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A New Approach to Non-Homogeneous Fuzzy Initial Value Problem

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Abstract: In this paper, we consider a high-order linear differential equation with fuzzy forcing function and with fuzzy initial values. We assume the forcing function be in a special form, which we call triangular fuzzy function. We present solution as a fuzzy set of real functions such that each real function satisfies the initial value problem by some membership degree. We propose a method to find the fuzzy solution. We present an example to illustrate applicability of the proposed method.

Keywords: Fuzzy initial value problem, fuzzy differential equation, fuzzy set.

1 Introduction

Fuzzy initial value problem for differential equations arises naturally during the modeling of dynamic systems with uncertainty. The term "Fuzzy differential equation" for the first time was put forward by Kandel and Byatt (1978), the initial value problem was studied by Seikkala (1987) and Kaleva (1987, 1990). Depending on the type of derivatives used in the equation, depending on whether the initial values and forcing function are fuzzy or not, the fuzzy initial value problem has been investigated by many authors so far [Hüllermeier (1997); Buckley and Feuring (2000, 2001); Buckley, Feuring, and Hayashi (2002); Lakshmikantham and Nieto (2003); Bede and Gal (2005); Bede, Rudas, and Bencsik (2007); Georgiou, Nieto, and Rodríguez-López (2005); Perfilieva et al. (2008); Chalco-Cano and Román-Flores (2008, 2009); Khastan, Bahrami, and Ivaz (2009); Gasilov, Amrahov, and Fatullayev (2011a); Khastan, Nieto, and Rodríguez-López (2011)].

The main difference between the present study and other studies is as follows: In all the studies, cited above, the forcing function and the solution function are assumed as fuzzy number-valued functions. In the present study, a fuzzy function is

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interpreted as a fuzzy set the elements of which are real functions, in other words as a fuzzy bunch of real functions. Each of these real functions has a certain membership degree. When we refer to value of such a fuzzy function at time t, we mean a fuzzy set (or, fuzzy number) consisting of values of the real functions at t. If the same value takes place for different functions, the higher membership degree of functions is assigned to be the membership degree of the value.

2 Preliminary: Basic concepts of fuzzy sets theory

In classical set theory, an element either belongs or does not belong to the given set. In contrast, in fuzzy set theory, an element has a degree of membership, which is a real number from $[0,\ 1]$, in the given fuzzy set. In fuzzy set theory, classical sets are called **crisp** sets.

A **fuzzy set** $\widetilde{A}$ can be defined as a pair of the **universal set** U and the **membership function** $\mu : U \rightarrow [0,\ 1]$. If the universal set U is fixed, a membership function fully determines a fuzzy set. We denote the membership function as $\mu_{\widetilde{A}}$ to emphasize that the fuzzy set $\widetilde{A}$ is under consideration.

For each $x \in U$, the number $\mu_{\widetilde{A}}(x)$ is called the **membership degree** of x in $\widetilde{A}$.

The **support** of $\widetilde{A}$ is a crisp set and is defined as $supp(\widetilde{A}) = \{x \in U \mid \mu_{\widetilde{A}}(x) > 0\}$.

Let $U = R$ (where R is the set of real numbers). Let also a, c and b be real numbers such that $a \leq c \leq b$. A set $\widetilde{u}$ with membership function

$$\mu(x) = \begin{cases} \frac{x-a}{c-a}, & a < x < c \\ 1, & x = c \\ \frac{b-x}{b-c}, & c < x < b \\ 0, & otherwise \end{cases}$$

is called a **triangular fuzzy number** and is denoted as $\widetilde{u} = (a,\ c,\ b)$. In geometric interpretations, we refer to the point c as a **vertex**. We denote $\underline{u} = a$ and $\overline{u} = b$ to indicate the left and the right boundaries of $\widetilde{u}$, respectively.

