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The Optimal Control Problem of Nonlinear Duffing Oscillator Solved by the Lie-Group Adaptive Method

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Abstract: In the optimal control theory, the Hamiltonian formalism is a famous one to find an optimal solution. However, when the performance index is complicated or for a degenerate case with a non-convexity of the Hamiltonian function with respect to the control force the Hamiltonian method does not work to find the solution. In this paper we will address this important issue via a quite different approach, which uses the optimal control problem of nonlinear Duffing oscillator as a demonstrative example. The optimally controlled vibration problem of nonlinear oscillator is recast into a nonlinear inverse problem by identifying the unknown heat source in a nonlinear parabolic partial differential equation (PDE). Then through a semi-discretization of the resultant PDE, the inverse problem is further reformulated to be a system of n-dimensional ODEs with n unknown point-wise sources, which allows a Lie-group adaptive method (LGAM) to recover the point-wise sources. The present method has three-fold advantages: it can easily minimize a complicated performance index to find an optimal control force of the nonlinear vibration system, it is effective for highly nonlinear optimal control problem, and it does not resort on the classical Hamiltonian formulation, *which provides only a necessary condition, but not a sufficient condition, for the optimality of the control law*. Numerical examples show that *the LGAM may find a better performance than the classical one*.

Keywords: Duffing oscillator, van der Pol oscillator, Optimal control problem, Lie-group adaptive method, Nonlinear inverse problem

1 Introduction

The structural mechanics is to analyze, as well as to determine, the responses of a given structure subject to external loading conditions. Based on the results analyzed, the structural engineers are able to check whether a proposed structural

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design can meet the adequate resistance requirements to a combination of loading conditions or not, and, if necessary, to revise the proposed design until all such requirements are satisfied. When the loading is quite large, the structure will respond non-linearly. Many nonlinear problems in structures can be modelled by hard spring or soft spring of a Duffing system. On the other hand, the dissipation of energy in a mechanical structure is often described by a viscous damping term, while the conservative part is described by a nonlinear spring element. The resulting equation of vibration is attractive because it can be mathematically treated.

We are frequently desired to control the response of a nonlinear structure to remain within a specified limit for the reason of safety. Sometimes we may encounter the problem that the external forces are not yet known, but service for a specific purpose of controlling the nonlinear structure to a desired state. Then the resulting problem is an optimal control problem. In this class of control problems, the control forces are intentionally designed such that a specified cost functional which weights the cost of control versus the allowed response is minimized. The control of nonlinear structural systems has gained much attention in the past several decades, and different controllers were proposed for the applications to different areas of disciplines [Suhardjo, Spencer and Sain (1992); Agrawal, Yang and Wu (1998)]. In the realm of nonlinear structural control, Davies (1972) has studied the time optimal control of the Duffing oscillator. Van Dooren and Vlassenbroeck (1982) have introduced a direct method by the Chebyshev series expansion to solve the controlled problem of the Duffing oscillator [El-Gindy, El-Hawary, Alim and El-Kady (1995); El-Kady and Elbarbary (2002)]. Razzaghi and Elnagar (1994) have applied a pseudospectral method to solve this problem, and Lakestani, Razzaghi and Dehghan (2006) have applied a semi-orthogonal spline wavelets to solve this problem. As a result, all the above methods required to solve a rather-complicated system of nonlinear algebraic equations. Recently, Dai, Schnoor and Atluri (2012) have applied a simple collocation method to reveal the complex oscillation behavior of the Duffing oscillator.

The Pontryagin's Maximum Principle and Bellman's Dynamic Programming had been the two main methods for solving the optimal control problems. For a general nonlinear system with specified minimized functional as being a performance index, the optimal state feedback control laws can be derived from the solution to the Hamilton-Jacobian-Bellman (HJB) equation, or from solving a system of two-point differential algebraic equations [Wang, Jhu, Yung and Wang (2011)]. In the above formulations one needs to assume that the Hamiltonian function be a convex and twice differentiable function of the control force. When the degenerate case is happened, the above optimal control theory becomes quite complicated. Moreover, the optimal control law derived from the Hamiltonian formulation is only a necessary condition, not a sufficient condition. In an attempt to overcome these difficulties, we present an alternative and yet useful approach based on the Lie-group adaptive method (LGAM), where we view the governing equation of nonlinear system to be the majority and the performance index as being a subsidiary target equation to be matched with a minimization.

Liu (2006a, 2006b, 2006c) has extended the group-preserving scheme (GPS) developed by Liu (2001) for initial value problems of ODEs to solve the boundary value problems (BVPs), namely the Lie-group shooting method (LGSM), and the numerical results revealed that the LGSM is a rather promising method to effectively solve the two-point BVPs.

In the construction of the Lie-group method for the solutions of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of the Lie group. It needs to stress that this one-step transformation property of the Lie-group cannot be shared by other numerical methods, because those methods do not belong to the Lie-group types. This important property as first pointed out by Liu (2006d) was employed to solve the backward in time Burgers equation. Liu (2008a, 2008b) has developed a Lie-group method to study the inverse vibration problem for estimating both the time-dependent damping and stiffness coefficients.

The Lie-group method possesses a greater advantage than other numerical methods due to its Lie-group structure, and it is a powerful technique to solve the direct problems and also the inverse problems of parameter identifications. Recently, Liu and Atluri (2010) have solved the Calderón's inverse problem by an effective combination of the Lie-group adaptive method (LGAM) and the finite-strip technique. By using the same idea, Liu (2012a) has solved the inverse vibration problem of the Euler-Bernoulli beam by identifying unknown external force. The LGAM views the Lie-group equation developed in the LGSM as a two-point Lie-group equation [Liu (2012b)], describing a nonlinear relation between the state quantities defined at two different times or at two different positions of an one-dimensional space. In this point of view of the LGAM we do not have a real target in the problem, and thus we can employ the Lie-group equation as a supplementary equation. It is interesting that Liu (2010) has applied the LGAM to identify the rigidity function of wave propagation problems without resorting on other data, besides those needed for the direct wave problem, Liu (2011a) has identified unknown initial condition and heat source by using the LGAM, and Liu (2011b, 2011c) has used the LGAM to solve the non-homogeneous heat conductivity identification problem. Liu and Chang (2011) have used the LGAM to identify the radiative coefficients in parabolic partial differential equations.

This paper is arranged as follows. We introduce a novel approach of the optimal control problem of a nonlinear mechanical oscillator in Section 2 by transforming it into a heat-source identification problem of a parabolic type PDE, and then wherein by discretizing the PDE into a system of ODEs at the discretized times, we face a problem with unknown point-wise sources for a system of n-dimensional ODEs. In Section 3 we give a brief sketch of the GPS for the system of nonlinear ODEs. Due to a good property of the Lie-group, we will propose an integration technique, such that the one-step GPS can be used to identify the point-wise control forces appeared in the resulting ODEs. The nonlinear algebraic equations are derived in Section 4 when we apply the one-step GPS to identify the control force. In Section 5 numerical examples are examined to test the Lie-group adaptive method (LGAM). Finally, we draw some conclusions in Section 6.

2 Two mathematical transformations

The purpose of this article is to compute the control force u in the following equation of motion of a nonlinear mechanical oscillator:

$$\ddot{\phi}+H(\phi,\dot{\phi})=u(t),\ \ t_0<t<t_f, \tag{1}$$

$$\phi(t_0)=A_0,\ \ \dot{\phi}(t_0)=B_0. \tag{2}$$

Here, H can be a quite general nonlinear function of displacement ϕ and velocity $\dot{\phi}$. We select an optimal control force by satisfying the following minimization of a specified performance index J:

$$\min\{J=g(\mathbf{x}(t_f))+\int_{t_0}^{t_f}L(\mathbf{x}(t),u(t),t)dt\}, \tag{3}$$

where $t\in[t_0,t_f]$ is a time interval we interest, and $\mathbf{x}=(\phi,\dot{\phi})$ is a state vector.

