0 \times L_0^2(\Omega_k)$ such that

$$\begin{cases} D_k(\vec{\omega}_k(\vec{\varphi}),\vec{v}) + b_k(\vec{v},\tau_k(\vec{\varphi})) &= 0, \qquad \forall \vec{v} \in W_k^{00}, \\ b_k(\vec{\omega}_k(\vec{\varphi}),q) &= 0, \qquad \forall q \in L_0^2(\Omega_k), \\ \vec{\omega}_k(\vec{\varphi}) &= \vec{\phi}, \qquad \text{on } \Gamma_1, \\ \vec{\omega}_k(\vec{\varphi}) &= 0, \qquad \text{on } \partial\Omega_k/\Gamma_1, \end{cases} \tag{3.29}$$

where

$$b_k(\vec{v},q) = \int\int_{\Omega_k} q \operatorname{div}\vec{v} dx_1 dx_2, \tag{3.30}$$

$$W_k^0 = \left\{\vec{v} \in W_0^1(\Omega_k)^2 | \vec{v} = 0,\ \text{on } \partial\Omega_k/\Gamma_1\right\}, \tag{3.31}$$

$$W_k^{00} = \left\{\vec{v} \in W_k^0 | \vec{v} = 0,\ \text{on } \Gamma_1\right\} \tag{3.32}$$

We call $(\vec{\omega}_k(\vec{\varphi}), \tau_k(\vec{\varphi}))$ as the Stokes extension from $\vec{\varphi}$ to $\Omega_k, k = 1,2$. Let

$$< S\vec{\varphi},\vec{\psi} > := S(\vec{\varphi},\vec{\psi}) := \sum_{k=1}^{2} [D_k(\vec{\omega}_k(\vec{\varphi}), R_k\vec{\psi}) + b_k(R_k\vec{\psi}, \tau_k(\vec{\varphi}))] \tag{3.33}$$

where the operator S is called the Skeklov-Poincar*é* operator on the auxiliary boundary Γ_1 and R_k denotes any extension operator from Φ to W_k^0:

$$R_k : \Phi \mapsto W_k^0 \tag{3.34}$$

$$R_k\vec{\varphi}|_{\Gamma_1} = \vec{\varphi}|_{\Gamma_1},\ \ R_k\vec{\varphi}|_{\partial\Omega_k/\Gamma_1} = 0. \tag{3.35}$$

Theorem 3.1

$$S(\vec{\varphi},\vec{\psi}) = D_1(\vec{\omega}_k(\vec{\varphi}),\vec{\omega}_k(\vec{\psi})) + D_2(\vec{\omega}_k(\vec{\varphi}),\vec{\omega}_k(\vec{\psi})). \tag{3.36}$$

Proof: From the definition (3.33)

$$\begin{aligned} S(\vec{\varphi},\vec{\psi}) &= \sum_{k+1}^{2} (D_k(\vec{\omega}_k(\vec{\varphi}),\vec{\omega}_k(\vec{\psi})) + D_k(\vec{\omega}_k(\vec{\varphi}), R_k\vec{\psi} - \vec{\omega}_k(\vec{\psi})) \\ &+ b_k(R_k\vec{\psi} - \vec{\omega}_k(\vec{\varphi}), \tau_k(\vec{\varphi})) + b_k(\vec{\omega}_k(\vec{\varphi}), \tau_k(\vec{\varphi}))). \end{aligned} \tag{3.37}$$

Because of $\vec{v} = R_k\vec{\psi} - \vec{\omega}_k(\vec{\psi}) \in W_k^{00}$, using (3.28) we obtain

$$b_k(\vec{\omega}_k(\vec{\varphi}), \tau_k(\vec{\varphi}))) = 0, \tag{3.38}$$

$$D_k(\vec{\omega}_k(\vec{\varphi}), R_k\vec{\psi} - \vec{\omega}_k(\vec{\psi})) + b_k(R_k\vec{\psi} - \vec{\omega}_k(\vec{\varphi}), \tau_k(\vec{\varphi}))) = 0. \tag{3.39}$$

The proof is complete.