We can express a triangular fuzzy number $\widetilde{u}$ as $\widetilde{u} = u_{cr} + \widetilde{u}_{un}$ (crisp part + uncertain part). Here $u_{cr} = c$ and $\widetilde{u}_{un} = (a-c, 0, b-c)$.

Fuzzy sets can be represented also via their α-cuts.

For each $\alpha \in (0,\ 1]$, the crisp set $A_\alpha = \left\{x \in U \mid \mu_{\widetilde{A}}(x) \geq \alpha\right\}$ is called the α-**cut** of $\widetilde{A}$. For $\alpha = 0$ we put $A_0 = closure(supp(\widetilde{A}))$.

It is easy to see that if α increases, A_α can only become narrower. Therefore, in the coordinate space, the α-cuts of a fuzzy set are bodies nested within one another.

For a triangular fuzzy number $\widetilde{u} = (a,\ c,\ b)$ the α-cuts are intervals $u_\alpha = [\underline{u_\alpha},\ \overline{u_\alpha}]$, where $\underline{u_\alpha} = a + \alpha(c-a)$ and $\overline{u_\alpha} = b + \alpha(c-b)$. The last formulas can be rewritten as $\underline{u_\alpha} = c + (1-\alpha)(a-c)$ and $\overline{u_\alpha} = c + (1-\alpha)(b-c)$. Consequently, $u_\alpha = [\underline{u_\alpha},\ \overline{u_\alpha}] = c + (1-\alpha)\ [a-c,\ b-c]$. From here one can see that an α-cut is homothetic and, in particular, similar to the interval $[a,\ b]$ (i.e. to the 0-cut) with ratio $(1-\alpha)$ if the vertex c is taken as a homothetic center.

3 A special kind of fuzzy functions

In this section we introduce a new kind of fuzzy functions.

Let $F_a(\cdot), F_c(\cdot), F_b(\cdot)$ be continuos functions on an interval I.

We call the fuzzy set $\widetilde{F}$, determined by the membership function

$$\mu_{\widetilde{F}}(y(\cdot)) = \begin{cases} \alpha, & y = F_a + \alpha(F_c - F_a) \text{ and } 0 < \alpha \leq 1 \\ \alpha, & y = F_b + \alpha(F_c - F_b) \text{ and } 0 < \alpha \leq 1 \\ 0, & otherwise \end{cases}$$

as **triangular fuzzy function** and denote it as $\widetilde{F} = \langle F_a, F_c, F_b \rangle$.

According to this definition a triangular fuzzy function is a fuzzy set (or, fuzzy bunch) of real functions. Among them only two functions have the membership degree α: the functions $y_1 = F_a + \alpha(F_c - F_a)$ and $y_2 = F_b + \alpha(F_c - F_b)$.

Example. In Fig. 1 we depict the triangular fuzzy function $\widetilde{F} = \langle F_a, F_c, F_b \rangle$, where
$F_a(t) = 4t^2 - 8t + 1$ (membership degree 0, bottom line);
$F_c(t) = 5t^2 - 10t + 3$ (membership degree 1, dashed line);
$F_b(t) = 7t^2 - 13t + \frac{9}{2}$ (membership degree 0, upper line). Functions with membership degrees 0.7 and 0.3 are depicted by dotted and dashed-dotted lines, respectively.

For each time $t \in I$, the value of a triangular fuzzy function is a triangular fuzzy number and can be expressed by the following formula:

$\widetilde{F}(t) = (\min\{F_a(t), F_c(t), F_b(t)\},\ F_c(t),\ \max\{F_a(t), F_c(t), F_b(t)\})$

4 A Fuzzy Initial Value Problem (FIVP)

In this section, we describe a fuzzy initial value problem (FIVP) and concept of solution which we propose. We investigate a fuzzy initial value problem for linear differential equation with triangular fuzzy forcing function and with fuzzy initial values. Such a FIVP can arise in modelling of a process the dynamics of which is crisp but there are uncertainties in forcing function and in initial values. Consider the second order fuzzy initial value problem:

$$\begin{cases} x'' + p(t)x' + q(t)x = \widetilde{F}(t) \\ x(0) = \widetilde{A} \\ x'(0) = \widetilde{B} \end{cases} \tag{1}$$