From this section on we start to develop a new method to compute the optimal control force $u(t)$, $t\in[t_0,t_f]$. Especially, when $H=\phi+\beta\phi^3$ we encounter an optimal control problem of the Duffing oscillator.

2.1 Transformation into a PDE

If we begin with

$$v(x,t)=(1+x)\phi(t), \tag{4}$$

then the ODE in Eqs. (1) and (2) can be transformed into a parabolic type nonlinear PDE:

$$v_x(x,t)=v_{tt}(x,t)+h(x,v,v_t)+\frac{v(x,t)}{1+x}-(1+x)u(t), \tag{5}$$

$$v(x,t_0)=A_0(1+x),\ \ v_t(x,t_0)=B_0(1+x). \tag{6}$$

In order to simplify the notations we let $v_t = \partial v(x,t)/\partial t$, $v_{tt} = \partial^2 v(x,t)/\partial t^2$, $v_x = \partial v(x,t)/\partial x$, as well as

$$h(x,v,v_t) := (1+x)H\left(\frac{v(x,t)}{1+x}, \frac{v_t(x,t)}{1+x}\right). \tag{7}$$

Eq. (5) is a nonlinear PDE of v, which is dependent on the function of h, and thus on the nonlinear mechanical system we consider.

The above transformation technique was first proposed by Liu (2008c) to treat an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Then, Liu (2008a, 2008b) and Liu, Chang, Chang and Chen (2008) have extended this idea to develop new methods for estimating the parameters in the inverse vibration problems. Because $u(t)$ in Eq. (5) is an unknown function, we have faced an inverse heat source problem with overspecified left boundary conditions. Indeed, it is a rather difficult inverse problem of PDE, giving no initial condition of $v(0,t)$ and no extra information of $v(x,t)$, but we need to recover $u(t)$ which minimizes the functional J in Eq. (3).

2.2 *Transformation into ODEs*

Applying a semi-discrete procedure to the PDE in Eq. (5) yields a coupled system of ODEs:

$$v_i'(x) = \mathscr{L}_i + h_i(x) + \frac{v_i(x)}{1+x} - (1+x)u_i, \tag{8}$$

where $\Delta t = (t_f - t_0)/(n+1)$ is a uniform time increment with $v_i(x) = v(x,t_i) = v(x, t_0 + i\Delta t)$ and $u_i = u(t_i)$ for simple notations. Also, $h_i(x) = h(x, v_i(x), \mathscr{K}_i)$ with $\mathscr{K}_i$ denoting the discretization of v_t at the point t_i, and $\mathscr{L}_i$ denoting the discretization of v_{tt} at the point t_i, by using the Differential Quadratures introduced in the Appendix.

Eq. (8) has totally n coupled ODEs for n variables $v_i(x), i = 1,\ldots,n$. Therefore, at the present time we have a set of underspecified conditions to solve the n-dimensional ODEs system (8), and n unknown point-wise source terms u_i. Below, we will develop a Lie-group adaptive method to find u_i.

3 The GPS for differential equations system

3.1 Group-preserving scheme

Upon letting $\mathbf{v} = (v_1, \ldots, v_n)^{\mathrm{T}}$ and denoting the right-hand side of Eq. (8) by $\mathbf{f}$, we can write that equation as a vector form:

$$\mathbf{v}' = \mathbf{f}(\mathbf{v}, x), \;\; \mathbf{v} \in \mathbb{R}^n, \;\; x \in \mathbb{R}. \tag{9}$$

Liu (2001) has embedded Eq. (9) into an augmented differential equations system:

$$\frac{d}{dx}\begin{bmatrix} \mathbf{v} \\ \|\mathbf{v}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{v},x)}{\|\mathbf{v}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{v},x)}{\|\mathbf{v}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{v} \\ \|\mathbf{v}\| \end{bmatrix}, \tag{10}$$

where the inclusion of the second row gives us a Minkowskian structure of the augmented state variables of $\mathbf{X} := (\mathbf{v}^{\mathrm{T}}, \|\mathbf{v}\|)^{\mathrm{T}}$, which automatically satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{11}$$

where

$$\mathbf{g} := \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{12}$$

is a Minkowski metric, $\mathbf{I}_n$ is the identity matrix of order n, and the superscript T stands for the transpose. In terms of $(\mathbf{v}^{\mathrm{T}}, \|\mathbf{v}\|)^{\mathrm{T}}$, Eq. (11) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{v}\cdot\mathbf{v} - \|\mathbf{v}\|^2 = \|\mathbf{v}\|^2 - \|\mathbf{v}\|^2 = 0, \tag{13}$$

where the dot between two vectors denotes the inner product. Therefore, the augmented state vector $\mathbf{X}$ is automatically located on the cone.

Consequently, we have an $(n+1)$-dimensional augmented differential equations system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \tag{14}$$

with a constraint (11), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{v},x)}{\|\mathbf{v}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{v},x)}{\|\mathbf{v}\|} & 0 \end{bmatrix} \tag{15}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$, because of

$$\mathbf{A}^{\mathrm{T}}\mathbf{g}+\mathbf{g}\mathbf{A}=\mathbf{0}. \tag{16}$$

This prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G}=\mathbf{g}, \quad \det\mathbf{G}=1, \quad G_0^0>0, \tag{17}$$

where G_0^0 is the 00th component of $\mathbf{G}$.

Although the dimension of the new system (14) is raised one more than the original system, it has been shown that the new system admits a group-preserving scheme (GPS) given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1}=\mathbf{G}(\ell)\mathbf{X}_{\ell}, \tag{18}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at x_ℓ, and $\mathbf{G}(\ell)\in SO_o(n,1)$ is the group value of $\mathbf{G}$ at x_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eq. (17), then $\mathbf{X}_\ell$ automatically satisfies the cone condition in Eq. (11).

The Lie-group element $\mathbf{G}(\ell)$ can be obtained from the constant $\mathbf{A}(\ell)\in so(n,1)$ by an exponential mapping:

$$\mathbf{G}(\ell)=\exp[\Delta x\mathbf{A}(\ell)]=\begin{bmatrix} \mathbf{I}_n+\frac{(a_\ell-1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & \frac{b_\ell\mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell\mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{19}$$

where

$$a_\ell:=\cosh\left(\frac{\Delta x\|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right), \quad b_\ell:=\sinh\left(\frac{\Delta x\|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right). \tag{20}$$

Substituting Eq. (19) for $\mathbf{G}(\ell)$ into Eq. (18), we can obtain

$$\mathbf{v}_{\ell+1}=\mathbf{v}_\ell+\eta_\ell\mathbf{f}_\ell, \tag{21}$$

$$\|\mathbf{v}_{\ell+1}\|=a_\ell\|\mathbf{v}_\ell\|+\frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell\cdot\mathbf{v}_\ell, \tag{22}$$

where

$$\eta_\ell:=\frac{b_\ell\|\mathbf{v}_\ell\|\|\mathbf{f}_\ell\|+(a_\ell-1)\mathbf{f}_\ell\cdot\mathbf{v}_\ell}{\|\mathbf{f}_\ell\|^2}. \tag{23}$$

3.2 One-step GPS

Throughout this paper the superscript f denotes the value at $x = x_f$, while the superscript 0 denotes the value at $x = 0$. Assume that the total length x_f is divided into K steps, that is, the stepsize we use in the GPS is $\Delta x = x_f/K$.