Set

$$S_k(\vec{\varphi},\vec{\psi}) = D_k(\vec{\omega}_k(\vec{\varphi}),\vec{\omega}_k(\vec{\psi})) \tag{3.40}$$

From the proof of Theorem 3.1 we derive

$$S_k(\vec{\varphi},\vec{\psi}) = D_k(\vec{\omega}_k(\vec{\varphi}),R_k\vec{\psi}) + b_k(R_k\vec{\psi},\tau_k(\vec{\psi})) \tag{3.41}$$

Therefore $S = S_1 + S_2$. We assume that the solution $(\vec{\omega}_k^*,\tau_k^*)$ satisfy the following equations

$$\begin{cases} D_k(\vec{\omega}_k^*,\vec{v}) + b_k(\vec{v},\tau_k^*) &= 0, \qquad \forall \vec{v} \in W_k^{00}, \\ b_k(\vec{\omega}_k^*,q) &= 0, \qquad \forall q \in L_0^2(\Omega_k), \\ \vec{\omega}_k^* &= 0, \qquad \text{on } \Gamma_1, \\ \vec{\omega}_k^* &= 0, \qquad \text{on } \partial\Omega_k/\Gamma_1, \end{cases} \tag{3.42}$$

Define $\chi_i \in \Phi'$ as

$$< \chi_i,\vec{\varphi} >:= F_i(R_i\vec{\varphi}) - D_i(\vec{\omega}_i^*,R_i\vec{\varphi}) - b_i(R_i\vec{\varphi},\tau_i^*) \ \ \forall \vec{\varphi} \in \Phi, \tag{3.43}$$

and also define $\chi := \chi_1 + \chi_2$, then we have

Lemma 3.1 *Assumed that $(\vec{u},p)$ is the solution of (2.6)and let$\lambda := \vec{u}|_{\Gamma_1}$, then λ the Steklov-Poincaré equation on the artificial boundary Γ_1:*

$$\lambda \in \Phi :< S\lambda,\mu >:=< \chi,\mu > \quad \forall \mu \in \Phi. \tag{3.44}$$

For its proof see Quarteroni and Valli (1999).

Theorem 3.2 *The DtN alternating method is equivalent to the associated preconditioned Richardson iterative method:*

$$S_1(\vec{\lambda}^{n+1} - \vec{\lambda}^n) = \theta_n(\chi - S\vec{\lambda}^n) \tag{3.45}$$

where $\chi = S\vec{\lambda}$.

Proof: The step (5) of Algorithm 3.1 is equivalent to solve the following Dirichlet problem in Ω_2:

$$\begin{cases} D_2(\vec{u}^{2n},\vec{v}) + b_2(\vec{v},p^{2n}) &= (\vec{f},\vec{v}), \qquad \forall \vec{v} \in W_2^{00}, \\ b_2(\vec{u}^{2n},q) &= 0, \qquad \forall q \in L_0^2(\Omega_2), \\ \vec{u}^{2n} &= \vec{\lambda}^n, \qquad \text{on } \Gamma_1, \end{cases} \tag{3.46}$$

(3.28) is equivalent to solve the mixed boundary value problem in Ω_1:

$$\begin{cases} D_1(\vec{u}^{2n+1},\vec{v})+b_1(\vec{v},p^{2n+1}) &= (\vec{f},\vec{v}), \qquad \forall \vec{v}\in W_1^{00},\\ b_1(\vec{u}^{2n+1},q) &= 0, \qquad \forall q\in L_0^2(\Omega_1),\\ \vec{u}^{2n+1} &= 0, \qquad \text{on } \Gamma_0,\\ \vec{t}^{2n+1} &= -\vec{t}^{2n}, \qquad \text{on } \Gamma_1. \end{cases} \tag{3.47}$$

Taking $k\to\infty$, (3.26) becomes

$$\vec{\lambda}^{n+1} = \theta\vec{u}^{2n+1}+(1-\theta_n)\vec{\lambda}^{n}, \ \text{ on } \Gamma_1. \tag{3.48}$$