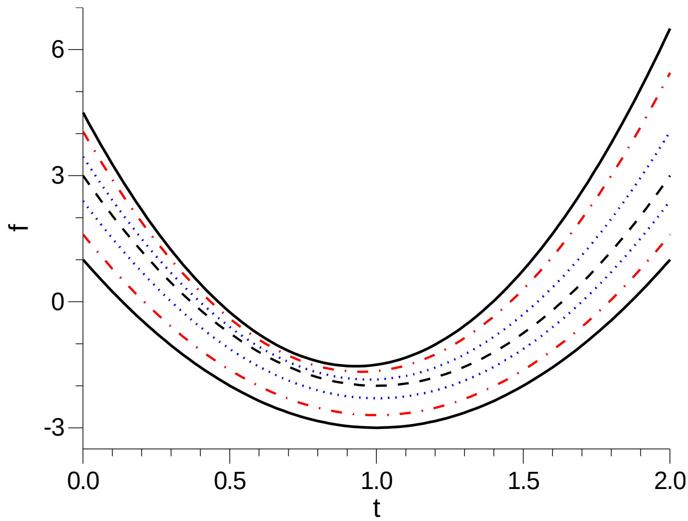

Figure 1: A triangular fuzzy function is a bunch of real functions.

where $p(t)$, $q(t)$ are continuous crisp functions; $\widetilde{F} = \langle F_a, F_c, F_b \rangle$ is triangular fuzzy function and $\widetilde{A}$, $\widetilde{B}$ are fuzzy numbers.

Let U (the universal set) be the set of functions that are continuous on interval $[0,+\infty)$ and twice differentiable on interval $(0,+\infty)$.

Definition 1. *The fuzzy set $\widetilde{X}$ on U with membership function*

$$\mu_{\widetilde{X}}(x(\cdot)) = \min\left\{\mu_{\widetilde{A}}(x(0)),\ \mu_{\widetilde{B}}(x'(0)),\ \mu_{\widetilde{F}}(x'' + p(t)x' + q(t)x)\right\} \tag{2}$$

we call the solution of the FIVP (1).

According to the definition, the solution $\widetilde{X}$ is a fuzzy set (or, bunch) of real functions such as $x(t)$. Each function $x(t)$ (with non-zero membership degree) satisfies the differential equation for some function $f \in supp(\widetilde{F})$ and has initial values $x(0) = a$ and $x'(0) = b$ from the intervals $\left(\underline{A},\ \overline{A}\right)$ and $\left(\underline{B},\ \overline{B}\right)$, respectively. We define the membership degree of the function $x(t)$ as the least membership degree of its initial values and the corresponding forcing function f.

One can interpret that we consider a FIVP as a set of crisp IVPs whose forcing function f belongs to the set $\widetilde{F}$ and initial values belong to the fuzzy sets $\widetilde{A}$ and $\widetilde{B}$.

The solution, defined above, of FIVP can be classified as united solution set (USS) [Kearfott (1996); Muzzioli and Reynaerts (2006)].

Let us represent $\widetilde{F} = f_{cr} + \widetilde{f}$, where $f_{cr} = F_c$, $\widetilde{f} = \langle f_a,\ 0,\ f_b \rangle = \langle F_a - F_c,\ 0,\ F_b - F_c \rangle$.

Let us also represent the initial values as $\widetilde{A} = a_{cr} + \widetilde{a}$ and $\widetilde{B} = b_{cr} + \widetilde{b}$, where a_{cr} and b_{cr} are crisp numbers, while $\widetilde{a}$ and $\widetilde{b}$ are fuzzy numbers.