Starting from $\mathbf{X}^0 = \mathbf{X}(0)$ and applying Eq. (18) step-by-step to integrate Eq. (14) we can calculate the value $\mathbf{X}^f$ at $x = x_f$ by

$$\mathbf{X}^f = \mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)\mathbf{X}^0. \tag{24}$$

However, let us recall that each $\mathbf{G}_i$, $i = 1,\ldots,K$, is an element of the Lie-group $SO_o(n,1)$, and by the closure property of the Lie group, $\mathbf{G}_K\cdots\mathbf{G}_1$ is also a Lie-group element of $SO_o(n,1)$ denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}^f = \mathbf{G}\mathbf{X}^0. \tag{25}$$

This is a one-step Lie-group transformation from $\mathbf{X}^0$ to $\mathbf{X}^f$, namely the one-step GPS.

3.2.1 A Generalized mid-point rule

We can approximately solve $\mathbf{G}$ by a generalized mid-point rule, which is obtained from an exponential mapping of a constant $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie-group element generated from such a constant matrix $\mathbf{A} \in so(n,1)$ has a closed-form solution:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^{\mathrm{T}} & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^{\mathrm{T}}}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{26}$$

where

$$\hat{\mathbf{v}} = r\mathbf{v}^0 + (1-r)\mathbf{v}^f, \quad \hat{\mathbf{f}} = \mathbf{f}(\hat{\mathbf{v}}, \hat{x}), \tag{27}$$

$$a = \cosh\left(\frac{x_f\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{v}}\|}\right), \quad b = \sinh\left(\frac{x_f\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{v}}\|}\right). \tag{28}$$

Here, we use the initial value $\mathbf{v}^0$ and the final value $\mathbf{v}^f$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $r \in [0,1]$ is a parameter to be determined and $\hat{x} = (1-r)x_f$. The above method applied a generalized mid-point rule to derive the solution of $\mathbf{G}$, and the resultant is a single-parameter Lie-group element $\mathbf{G}(r)$. After developing the LGAM in Section 4, we can determine the correct value of r by adapting the given J to a minimization.

3.2.2 A Lie-group mapping between two points on the cone

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{v}}\|}, \tag{29}$$

such that Eqs. (26) and (28) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{30}$$

$$a = \cosh(x_f\|\mathbf{F}\|), \quad b = \sinh(x_f\|\mathbf{F}\|). \tag{31}$$

From Eqs. (25) and (30) it follows that

$$\mathbf{v}^f = \mathbf{v}^0 + \eta\mathbf{F}, \tag{32}$$

$$\|\mathbf{v}^f\| = a\|\mathbf{v}^0\| + b\frac{\mathbf{F}\cdot\mathbf{v}^0}{\|\mathbf{F}\|}, \tag{33}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{v}^0 + b\|\mathbf{v}^0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{34}$$

Substituting Eq. (32), which is written as

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{v}^f - \mathbf{v}^0), \tag{35}$$

into Eq. (33) and dividing both sides by $\|\mathbf{v}^0\|$ we can obtain

$$\frac{\|\mathbf{v}^f\|}{\|\mathbf{v}^0\|} = a + b\frac{(\mathbf{v}^f - \mathbf{v}^0)\cdot\mathbf{v}^0}{\|\mathbf{v}^f - \mathbf{v}^0\|\|\mathbf{v}^0\|}, \tag{36}$$

where

$$a = \cosh\left(\frac{x_f\|\mathbf{v}^f - \mathbf{v}^0\|}{\eta}\right), \quad b = \sinh\left(\frac{x_f\|\mathbf{v}^f - \mathbf{v}^0\|}{\eta}\right) \tag{37}$$

are obtained from Eq. (31) by inserting Eq. (35) for **F**.

Let

$$\cos\theta := \frac{[\mathbf{v}^f - \mathbf{v}^0]\cdot\mathbf{v}^0}{\|\mathbf{v}^f - \mathbf{v}^0\|\|\mathbf{v}^0\|}, \quad S := x_f\|\mathbf{v}^f - \mathbf{v}^0\|, \tag{38}$$

and from Eq. (36) it follows that

$$\frac{\|\mathbf{v}^f\|}{\|\mathbf{v}^0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\theta \sinh\left(\frac{S}{\eta}\right). \tag{39}$$

By defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{40}$$

and from Eq. (39) we can obtain a quadratic equation for Z:

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{v}^f\|}{\|\mathbf{v}^0\|}Z + 1 - \cos\theta = 0. \tag{41}$$

On the other hand, by inserting Eq. (35) for $\mathbf{F}$ into Eq. (34) we can obtain

$$\|\mathbf{v}^f - \mathbf{v}^0\|^2 = (a-1)(\mathbf{v}^f - \mathbf{v}^0)\cdot\mathbf{v}^0 + b\|\mathbf{v}^0\|\|\mathbf{v}^f - \mathbf{v}^0\|. \tag{42}$$

Dividing both sides by $\|\mathbf{v}^0\|\|\mathbf{v}^f - \mathbf{v}^0\|$ and using Eqs. (37), (38) and (40) we can obtain another quadratic equation for Z:

$$(1+\cos\theta)Z^2 - 2\left(\cos\theta + \frac{\|\mathbf{v}^f - \mathbf{v}^0\|}{\|\mathbf{v}^0\|}\right)Z + \cos\theta - 1 = 0. \tag{43}$$

From Eqs. (41) and (43), the solution of Z is found to be

$$Z = \frac{(\cos\theta - 1)\|\mathbf{v}^0\|}{\cos\theta\|\mathbf{v}^0\| + \|\mathbf{v}^f - \mathbf{v}^0\| - \|\mathbf{v}^f\|}, \tag{44}$$

and then from Eqs. (40) and (38) we can obtain

$$\eta = \frac{x_f\|\mathbf{v}^f - \mathbf{v}^0\|}{\ln Z}. \tag{45}$$

Therefore, between any two points $(\mathbf{v}^0, \|\mathbf{v}^0\|)$ and $(\mathbf{v}^f, \|\mathbf{v}^f\|)$ on the cone, there exists a Lie-group element $\mathbf{G} \in SO_o(n,1)$ mapping $(\mathbf{v}^0, \|\mathbf{v}^0\|)$ onto $(\mathbf{v}^f, \|\mathbf{v}^f\|)$, which is given by

$$\begin{bmatrix} \mathbf{v}^f \\ \|\mathbf{v}^f\| \end{bmatrix} = \mathbf{G}\begin{bmatrix} \mathbf{v}^0 \\ \|\mathbf{v}^0\| \end{bmatrix}, \tag{46}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{v}^0$ and $\mathbf{v}^f$ through the following equations:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{47}$$

$$a = \cosh(x_f\|\mathbf{F}\|), \quad b = \sinh(x_f\|\mathbf{F}\|), \quad \mathbf{F} = \frac{1}{\eta}(\mathbf{v}^f - \mathbf{v}^0). \tag{48}$$

4 Computing the control force by the LGAM

In this section we compute the optimal control force with the point-wise u_i. From Eqs. (29) and (32) we have a very useful *Lie-group equation*:

$$\mathbf{v}^f = \mathbf{v}^0 + \eta \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{v}}\|}. \tag{49}$$

Up to here we have constructed a Lie-group equation (49), which is a universal algebraic equation applicable to any vector field $\mathbf{f}$. This equation involves four quantities of $\mathbf{v}^0$, $\mathbf{v}^f$, $\mathbf{f}$ and r together, where the last is a single parameter to be determined.