We consider the error $\vec{e}^i = \vec{u}-\vec{u}^i$, $e_p^i = p-p^i$, $\vec{\mu}^n = \vec{\lambda}-\vec{\lambda}^n$ and compare (2.6) with (3.46), (3.47). We derive that $\vec{e}^i, e_p^i, \vec{\mu}^i$ satisfy

$$\begin{cases} D_2(\vec{e}^{2n},\vec{v})+b_2(\vec{v},e_p^{2n}) &= 0, \qquad \forall \vec{v}\in W_2^{00},\\ b_2(\vec{e}^{2n},q) &= 0, \qquad \forall q\in L_0^2(\Omega_2),\\ \vec{e}^{2n} &= \vec{\mu}^n, \qquad \text{on } \Gamma_1, \end{cases} \tag{3.49}$$

$$\begin{cases} D_1(\vec{e}^{2n+1},\vec{v})+b_1(\vec{v},e_p^{2n+1}) &= 0, \qquad \forall \vec{v}\in W_1^{00},\\ b_1(\vec{e}^{2n+1},q) &= 0, \qquad \forall q\in L_0^2(\Omega_1),\\ \vec{e}^{2n+1} &= 0, \qquad \text{on } \Gamma_0,\\ \vec{t}_1-\vec{t}^{2n+1} &= -(t_2-\vec{t}^{2n}), \quad \text{on } \Gamma_1, \end{cases} \tag{3.50}$$

$$\vec{\mu}^{n+1} = \theta_n\vec{e}^{2n+1}+(1-\theta_n)\vec{\mu}^{n}, \text{ on } \Gamma_1. \tag{3.51}$$

From (3.49) we obtain

$$\vec{e}^{2n} = \vec{\omega}_2(\vec{\mu}^n), \quad e_p^{2n} = \tau_2(\vec{\tau}^n). \tag{3.52}$$

Also From (3.50) we get

$$\vec{e}^{2n+1} = \vec{\omega}_1(\vec{e}^{2n+1}|_{\Gamma_1}), \quad e_p^{2n+1} = \tau_1(\vec{e}^{2n+1}|_{\Gamma_1}). \tag{3.53}$$

By the fourth equation of (3.50) we have

$$\int_{\Gamma_1}(\vec{t}_1-\vec{t}^{2n+1})R_1\vec{\varphi}ds = -\int_{\Gamma_1}(\vec{t}_2-\vec{t}^{2n})R_2\vec{\varphi}ds \quad \forall\vec{\varphi}\in\Phi. \tag{3.54}$$

Using Green Formula we derive

$$D_1(\vec{e}^{2n+1}, R_1\vec{\varphi}) + b_1(R_1\vec{\varphi}, e_p^{2n+1}) = -(D_2(\vec{e}^{2n}, R_2\vec{\varphi}) + b_2(R_2\vec{\varphi}, e_p^{2n})), \tag{3.55}$$

i.e. $S_1(\vec{e}^{2n+1}|_{\Gamma_1}, \vec{\varphi}) = -S_2(\vec{\mu}^n, \vec{\varphi})$. So

$$\vec{e}^{2n+1}|_{\Gamma_1} = -S_1^{-1}S_2\vec{\mu}^n. \tag{3.56}$$

Then we obtain

$$\begin{aligned} \vec{\mu}^{n+1} - \vec{\mu}^n &= \theta_n\vec{e}^{2n+1}|_{\Gamma_1} + (1-\theta_n)\vec{\mu}^n - \vec{\mu}^n \\ &= \theta_n\vec{e}^{2n+1}|_{\Gamma_1} - \theta_n\vec{\mu}^n \\ &= -\theta_n S_1^{-1}S_2\vec{\mu}^n - \theta_n\vec{\mu}^n \end{aligned} \tag{3.57}$$

or

$$S_1(\vec{\mu}^{n+1} - \vec{\mu}^n) = -\theta_n(S_2 + S_1)\vec{\mu}^n = -\theta_n S\vec{\mu}^n. \tag{3.58}$$

By $\vec{\lambda} = \vec{\mu}|_{\Gamma_1}$ and $S\vec{\lambda} = \chi$, we have the following Richardson iteration formulation

$$S_1(\vec{\lambda}^{n+1} - \vec{\lambda}^n) = \theta_n(\chi - S\vec{\lambda}^n). \tag{3.59}$$

The proof is complete.