We split the FIVP (1) to the following problems:

1) Associated crisp problem (which is non-homogeneous)

$$\begin{cases} x'' + p(t)x' + q(t)x = f_{cr}(t) \\ x(0) = a_{cr} \\ x'(0) = b_{cr} \end{cases} \tag{3}$$

2) Homogeneous problem with fuzzy initial values

$$\begin{cases} x'' + p(t)x' + q(t)x = 0 \\ x(0) = \widetilde{a} \\ x'(0) = \widetilde{b} \end{cases} \tag{4}$$

3) Non-homogeneous problem with fuzzy forcing function and zero initial values

$$\begin{cases} x'' + p(t)x' + q(t)x = \widetilde{f}(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases} \tag{5}$$

If $p(t), q(t)$ and $f_{cr}(t)$ are continuous functions the crisp problem (3) has a unique solution $x_{cr}(t)$. This solution can be computed by means of analytical or numerical methods.

To solve the problem (4) we modify the method, proposed by Gasilov, Amrahov, and Fatullayev (2011b). Let $x_1(t)$ and $x_2(t)$ be linear independent solutions of the crisp differential equation

$$x'' + p(t)x' + q(t)x = 0 \tag{6}$$

One can constitute the following functions

$$\begin{aligned} w_1(t) &= (x_2'(0)x_1(t) - x_1'(0)x_2(t)) \,/\, (x_2'(0)x_1(0) - x_1'(0)x_2(0)) \\ w_2(t) &= (x_1(0)x_2(t) - x_2(0)x_1(t)) \,/\, (x_2'(0)x_1(0) - x_1'(0)x_2(0)) \end{aligned} \tag{7}$$

We note that $w_1(t)$ and $w_2(t)$ are the solutions of (6) corresponding to the initial values $(x(0),\ x'(0)) = (1,\ 0)$ and $(x(0),\ x'(0)) = (0,\ 1)$, respectively.

Then the value of the solution to the problem (4) at a given time t is

$$\widetilde{x}_{iv}(t) = w_1(t)\ \widetilde{a} + w_2(t)\ \widetilde{b} \tag{8}$$

where the operations are assumed to be multiplication of real number with fuzzy one, and addition of fuzzy numbers.

According to Definition 1, the solution $\widetilde{X}_f$ of the fuzzy problem (5) is determined by the following membership function:

$$\mu_{\widetilde{X}_f}(x(\cdot)) = \begin{cases} \mu_{\widetilde{f}}(x'' + p(t)x' + q(t)x), & \text{if } x(0) = 0;\ x'(0) = 0 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Since the given problem (1) is linear, the solution is of the form $\widetilde{X} = x_{cr} + \widetilde{x}_{iv} + \widetilde{X}_f$ (crisp solution + uncertainty due to initial values + uncertainty due to forcing function). There are not essential difficulties with solving (3) and (4). Therefore, we have to solve the problem (5).

Formally, the formula (9) determines the solution $\widetilde{X}_f$ of (5), but does not give the answers to some questions: 1) What kind of structure has the value $\widetilde{X}_f(t)$ of the solution at a given time t? How to calculate this value? 2) Is $\widetilde{X}_f(t)$ a triangular fuzzy number or not if $\widetilde{f}$ is triangular fuzzy function? We give answers below.

To establish the solution we use the following property of linear differential equations. If a function $x(t)$ is a solution of the equation $x'' + p(t)x' + q(t)x = f(t)$, then the function $kx(t)$ (here k is a constant number) is a solution of the equation $x'' + p(t)x' + q(t)x = kf(t)$.