We can write $\hat{\mathbf{f}}$ explicitly,

$$\hat{\mathbf{f}} = \begin{bmatrix} \hat{\mathscr{L}}_1 + \frac{\hat{v}_1}{1+\hat{x}} + \hat{h}_1 - (1+\hat{x})u_1 \\ \hat{\mathscr{L}}_2 + \frac{\hat{v}_2}{1+\hat{x}} + \hat{h}_2 - (1+\hat{x})u_2 \\ \vdots \\ \hat{\mathscr{L}}_n + \frac{\hat{v}_n}{1+\hat{x}} + \hat{h}_n - (1+\hat{x})u_n \end{bmatrix}, \tag{50}$$

where $\hat{x} = (1-r)x_f$, $\hat{v}_i = rv_i^0 + (1-r)v_i^f$ and $\hat{\mathscr{L}}_i = \mathscr{L}_i(\hat{v}_i)$. The term $\hat{\mathscr{L}}_i$ is calculated by the Differential Quadrature (see the Appendix) with $\hat{\mathscr{L}}_i = b_{ij}\hat{v}_j$.

From Eqs. (49) and (50) we can derive a formula to calculate u_i:

$$u_i = \frac{1}{1+\hat{x}} \left[\hat{\mathscr{L}}_i + \frac{\hat{v}_i}{1+\hat{x}} + \hat{h}_i - \frac{\|\hat{\mathbf{v}}\|}{\eta}(v_i^f - v_i^0) \right]. \tag{51}$$

Now, the numerical procedures for computing u_i are described as follows. We assume the initial values of u_i. Substituting them into Eq. (5), using the initial conditions in Eq. (6), and integrating Eq. (5) from $t = t_0$ to $t = t_f$, we can calculate v_i^0 and v_f^0. Then, inserting v_i^0 and v_f^0 into Eq. (51) we can calculate a new u_i, which is then compared with the old u_i. If the difference of these two sets of u_i is smaller than a given stopping criterion, then the iteration is terminated, and thus the final u_i is obtained. The processes are summarized as follows:

Step 1: Select a value of $r \in [0,1]$.
Step 2: Give an initial guess of u_i.
Step 3: For $j = 1,2\ldots$, we repeat the following computations. Calculate v_i^0 and v_i^f by Eqs. (5) and (6). Insert the above v_i^0 and v_i^f denoted, respectively, by $v_i^0(j)$ and

$v_i^f(j)$ into

$$u_i^j = \frac{1}{1+\hat{x}}\left[\hat{\mathscr{L}}_i(j) + \frac{\hat{v}_i(j)}{1+\hat{x}} + \hat{h}_i(j) - \frac{\|\hat{\mathbf{v}}(j)\|}{\eta^j}[v_i^f(j) - v_i^0(j)]\right], \tag{52}$$

where η^j is calculated from Eq. (45) by inserting $v_i^0(j)$ and $v_i^f(j)$. If u_i^j converges according to a given convergence criterion:

$$C_j := \sqrt{\frac{1}{n}\sum_{i=1}^{n}(u_i^{j+1} - u_i^j)^2} < \varepsilon, \tag{53}$$

then stop; otherwise, go to **Step 3**.
Step 4: Finally, we search a suitable value of r by

$$\min_{r\in[0,1]} J = g(v_n^0, v_{t,n}^0) + \sum_{j=1}^{n} b_j L(v_j^0, v_{t,j}^0, u_j, t_j), \tag{54}$$

which is an approximation of J defined in Eq. (3), and which can be evaluated by using the Integral Quadrature introduced in the Appendix. When r is selected we can insert it into Eq. (51) to calculate u_i.

The present LGAM has used a fictitious dimension of x to derive Eq. (52) by supposing a fictitious target $v_i^f(j)$ at x_f. We can repeatedly use the time direction integration of Eqs. (5) and (6) to obtain the new data of $v_i^0(j)$ and $v_i^f(j)$, and then we can adjust u_i by Eq. (52) until they are convergent.

5 Numerical tests

5.1 Example 1

We first consider an optimal control problem treated by Feldbaum (1973):

$$\begin{aligned} &J = \frac{1}{2}\int_0^1 [\phi^2(t) + u^2(t)]dt, \\ &\dot{\phi} = -\phi + u, \\ &\phi(0) = 1. \end{aligned} \tag{55}$$

The exact solutions are

$$\phi = e^{(\alpha-1)t}, \;\; u = \alpha\phi = \alpha e^{(\alpha-1)t}, \tag{56}$$

where α is solved from the following algebraic equation:

$$(2\alpha^3 - \alpha^2 - 3)e^{2(\alpha-1)} - \alpha^2 + 2\alpha + 1 = 0, \tag{57}$$

which is about $\alpha = -0.333724$. The minimization of $J = (1+\alpha^2)[e^{2(\alpha-1)} - 1]/[4(\alpha-1)]$ is about 0.19385759.

Under the following parameters $\Delta t = 1/200$, $\Delta x = x_f/20 = 0.5/20$, and $\varepsilon = 10^{-3}$, we let r run in an interval of $r \in [0.4, 0.6]$ within 20 iterations to find the best r as shown in Fig. 1(a), which is happened at $r = 0.51$. The control force computed from the LGAM is compared with the exact one in Fig. 1(b), revealing that the numerical control force is very accurate with the maximum error being 1.42×10^{-2}. The computed J is about 0.193958, which is slightly larger than the exact one $J_e = 0.19385759$. The time history of ϕ is compared with the exact one in Fig. 1(c), and they are almost coincident, with the maximum error about 2.386×10^{-3}.

It is interesting that for this example, we can find a sub-optimal solution under a constant control force $u = \beta$. Through some algebraic operations we can derive

$$u = \beta = -\frac{e^2 - 2e + 1}{e^2 + 4e - 1} \approx -0.1710382, \quad \phi = \beta + (1-\beta)e^{-t},$$
$$J = \frac{1}{4}\left[1 - \frac{1}{e^2}\right](1 - 2\beta + \beta^2) + \left[1 - \frac{1}{e}\right](\beta - \beta^2) + \beta^2 \approx 0.19908047. \quad (58)$$

In Fig. 2 we compare the above two closed-form solutions. When the maximum difference of ϕ is 2.74×10^{-2}, the maximum difference of u is 0.163. This fact demonstrates that when the control forces have a moderate difference, the displacements and the performance indices have a little difference.

In the previous computation the initial guess of u_i is given by $u_i = 0.5\alpha e^{(\alpha-1)t_i}$; however, we can try another initial guess of u_i by $u_i = -0.01$. We let r run in an interval of $r \in [0.5, 0.6]$ to find the best r as shown in Fig. 3(a). The control force is compared with the exact one in Fig. 3(b), which revealing the numerical control force is very accurate with a maximum error being 1.948×10^{-2}, where J is about 0.19407. In Fig. 3(c) the time history of ϕ is compared with the exact one in Eq. (58), and they are quite close, with the maximum difference being 8.41×10^{-3}.