Remark 3.2 *When we prove that the solution $\vec{u}$ is convergent, we get the convergence of p, e.g. from (3.49), (3.50), as $n \to \infty$, $\vec{e}^{2n}, \vec{e}^{2n+1} \to 0$, therefore we obtain $e_p^{2n}, e_p^{2n+1} \to 0$, i.e. p is also convergent.*

Let $\vec{e}_n = \vec{\lambda}^n - \vec{\lambda}$ and then we have

$$\vec{e}_{n+1} = (I - \theta_n S_1^{-1}S)\vec{e}_n \tag{3.60}$$

Define $H_k(\vec{\varphi})$ as a Stokes extension from $\vec{\varphi} \in \Phi$ to Ω_k:

$$\begin{aligned} H_k(\vec{\varphi}) \in W_k^0 : D_k(H_k(\vec{\varphi}), \vec{v}) &= 0 \quad \forall \vec{v} \in W_k^{00}, \\ H_k(\vec{\varphi}) &= \vec{\varphi} \quad \text{on } \Gamma_1, \\ H_k(\vec{\varphi}) &= 0 \quad \text{on } \partial\Omega_k/\Gamma_1. \end{aligned} \tag{3.61}$$

$$||H_k(\vec{\lambda})||_k^2 = D_k(H_k(\vec{\lambda}), H_k(\vec{\lambda})), k = 1, 2. \tag{3.62}$$

Then by the priori estimate we obtain

$$||H_k(\vec{\varphi})||_{1,\Omega_k} \le ||\vec{\varphi}||_\Phi, \ \ \forall \vec{\varphi} \in \Phi. \tag{3.63}$$

In order to get the convergence of Algorithm 3.1, the following lemmas are given first.

Lemma 3.2 *For $k=1,2$, $C_1||H_k(\vec{\lambda})||_k \le ||\vec{\omega}_k(\vec{\lambda})||_k \le C_2||H_k(\vec{\lambda})||_k \ \ \forall\vec{\lambda}\in\Phi$.*

Proof: Applying the trace theorem and (3.63), we have

$$||H_k(\vec{\lambda})||_1 \le C||\vec{\lambda}||_\Phi \le C'||\vec{\omega}_k(\vec{\lambda})||_k. \tag{3.64}$$

So

$$C_1||H_k(\vec{\lambda})||_k \le ||\vec{\omega}_k(\vec{\lambda})||_k. \tag{3.65}$$

Due to $\vec{\omega}_k(\vec{\lambda})-H_k(\vec{\lambda})\in W_k^{00}$, by the definition (3.61) we get

$$D_k(H_k(\vec{\lambda}),H_k(\vec{\lambda})-\vec{\omega}_k(\vec{\lambda}))=0, \tag{3.66}$$

Therefore,

$$D_k(H_k(\vec{\lambda}),H_k(\vec{\lambda}))=D_k(H_k(\vec{\lambda}),\vec{\omega}_k(\vec{\lambda})). \tag{3.67}$$

Finally we derive

$$||H_k(\vec{\lambda})||_k^2=D_k(H_k(\vec{\lambda}),\vec{\omega}_k(\vec{\lambda}))\le C_2||H_k(\vec{\lambda})||_k||\vec{\omega}_k(\vec{\lambda})||_k \tag{3.68}$$

Then the lemma is proved. Using the trace theorem we have the following lemma.

Lemma 3.3 *There exists positive constants σ and ζ such that*

$$\sigma=\sup_{\vec{\lambda}}\frac{||H_1(\vec{\lambda})||_1^2}{H_2(\vec{\lambda})||_2^2}, \quad \zeta=\sup_{\vec{\lambda}}\frac{H_2(\vec{\lambda})||_2^2}{H_1(\vec{\lambda})||_1^2}, \tag{3.69}$$

where $H_k(\vec{\lambda})||_k^2=D_k(H_k(\vec{\lambda}),H_k(\vec{\lambda})),k=1,2$.

Then we have

$$C_1||H_1(\vec{\lambda})||_1||\le H_2(\vec{\lambda})||_2\le C_2||H_1(\vec{\lambda})||_1 \quad \forall\vec{\lambda}\in\Phi. \tag{3.70}$$

From Lemma 3.2 and (3.70), we obtain

$$K_1||\vec{\omega}_1(\vec{\lambda})||_1\le||\vec{\omega}_2(\vec{\lambda})||_2\le K_1||\vec{\omega}_1(\vec{\lambda})||_1, \tag{3.71}$$

where K_1,K_2 are positive constants. So $||\vec{\omega}_1(\vec{\lambda})||_1$ is equivalent with $||\vec{\omega}_2(\vec{\lambda})||_2$.