Let $x_a(t)$ be the solution of the problem

$$x'' + p(t)x' + q(t)x = f_a(t);\ x(0) = 0;\ x'(0) = 0 \tag{10}$$

Let also $x_b(t)$ be the solution of the same problem with $f_b(t)$ on the right-hand side, i.e. the problem

$$x'' + p(t)x' + q(t)x = f_b(t);\ x(0) = 0;\ x'(0) = 0 \tag{11}$$

Then the solution of the problem (5) is a triangular fuzzy function given by the formula

$$\widetilde{X}_f = \langle x_a,\ 0,\ x_b \rangle \tag{12}$$

We note that

$$\widetilde{X}_f(t) = (\min\{x_a(t), 0, x_b(t)\},\ 0,\ \max\{x_a(t), 0, x_b(t)\})$$

The result obtained above can be summarized in the following theorem.

Theorem 1. *Let the forcing function of FIVP (5) be a triangular fuzzy function* $\widetilde{f} = \langle f_a,\ 0,\ f_b \rangle$ *and* $p(t), q(t), f_a(t), f_b(t)$ *be continuous functions. Also let* $x_a(t)$ *and* $x_b(t)$ *be solutions of the problems (10) and (11), respectively.*
Then FIVP (5) has unique solution $\widetilde{X}_f$ *in the sense of Definition 1, which is the triangular fuzzy function given by* $\widetilde{X}_f = \langle x_a,\ 0,\ x_b \rangle$.

4.1 Solution algorithm

Based on the arguments above, we propose the following algorithm to solve FIVP (1):

1. Represent the forcing function as $\widetilde{F} = f_{cr} + \langle f_a,\ 0,\ f_b\rangle$; the initial values as $\widetilde{A} = a_{cr} + \widetilde{a}$ and $\widetilde{B} = b_{cr} + \widetilde{b}$.

2. Find linear independent solutions $x_1(t)$ and $x_2(t)$ of the crisp differential equation $x'' + p(t)x' + q(t)x = 0$. Define the functions $w_1(t)$ and $w_2(t)$ by formula (7).

3. Find the solution $x_{cr}(t)$ of the corresponding crisp problem (3).

4. Find the solutions $x_a(t)$ and $x_b(t)$ of the crisp problems (10) and (11).

5. The value of the solution to (1) is

$$\widetilde{x}(t) = x_{cr}(t) + w_1(t)\,\widetilde{a} + w_2(t)\,\widetilde{b} + (\min\{x_a(t), 0, x_b(t)\},\ 0,\ \max\{x_a(t), 0, x_b(t)\}) \tag{13}$$

Remark: The approach is valid also for the general case, when n-th order initial value problem is considered.

5 Example

Example 1. Let us consider the second order fuzzy initial value problem:

$$\begin{cases} x'' - 2x' + 5x = \left\langle 4t^2 - 8t + 1,\ 5t^2 - 10t + 3,\ 7t^2 - 13t + \frac{9}{2}\right\rangle \\ x(0) = (\ 1,\ 2,\ 2.5) \\ x'(0) = (3.5,\ 4,\ 4.5) \end{cases} \tag{14}$$

We represent the forcing function, which is depicted in Fig. 1, as $\widetilde{F} = f_{cr} + \widetilde{f} = 5t^2 - 10t + 3 + \left\langle -t^2 + 2t - 2,\ 0,\ 2t^2 - 3t + \frac{3}{2}\right\rangle$; the initial values as $\widetilde{A} = 2 + (-1,\ 0,\ 0.5)$, $\widetilde{B} = 4 + (-0.5,\ 0,\ 0.5)$.

Firstly, we solve associated crisp non-homogeneous problem

$$\begin{cases} x'' - 2x' + 5x = 5t^2 - 10t + 3 \\ x(0) = 2 \\ x'(0) = 4 \end{cases} \tag{15}$$

and find the crisp solution

$$x_{cr}(t) = t^2 - 1.2t - 0.28 + e^t\ (2.28\cos 2t + 1.46\sin 2t) \tag{16}$$

(the dashed line in Fig. 4).