5.2 *Example 2*

We consider an optimal control problem of a simple harmonic oscillator:

$$J = \frac{1}{2}\int_{-T}^{0} u^2(t)dt,$$
$$\ddot{\phi} + \phi = u,$$
$$\phi(-T) = A_0,\ \dot{\phi}(-T) = B_0,\ \ \phi(0) = 0,\ \dot{\phi}(0) = 0, \quad (59)$$

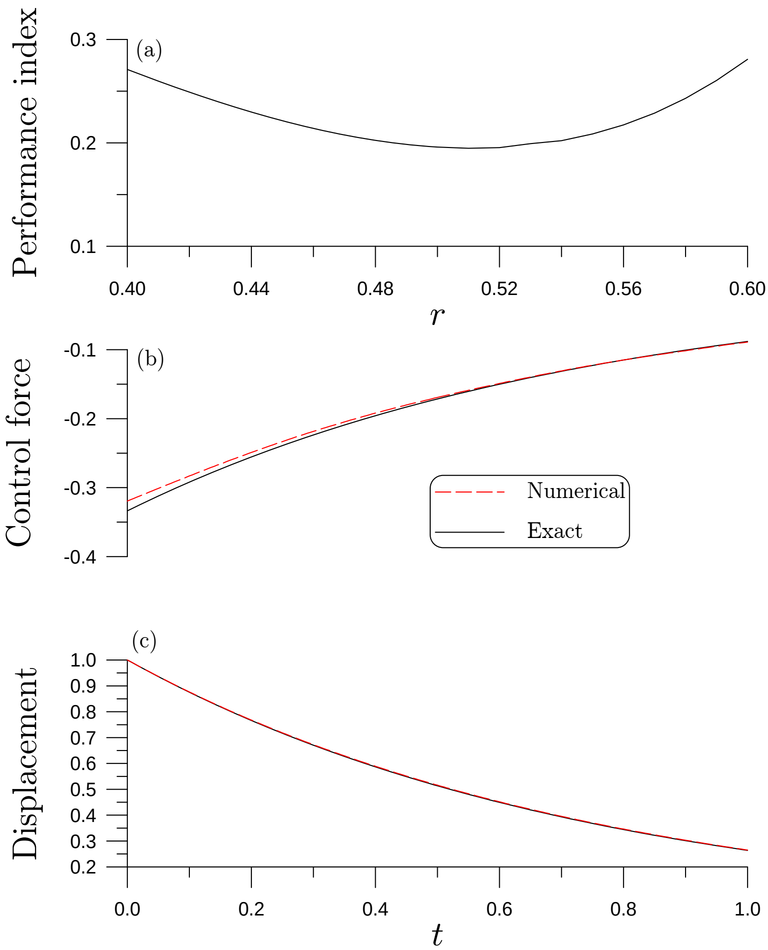

Figure 1: For the Feldbaum optimal control problem solved by the LGAM: (a) showing the performance index, (b) comparing the control forces, and (c) comparing the displacements.

where we fix $A_0 = 0.5$, $B_0 = -0.5$ and $T = 2$. The exact solutions derived from the optimality condition are

$$
\begin{aligned}
\phi &= \frac{1}{2}[At\sin t + B(\sin t - t\cos t)], \quad u = A\cos t + B\sin t, \\
J &= \frac{1}{8}[2T(A^2+B^2) + (A^2-B^2)\sin(2T) - 4AB\sin^2 T], \\
A &= \frac{2[A_0 T\sin T - B_0(T\cos T - \sin T)]}{T^2 - \sin^2 T}, \\
B &= \frac{2[B_0 T\sin T + A_0(\sin T + T\cos T)]}{T^2 - \sin^2 T}.
\end{aligned}
\tag{60}
$$

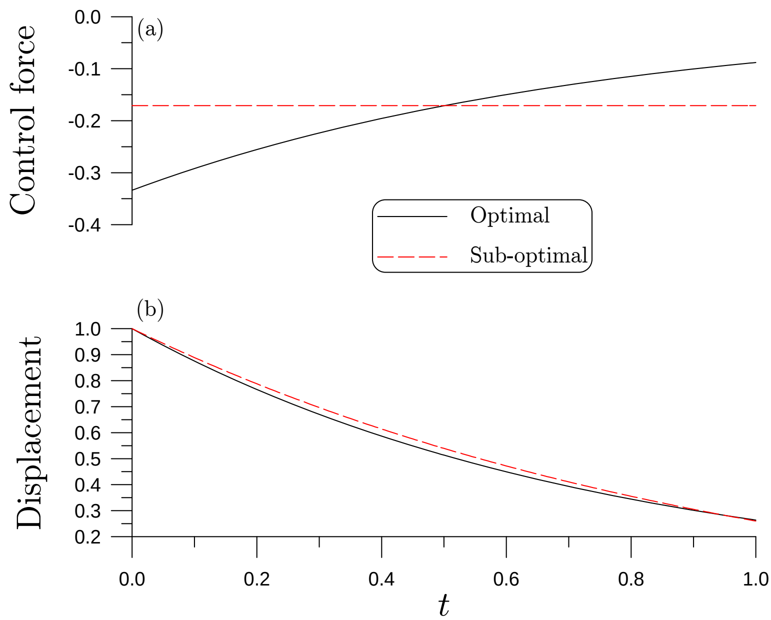

Figure 2: For the Feldbaum control problem comparing the optimal and sub-optimal solutions.

Here the minimization of J is about 0.184858542 [Vlassenbroeck and Van Dooren (1988)].

In the search of the best r we have added a penalty term $w[(v_n^0)^2 + (v_{t,n}^0)^2]$ with $w = 50$ to the minimized function of J, because we have to take the final time conditions in Eq. (59) into account.

Under the following parameters $\Delta t = 2/200$, $\Delta x = 0.5/20$, and $\varepsilon = 10^{-3}$ we let r run in the interval of $r \in [0.6, 0.65]$ within 10 iterations to find the best r as shown in Fig. 4(a). The control force is compared with the exact one in Fig. 4(b), where J is about 0.18356, which is slightly smaller than the exact one. The time history of ϕ is compared with the exact one in Fig. 4(c), and they are rather close, with the maximum error being 1.72×10^{-2}.

We must emphasize that the $J = 0.18356$ we obtained by the LGAM is smaller than that $J = 0.184858542$ derived from the optimality condition. This is possible, because the optimality condition is only a *necessary condition*, not a *sufficient condition*.

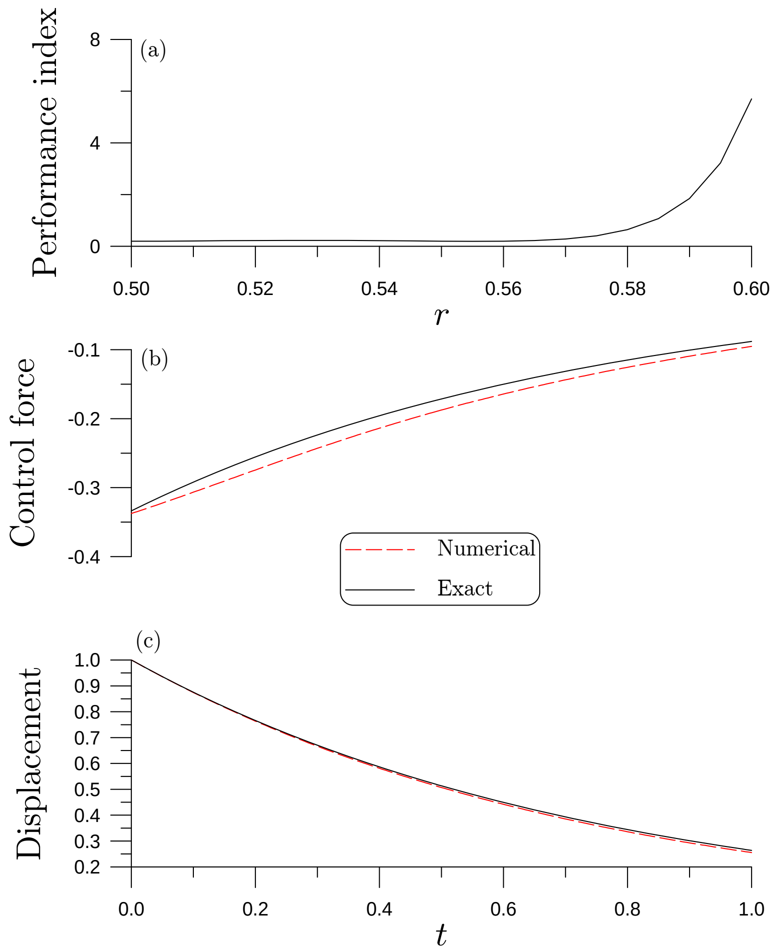

Figure 3: For the Feldbaum optimal control problem solved by the LGAM with a constant initial guess: (a) showing the performance index, (b) comparing the control forces, and (c) comparing the displacements.