Theorem 3.3 *There exists $\theta^*\in(0,1)$ such that $\forall\theta_n\in(0,\theta^*)$, Algorithm 3.1 is convergent.*

Proof: Since

$$\frac{(S\vec{\lambda},\vec{\lambda})}{(S_1\vec{\lambda},\vec{\lambda})} = 1 + \frac{(S_2\vec{\lambda},\vec{\lambda})}{(S_1\vec{\lambda},\vec{\lambda})} = \frac{D_2(\vec{\omega}_2(\vec{\lambda}),\vec{\omega}_2(\vec{\lambda}))}{D_1(\vec{\omega}_1(\vec{\lambda}),\vec{\omega}_1(\vec{\lambda}))} \tag{3.72}$$

Connecting (3.71) we obtain

$$1 + K_1^2 \leq \frac{(S\vec{\lambda},\vec{\lambda})}{(S_1\vec{\lambda},\vec{\lambda})} \leq 1 + K_2^2 \tag{3.73}$$

Due to the matrix $B = I - \theta_n S_1^{-1} S_2$, we can deduce that

$$\rho(B) = \max_i |1 - \theta_n \lambda_i|, \tag{3.74}$$

where λ_i is the eigenvalue of $S_1^{-1}S_2$, $\rho(b)$ is spectral radius. We choose the value of θ_n such that

$$0 < \theta_n < \frac{2}{2 + K_1^2 + K_2^2} < \frac{2}{1 + K_2^2} \leq \min_i \frac{2}{\lambda_i}. \tag{3.75}$$

So we have $\rho(b) < 1$ and deduce the iteration is convergent. We may choose $\theta^* = \frac{2}{2+K_1^2+K_2^2}$, then there exists a positive constant $\theta^* \in (0,1)$ such that $\forall \theta_n \in (0,\theta^*)$, $\rho(B) < 1$, that's means Algorithm 3.1 is convergent. The proof is complete.

4. Numerical Experiments

Let Ω^c be a unbounded domain outside a square Ω

$$\Omega = \{(x,y)| -L \leq x \leq L, -L \leq y \leq L, L > 0\}, \tag{4.76}$$

Γ_0 denotes the boundary of Ω, i.e.

$$\Gamma_0 = \{(x,y)| -L \leq x \leq L, y = \pm L; x = \pm L, -L \leq y \leq L\}. \tag{4.77}$$

The artificial boundary Γ_1 is a circumference with radius R, $R > \sqrt{2}L$, which dividing the unbounded domain Ω^c into one bounded sub-domain Ω_1 and another unbounded sub-domain Ω_2. We divide Γ_1 into equal segmental arcs and denote the triangulation in Γ_0 and Γ_1 as Γ_{0h} and Γ_{1h}, respectively. We take piecewise linear boundary element on Γ_1:

$$L_i(\theta) = \begin{cases} \dfrac{N}{\theta - \theta_{i-1}} & \theta_{i-1} \leq \theta \leq \theta_i \\ \dfrac{N}{\theta_{i+1} - \theta} & \theta_i \leq \theta \leq \theta_{i+1}, i = 1, \cdots, N. \\ 0 & \text{otherwise} \end{cases} \tag{4.78}$$

Then we have the discrete formulation of the natural integral operator $\mathscr{K}$ (2.11):

$$\mathscr{K}_h = \frac{1}{R}\begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} \tag{4.79}$$

where

$$Q_{11} = Q_{22} = 2\eta \begin{bmatrix} a_0 & a_1 & \cdots & a_{N-2} & a_{N-1} \\ a_{N-1} & a_0 & \cdots & a_{N-3} & a_{N-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{N-1} & a_0 \end{bmatrix}, \tag{4.80}$$

Q_{11}, Q_{22} are symmetric circular matrix of rank $N-1$ and

$$a_k = \frac{4N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\sin^4\frac{j}{N}\pi\cos\frac{jk}{N}2\pi, \quad k = 0, 1, \cdots, N-1. \tag{4.81}$$