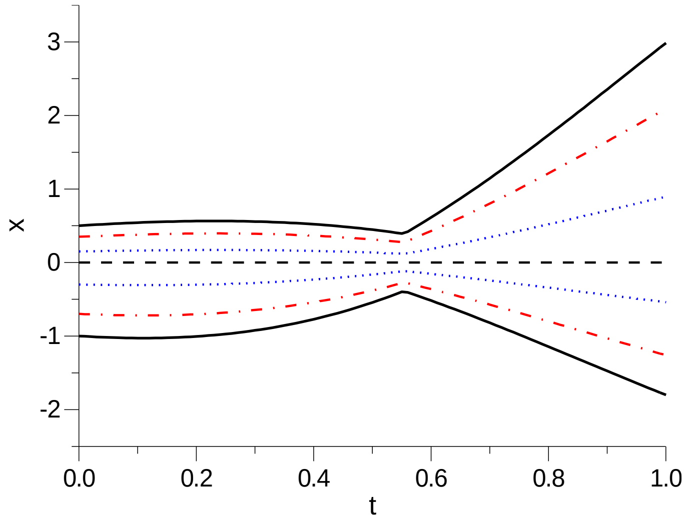

Figure 2: Uncertainty of the solution due to initial values.

Secondly, to find the uncertainty of the solution due to initial values we solve fuzzy homogeneous problem

$$\begin{cases} x'' - 2x' + 5x = 0 \\ x(0) = (\,-1,\ 0,\ 0.5) \\ x'(0) = (-0.5,\ 0,\ 0.5) \end{cases} \tag{17}$$

The solution can be calculated by the formula (8). $x_1(t) = e^t \cos 2t$ and $x_2(t) = e^t \sin 2t$ are linear independent solutions for the differential equation $x'' - 2x' + 5x = 0$. Then from (7) and (8) we have the following formula for the solution to the problem (17):

$$\widetilde{x}_{iv}(t) = \frac{1}{2} e^t (2\cos 2t - \sin 2t)\ (-1,\ 0,\ 0.5) + \frac{1}{2} e^t \sin 2t\ (-0.5,\ 0,\ 0.5) \tag{18}$$

where the arithmetical operations are considered to be fuzzy operations. The fuzzy solution $\widetilde{x}_{iv}(t)$ forms a band in the tx-coordinate space (Fig. 2). From (18) one can find the lower and upper boundaries of this band:

$\underline{x_{iv}}(t) = \frac{1}{4} e^t\ (\min\{-2(2\cos 2t - \sin 2t),\ 2\cos 2t - \sin 2t\} + \min\{-\sin 2t,\ \sin 2t\})$

$\overline{x_{iv}}(t) = \frac{1}{4} e^t\ (\max\{-2(2\cos 2t - \sin 2t),\ 2\cos 2t - \sin 2t\} + \max\{-\sin 2t,\ \sin 2t\})$

Since the initial values are triangular fuzzy numbers, an α-cut of the solution can be determined by similarity with the coefficient $(1-\alpha)$, i.e.

$$x_{iv,\ \alpha}(t) = (1-\alpha)\ \left[\underline{x_{iv}}(t),\ \overline{x_{iv}}(t)\right] \tag{19}$$

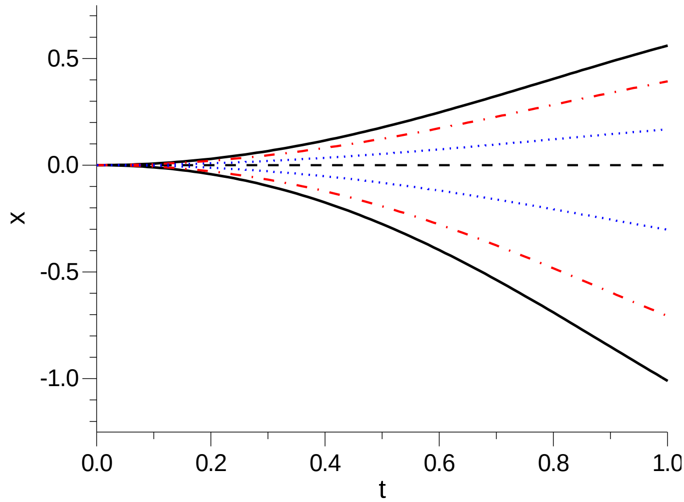

Figure 3: Uncertainty of the solution due to forcing function.