5.3 Example 3

We consider the following performance index for the above oscillator:

$$J = \frac{1}{2}\phi^2(t_f) + \frac{1}{2}[\dot{\phi}(t_f)]^2 + \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt, \tag{61}$$

where we fix $t_0 = 0$, $t_f = 2$, $\phi(0) = 0.5$ and $\dot{\phi}(0) = -0.5$. Through some derivations we can obtain the exact solutions of u and ϕ.

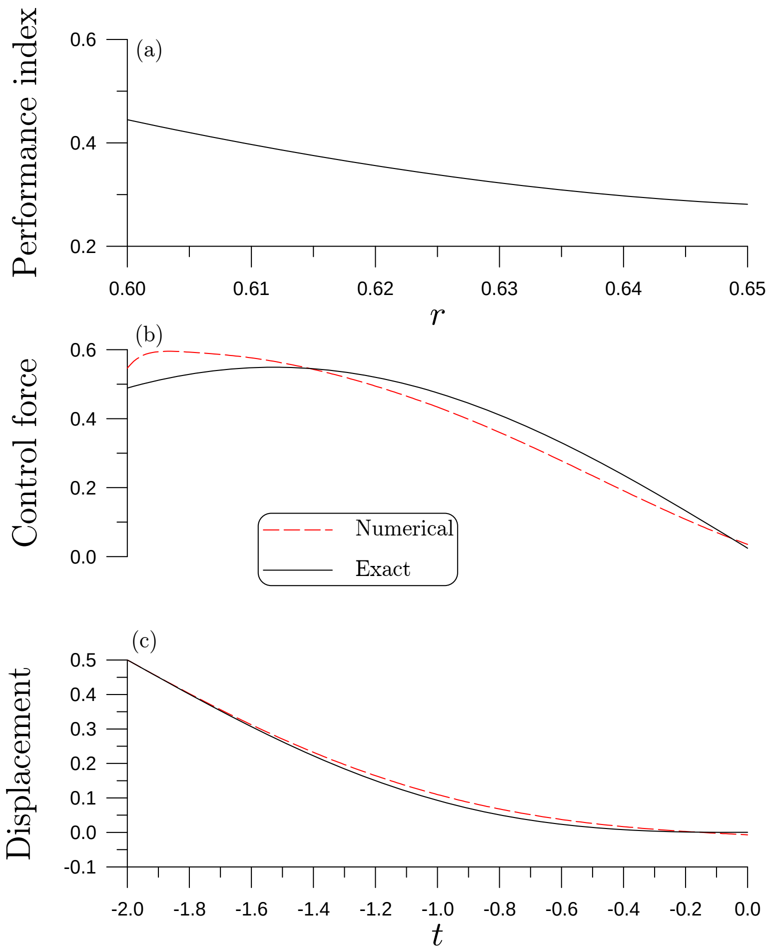

Figure 4: For an optimal control problem of a simple oscillator with end constraints solved by the LGAM: (a) showing the performance index, (b) comparing the control forces, and (c) comparing the displacements.

Here the exact minimization of J is about 0.104565. Under the following parameters $\Delta t = 2/200$, $\Delta x = 0.2/20$, and $\varepsilon = 10^{-3}$ we let r run in the interval of $r \in [0.2, 0.7]$ within 10 iterations to find the best r as shown in Fig. 5(a). The control force obtained is compared with the exact one in Fig. 5(b), where J is about 0.11102. The time history of ϕ is compared with the exact one in Fig. 5(c), which is close to the exact one, with the maximum error being 1.72×10^{-2}.

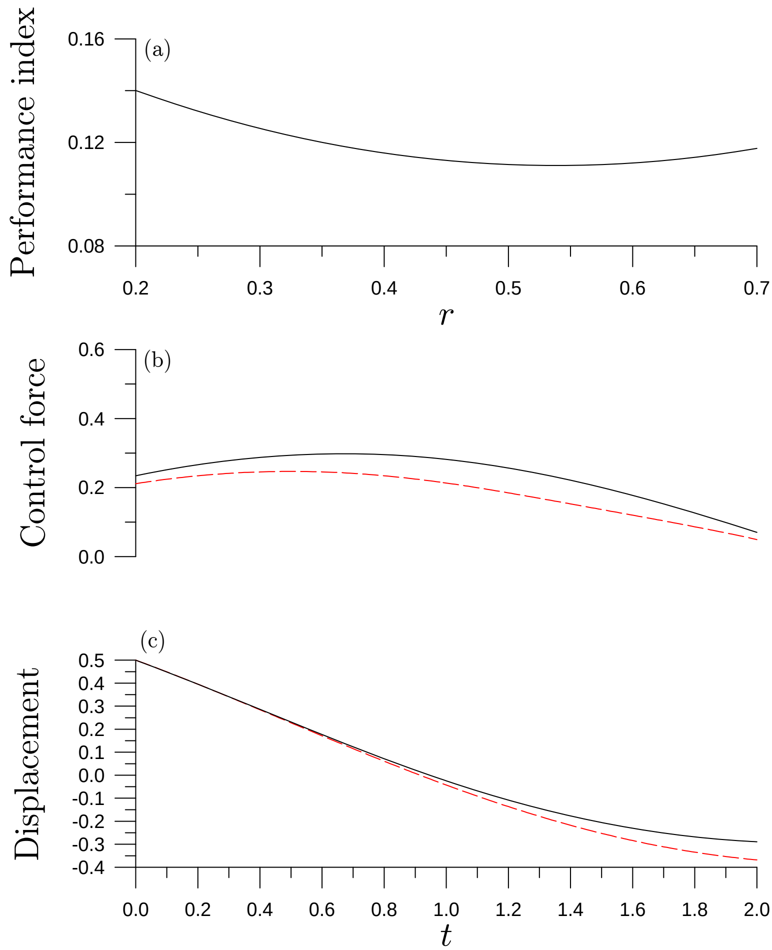

Figure 5: For an optimal control problem of a simple oscillator without end constraints solved by the LGAM: (a) showing the performance index, (b) comparing the control forces, and (c) comparing the displacements.

5.4 Example 4

In this example we solve the optimal control problem of the Duffing oscillator [Davies (1972); van Dooren and Vlassenbroeck (1982); El-Gindy, El-Hawary, Alim and El-Kady (1995); El-Kady and Elbarbary (2002); Lakestani, Razzaghi and De-