When computing, we substitute $\sum\limits_{j=1}^{\infty}$ by $\sum\limits_{j=1}^{M}$ and we choose $M = 20$ here. We make the finite element subdivision in Ω_1 and apply the mixed finite element method to solve the discrete system in Ω_1; As to the unbounded subdomain we directly use the following Poisson integral formula:

$$\begin{pmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \\ p(r,\theta) \end{pmatrix} = \begin{pmatrix} p_{ss} & p_{st} \\ p_{ts} & p_{tt} \\ p_{sss} & p_{ttt} \end{pmatrix} * \begin{pmatrix} u_r(R,\theta) \\ u_\theta(R,\theta) \end{pmatrix}, \quad r > R, \tag{4.82}$$

where

$$
\begin{aligned}
p_{ss} &= \cos\theta P(r,\theta) + \frac{r^2-R^2}{2r^2}\left[\cos\theta\left(-r\frac{\partial P(r,\theta)}{\partial r}\right) + \sin\theta\frac{\partial P(r,\theta)}{\partial\theta}\right],\\
p_{st} &= \sin\theta P(r,\theta) + \frac{r^2-R^2}{2r^2}\left[\sin\theta\left(-r\frac{\partial P(r,\theta)}{\partial r}\right) - \cos\theta\frac{\partial P(r,\theta)}{\partial\theta}\right],\\
p_{ts} &= -\sin\theta P(r,\theta) + \frac{r^2-R^2}{2r^2}\left[\sin\theta\left(-r\frac{\partial P(r,\theta)}{\partial r}\right) - \cos\theta\frac{\partial P(r,\theta)}{\partial\theta}\right],\\
p_{tt} &= \cos\theta P(r,\theta) - \frac{r^2-R^2}{2r^2}\left[\cos\theta\left(-r\frac{\partial P(r,\theta)}{\partial r}\right) + \sin\theta\frac{\partial P(r,\theta)}{\partial\theta}\right],\\
p_{sss} &= \frac{2\eta}{r}\left[\cos\theta\left(-r\frac{\partial P(r,\theta)}{\partial r}\right) + \sin\theta\frac{\partial P(r,\theta)}{\partial\theta}\right],\\
p_{ttt} &= \frac{2\eta}{r}\left[\sin\theta\left(-r\frac{\partial P(r,\theta)}{\partial r}\right) - \cos\theta\frac{\partial P(r,\theta)}{\partial\theta}\right],
\end{aligned}
$$

and

$$
P(r,\theta) = \frac{r^2-R^2}{2\pi(R^2+r^2-2rR\cos\theta)}, \quad r > R. \tag{4.83}
$$

Here $P(r,\theta-\theta')$ is just the Poisson kernel for the harmonic equation in an exterior circular domain of radius R. Denote the maximum error of the exact solution and approximation solution in Ω_{1h} as:

$$
\begin{aligned}
E_1(n) &= \max_{\Omega_{1h}}\left|u_1 - u_{1h}^{2n+1}\right|, \qquad (4.84)\\
E_2(n) &= \max_{\Omega_{1h}}\left|u_2 - u_{2h}^{2n+1}\right],\\
E_p(n) &= \max_{\Omega_{1h}}\left|p - p_h^{2n+1}\right|
\end{aligned}
$$