In Fig. 2 we show the fuzzy solution $\widetilde{x}_{iv}$ via its α-cuts: 1-cut or crisp solution (dashed line), 0.7-cut (dotted lines), 0.3-cut (dashed-dotted lines), 0-cut or boundaries of the solution (continues lines).

Thirdly, to find the uncertainty of the solution due to the forcing function we solve fuzzy non-homogeneous problem

$$\begin{cases} x'' - 2x' + 5x = \left\langle -t^2 + 2t - 2,\ 0,\ 2t^2 - 3t + \frac{3}{2} \right\rangle \\ x(0) = 0 \\ x'(0) = 0 \end{cases} \tag{20}$$

The solution of the equation $x'' - 2x' + 5x = f(t)$ with $f(t) = f_a(t) = -t^2 + 2t - 2$ is
$x_a(t) = -0.2t^2 + 0.24t - 0.224 + e^t\ (0.224\cos 2t - 0.232\sin 2t)$.
For $f(t) = f_b(t) = 2t^2 - 3t + \frac{3}{2}$ we have
$x_b(t) = 0.4t^2 - 0.28t + 0.028 + e^t\ (-0.028\cos 2t + 0.154\sin 2t)$.
The solution of (20) is the triangular fuzzy function $\widetilde{X}_f = \langle x_a,\ 0,\ x_b \rangle$ which is graphed in Fig. 3 and for which we have

$$\widetilde{X}_f(t) = (\min\{x_a(t), 0, x_b(t)\},\ 0,\ \max\{x_a(t), 0, x_b(t)\}) \tag{21}$$

The sum of 3 functions ((16), (18) and (21)) gives the solution of the given problem (14) (See, Fig. 4).

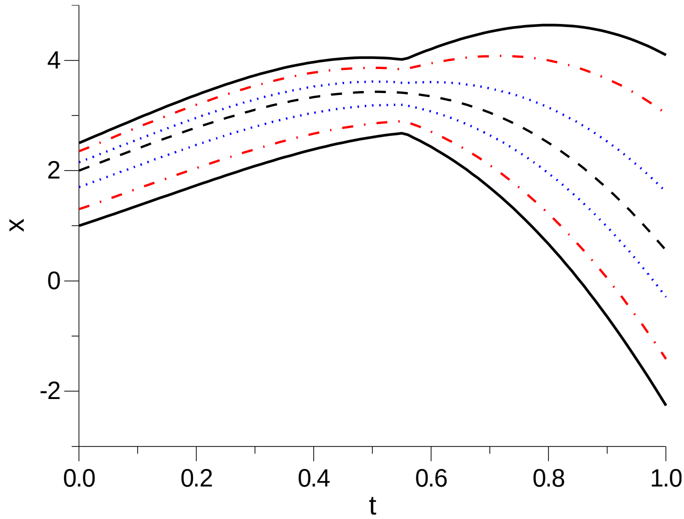

Figure 4: The fuzzy solution, obtained by the proposed method and its α-cuts. Dashed line represents the crisp solution.

Remark: We would like to emphasize the following: The solution of the problem (17) is not a triangular fuzzy function, but its value at each time t is a triangular fuzzy number. Fig. 2 presents the solution function via dependence of its value on time. In contrast, the solution of the problem (20) is a triangular fuzzy function, according to Theorem 1. Fig. 3 plots this triangular fuzzy function as a bunch of real functions. For Fig. 4 the situation is the same as for Fig. 2.

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