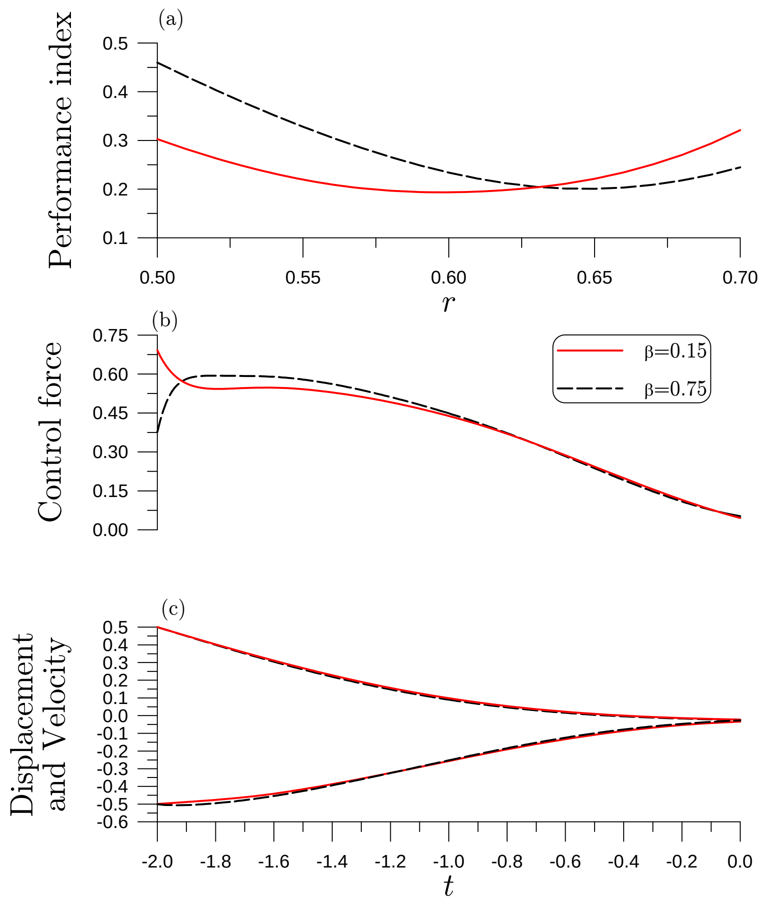

Figure 6: For an optimal control problem of a Duffing oscillator with end constraints solved by the LGAM, showing (a) the performance index, (b) the control force, and (c) the displacement and velocity.

hghan (2006)]. The performance index and equation of motion are given by

$$J = \frac{1}{2}\int_{-T}^{0} u^2(t)dt,$$
$$\ddot{\phi}(t) + \phi(t) + \beta\phi^3(t) = u(t),$$
$$\phi(-T) = A_0,\ \dot{\phi}(-T) = B_0,\ \phi(0) = \dot{\phi}(0) = 0, \tag{62}$$

where we fix $A_0 = 0.5$, $B_0 = -0.5$ and $T = 2$.

In the search of the best r we have added a penalty term $w[(v_n^0)^2 + (v_{t,n}^0)^2]$ with

$w = 10$. Under the following parameters $\Delta t = 2/200$, $\Delta x = 0.2/30$, and $\varepsilon = 10^{-3}$ we let r run in the interval of $r \in [0.5, 0.7]$ within 20 iterations to find the best r as shown in Fig. 6(a) for the case of $\beta = 0.15$, where the minimal point is located at $r = 0.6$. The control force solved from the LGAM is shown in Fig. 6(b) by the solid line, where the value of J is about 0.17727. The time histories of ϕ and $\dot{\phi}$ are displayed in Fig. 6(c) by the solid lines. It can be seen that the curves of displacement and velocity both match the terminal conditions $\phi(0) = 0$ and $\dot{\phi}(0) = 0$ very well. It deserves to note that the value of J we obtained is slightly smaller than 0.1874, which was obtained by other methods [van Dooren and Vlassenbroeck (1982); Razzaghi and Elnagar (1994); Lakestani, Razzaghi and Dehghan (2006)]. It shows that the present method can achieve a better control strategy than other methods.

Similarly, we consider a strongly nonlinear case with $\beta = 0.75$. The minimal point is happened at $r = 0.65$, and the numerical results are shown in Figs. 6(a)-6(c) by the dashed lines. The value of J is about 0.1879, which is also smaller than 0.1979 obtained by Razzaghi and Elnagar (1994). It shows again that the present method can achieve a better optimal control than the method of Razzaghi and Elnagar (1994). Although the curves of control force are quite different, the curves of displacement and velocity are close.

5.5 *Example 5*

In this example we solve the optimal control problem of the Duffing oscillator under a more complex performance index:

$$J = \frac{1}{2}\int_{-T}^{0} [\phi^2(t) + \dot{\phi}^2(t) + \exp(u^2(t))]dt, \tag{63}$$

which is subjected to the initial conditions with $A_0 = 0.5$, $B_0 = -0.5$ and the ends are free.

In the Hamiltonian formulation, it is difficult to express u as a function of the co-state variables; hence, many methods based on the Hamiltonian formulation cannot be applied to this problem.

Under the following parameters $\Delta t = 2/200$, $\Delta x = 0.2/30$, $\varepsilon = 10^{-3}$ and $\beta = 0.75$ we let r run in the interval of $r \in [0.2, 0.7]$ within 20 iterations to find the best r as shown in Fig. 7(a), where the minimal point is located at $r = 0.525$. The control force solved from the LGAM is shown in Fig. 7(b), where the value of J is about 1.466. The time histories of ϕ and $\dot{\phi}$ are displayed in Fig. 7(c). This example reveals that the present method of LGAM is quite easily used to treat other optimal

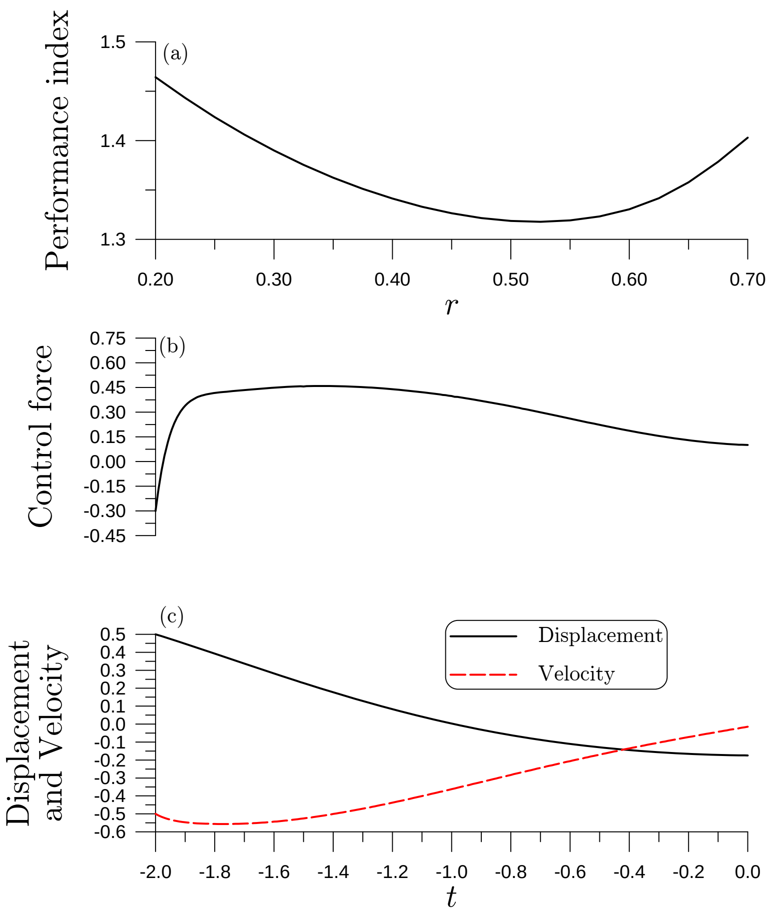

Figure 7: For an optimal control problem of a Duffing oscillator with free ends solved by the LGAM, showing (a) the performance index, (b) the control force, and (c) the displacement and velocity.