and denote the maximum error of the approximation solutions between two itera-

Table 1: The maximum error of Example 1, $R = 6$, $\eta = 1.0$

m	N	iteration steps						
		n	1	2	3	4	5	6
		k	15	13	9	5	3	2
8	16	$E_1(n)$	6.1258E-2	2.4928E-2	2.5451E-2	2.3407E-2	2.5136E-2	2.3776E-2
		$E_2(n)$	3.3527E-2	1.0444E-2	2.2920E-2	4.8594E-3	4.4760E-3	3.8078E-3
		$E_p(n)$	3.9747E-2	2.8606E-2	3.2609E-2	3.1652E-2	3.2228E-2	3.1915E-2
		$\varepsilon_1(n)$	8.9036E-2	8.3953E-3	3.6476E-3	4.7622E-3	3.93963E-3	3.3723E-3
		$\varepsilon_2(n)$	6.1305E-2	1.3319E-2	6.0768E-3	4.1130E-3	3.2505E-3	2.6302E-3
		$\varepsilon_p(n)$	3.9747E-2	2.2763E-2	4.8224E-3	9.5640E-4	5.9312E-4	4.6203E-4
		k	12	11	7	3	1	1
16	64	$E_1(n)$	5.3999E-2	1.4930E-2	1.3687E-2	1.4311E-2	1.3920E-2	1.4157E-2
		$E_2(n)$	2.2774E-2	9.5596E-3	4.5125E-3	5.9324E-3	5.1798E-3	5.6665E-3
		$E_p(n)$	3.2187E-2	1.3863E-2	1.5544E-2	1.5355E-2	1.5392E-2	1.5370E-2
		$\varepsilon_1(n)$	8.1777E-2	1.5129E-3	2.9905E-3	1.9736E-3	8.2650E-4	5.4434E-4
		$\varepsilon_2(n)$	5.0552E-2	7.0578E-3	1.5182E-3	6.6081E-4	4.2220E-4	2.7555E-4
		$\varepsilon_p(n)$	3.2187E-2	2.9068E-2	3.4258E-3	4.7867E-4	1.0273E-4	6.8058E-5

tion steps as:

$$\begin{aligned} \varepsilon_1(n) &= \max_{\Omega_{1h}} \left| u_{1h}^{2n-1} - u_{1h}^{2n+1} \right|, \qquad (4.85) \\ \varepsilon_2(n) &= \max_{\Omega_{1h}} \left| u_{2h}^{2n-1} - u_{2h}^{2n+1} \right|, \\ \varepsilon_p(n) &= \max_{\Omega_{1h}} \left| p_h^{2n-1} - p_h^{2n+1} \right|. \end{aligned}$$

In the following numerical examples, we take the side length $R = 6$, $\eta = 1.0$, $L = 3.0$ and denote m as the numbers of arcs on the artificial boundary, N as the number of elements in Ω_1, n as the iteration steps of the DtN alternating algorithm, k as the iteration steps of the mixed finite element in the bounded domain Ω_1. In computation we choose $\theta_n = 0.5, \rho = 0.5$. $\varepsilon_1 = 10^{-4}, \varepsilon_2 = 10^{-3}$ are denoted as the bound of error. The computational results are listed in Table 1 – Table 3.

Example 1. Let Ω^c be the exterior domain outside of the square with side length

Table 2: The maximum error of Example 2, $R=6$, $\eta=1.0$

m	N	iteration steps						
		n	1	2	3	4	5	6
		k	14	13	8	5	3	2
8	16	$E_1(n)$	4.8023E-3	3.0654E-3	1.2607E-3	1.0116E-3	8.1233E-4	6.5283E-4
		$E_2(n)$	4.8023E-3	3.0654E-3	1.2607E-3	1.0116E-3	8.1233E-4	6.5283E-4
		$E_p(n)$	1.5489E-2	9.6173E-3	2.9424E-3	2.3247E-3	2.3247E-3	1.8374E-3
		$\varepsilon_1(n)$	1.7369E-3	1.1026E-3	4.4837E-4	3.5877E-4	2.8726E-4	2.3019E-4
		$\varepsilon_2(n)$	1.7369E-3	1.1026E-3	4.4837E-3	3.5877E-4	2.8726E-4	2.3019E-4
		$\varepsilon_p(n)$	5.8720E-3	3.6361E-3	1.4011E-3	1.1053E-3	8.7221E-4	6.8820E-4
		k	10	9	5	2	1	1
16	64	$E_1(n)$	3.5749E-3	1.8478E-3	9.6059E-4	6.2481E-4	4.0722E-4	3.2902E-4
		$E_2(n)$	3.5749E-3	1.8478E-3	9.6059E-4	6.2481E-4	4.0722E-4	3.2902E-4
		$E_p(n)$	1.2253E-2	6.7483E-3	3.7826E-3	2.5941E-3	1.7903E-3	1.4906E-3
		$\varepsilon_1(n)$	1.2731E-3	6.5593E-4	4.5631E-4	2.1759E-4	1.4122E-4	1.1383E-4
		$\varepsilon_2(n)$	1.2731E-3	6.5593E-4	4.5631E-4	2.1759E-4	1.4122E-4	1.1383E-4
		$\varepsilon_p(n)$	4.0344E-3	2.1692E-3	1.6292E-3	8.0381E-4	5.4760E-4	4.5318E-4

$2L$, solve the following Stokes equation:

$$\begin{cases} -\eta\Delta\vec{u}+\nabla p &= 0 \quad \text{in } \Omega^c \\ \mathrm{div}\vec{u} &= 0 \quad \text{in } \Omega^c \\ \vec{u} &= 0 \quad \text{on } \Gamma_0 \end{cases} \tag{4.86}$$

with the exact solution

$$\begin{cases} u_1(r,\theta) &= \dfrac{\cos 2\theta}{r^2} = \dfrac{x^2-y^2}{(x^2+y^2)^2}, \\ u_2(r,\theta) &= \dfrac{\sin 2\theta}{r^2} = \dfrac{2xy}{(x^2+y^2)^2}, \\ p(r,\theta) &= 0 \end{cases} \tag{4.87}$$

Example 2. Let Ω^c be the exterior domain outside of the square with side length

Table 3: The maximum error of Example 3, $R = 6$, $\eta = 1.0$

m	N	iteration steps						
		n	1	2	3	4	5	6
		k	15	14	10	5	3	2
8	16	$E_1(n)$	2.7909E-3	1.9393E-3	9.3822E-4	3.8010E-4	1.8513E-4	1.5474E-4
		$E_2(n)$	1.1149E-3	7.6882E-4	3.6803E-4	1.4819E-4	7.2142E-5	6.0321E-5
		$E_p(n)$	8.9105E-3	6.1023E-3	2.8593E-3	1.1073E-3	5.1823E-4	4.2864E-4
		$\varepsilon_1(n)$	1.1739E-3	5.9094E-4	2.8494E-4	1.1494E-4	5.5773E-5	4.6264E-5
		$\varepsilon_2(n)$	4.7591E-4	2.3745E-4	1.1262E-4	4.4886E-5	2.1686E-5	1.8101E-5
		$\varepsilon_p(n)$	3.8612E-3	1.9246E-3	9.0270E-4	3.4978E-4	1.6368E-4	1.3538E-4
		k	12	11	8	4	3	2
16	64	$E_1(n)$	1.1391E-3	7.6308E-4	3.8264E-4	1.8882E-4	1.2225E-4	9.1144E-5
		$E_2(n)$	1.1391E-3	7.6308E-4	3.4255E-4	1.5297E-4	9.6742E-5	7.1249E-5
		$E_p(n)$	1.1933E-2	8.8453E-3	4.8368E-3	2.2625E-3	1.4322E-3	1.0557E-3
		$\varepsilon_1(n)$	2.0683E-4	2.2487E-4	9.3762E-5	4.7381E-5	1.6662E-5	1.2517E-5
		$\varepsilon_2(n)$	2.0683E-4	2.5182E-4	1.0092E-4	4.0254E-5	1.3717E-5	1.0108E-5
		$\varepsilon_p(n)$	1.6571E-3	2.3001E-3	1.2658E-3	5.9433E-4	2.0264E-4	1.4941E-4

$2L$, $R = 6$, $\eta = 1.0$, solve the Stoke's problem (4.86) with the exact solution

$$\begin{cases} u_1(r,\theta) &= \dfrac{\cos 3\theta + \cos\theta}{r} = \dfrac{2x^3 - 2xy^2}{(x^2+y^2)^2}, \\ u_2(r,\theta) &= \dfrac{\sin 3\theta - \sin\theta}{r} = \dfrac{2x^2y - 2y^3}{(x^2+y^2)^2}, \\ p(r,\theta) &= \dfrac{4\eta}{r^2}\cos 2\theta = 4\eta\dfrac{x^2-y^2}{(x^2+y^2)^2} \end{cases} \tag{4.88}$$

Example 3. Let Ω^c be the exterior domain outside of the square with side length $2L$, solve the Stoke's problem (4.86) with the exact solution

$$\begin{cases} u_1(x,y) &= x^2 + y^2, \\ u_2(x,y) &= -2xy, \\ p(x,y) &= 4\eta x \end{cases} \tag{4.89}$$

The results of the numerical examples in Table 1 - Table 3 show us Algorithm 3.1 is convergent for good relaxation factors θ_n, ρ and when $\theta_n \approx 0.5$ the convergence is faster and the rate of convergence is independent of the parameter h of meshes.

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