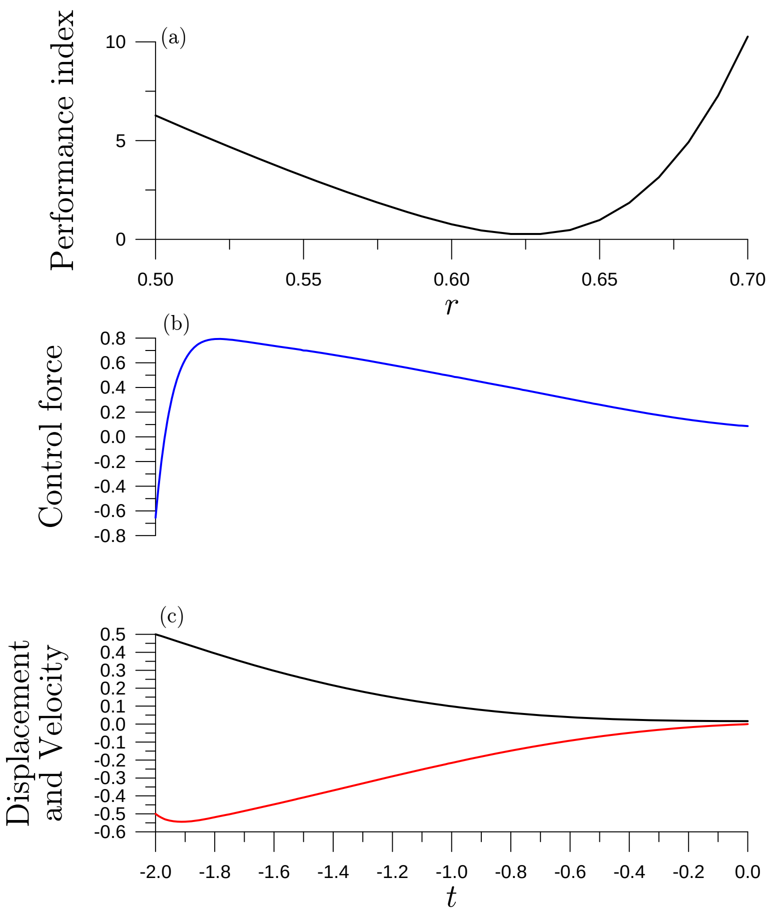

Figure 8: For an optimal control problem of a van der Pol oscillator with end constraints solved by the LGAM, showing (a) the performance index, (b) the control force, and (c) the displacement and velocity.

control problems with a complex functional of J.

5.6 Example 6

In this example we solve the optimal control problem of the following van der Pol oscillator:

$$\begin{aligned}&\ddot{\phi}(t)+\phi(t)+\beta(1-\phi^2)\dot{\phi}(t)=u(t),\\&\phi(-T)=A_0,\ \dot{\phi}(-T)=B_0,\ \phi(0)=\dot{\phi}(0)=0,\end{aligned} \tag{64}$$

where we fix $\beta=0.15$, $A_0=0.5$, $B_0=-0.5$ and $T=2$. The performance index used in this optimal control is also given by Eq. (62).

Under the following parameters $\Delta t=2/200$, $\Delta x=0.5/20$, and $\varepsilon=10^{-3}$ we let r run in the interval of $r\in[0.5,0.7]$ within 30 iterations to find the best r as shown in Fig. 8(a). In the search of the best r we have added a penalty term $w[(v_n^0)^2+(v_{t,n}^0)^2]$ with $w=30$ to the minimized function of J. The control force solved from the LGAM is shown in Fig. 8(b), where the value of J is about 0.256622, of which the minimum is happened at $r=0.63$. The time histories of ϕ and $\dot{\phi}$ are displayed in Fig. 8(c). It can be seen that the curves of displacement and velocity both match the terminal conditions $\phi(0)=0$ and $\dot{\phi}(0)=0$ very well.

6 Conclusions

For an optimally controlled vibration problem of nonlinear oscillators to find an optimal control force, we have transformed the equation of motion into a parabolic PDE with an unknown heat-source to be identified. Hence, it became a nonlinear inverse problem. By a semi-discretization of the PDE, the optimal control problem was further reformulated as being a system of n-dimensional ODEs with n unknown point-wise sources. However, based-on the Lie-group method in a fictitious dimension we have developed a Lie-group adaptive method (LGAM) to easily and correctly find the optimal control force. The present LGAM can handle the minimization problem with a complex performance index, where the control force can be computed accurately. In the LGAM, the minimization of the performance index was used as a target equation to select a suitable value of the parameter r, such that the optimal control problem, upon being formulated in the framework of the LGAM, becomes quite easy to find the optimal control force, and the computational cost is saving. Numerical examples disclosed that the LGAM can obtain a better value of the performance index than other methods.

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Appendix

In this appendix we provide some backgrounds of the Differential Quadrature (DQ) and Integral Quadrature (IQ).

Bellman and Casti (1971), and Bellman, Kashef and Casti (1972) first proposed the Differential Quadrature (DQ) approximation of derivatives to mimic the integral quadrature. Here, we consider a scalar function $f(x)$ defined in a closed interval $x \in [a,b]$. It is supposed that there are n grid points with coordinates $x_1 = a, x_2, \ldots, x_n = b$. The function $f(x)$ is assumed to be differentiable at any grid point, so that its first-order derivative $f'(x)$ at any grid point x_i can be approximated by

$$f'(x_i) = \sum_{j=1}^{n} a_{ij} f(x_j). \tag{A1}$$

In the first approach of Bellman, Kashef and Casti (1972), the test functions are chosen as

$$g_k(x) = x^k, \ \ k = 0, 1, \ldots, n-1, \tag{A2}$$

such that we have the following algebraic equations to determine the weighting coefficients a_{ij}:

$$\begin{cases} \sum_{j=1}^n a_{ij} = 0, \\ \sum_{j=1}^n a_{ij} x_j = 1, \\ \sum_{j=1}^n a_{ij} x_j^k = k x_i^{k-1}, \ \ k = 2, \ldots, n-1. \end{cases} \tag{A3}$$

Similarly, for the integral quadrature:

$$\int_a^b f(x) dx = \sum_{i=1}^n b_i f(x_i), \tag{A4}$$

we can derive

$$\begin{cases} \sum_{i=1}^n b_i = b - a, \\ \sum_{i=1}^n b_i x_i^k = \frac{b^{k+1} - a^{k+1}}{k+1}, \ \ k = 1, \ldots, n-1. \end{cases} \tag{A5}$$

By inspection, we can see that the above systems are with the Vandermonde matrix as the coefficient matrix. Therefore, we can apply the technique described by Liu and Atluri (2009) to solve the above system, i.e., we solve

$$\begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ \frac{x_1}{R_0} & \frac{x_2}{R_0} & \ldots & \frac{x_{n-1}}{R_0} & \frac{x_n}{R_0} \\ \left(\frac{x_1}{R_0}\right)^2 & \left(\frac{x_2}{R_0}\right)^2 & \ldots & \left(\frac{x_{n-1}}{R_0}\right)^2 & \left(\frac{x_n}{R_0}\right)^2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \left(\frac{x_1}{R_0}\right)^{n-2} & \left(\frac{x_2}{R_0}\right)^{n-2} & \ldots & \left(\frac{x_{n-1}}{R_0}\right)^{n-2} & \left(\frac{x_n}{R_0}\right)^{n-2} \\ \left(\frac{x_1}{R_0}\right)^{n-1} & \left(\frac{x_2}{R_0}\right)^{n-1} & \ldots & \left(\frac{x_{n-1}}{R_0}\right)^{n-1} & \left(\frac{x_n}{R_0}\right)^{n-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b - a \\ \frac{b^2 - a^2}{2R_0} \\ \vdots \\ \frac{b^{k+1} - a^{k+1}}{(k+1) R_0^k} \\ \vdots \\ \frac{b^n - a^n}{n R_0^{n-1}} \end{bmatrix}, \tag{A6}$$

where R_0 is a scaling factor. When the coefficient matrix of the first-order differential DQ is obtained, we can obtain the second-order differential [Shen and Liu (2011)]:

$$f''(x_i) = \sum_{j=1}^n b_{ij} f(x_j), \tag{A7}$$

where $b_{ij} = a_{ik} a_{kj}$. In the context the matrix a_{ij} will be denoted by $\mathscr{K}$, while the matrix b_{ij} will be denoted by $\mathscr{L}$.