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An Inverse Problem for Two Spectra of Complex Finite Jacobi Matrices

Gusein Sh. Guseinov[1]

Abstract: This paper deals with the inverse spectral problem for two spectra of finite order complex Jacobi matrices (tri-diagonal symmetric matrices with complex entries). The problem is to reconstruct the matrix using two sets of eigenvalues, one for the original Jacobi matrix and one for the matrix obtained by replacing the first diagonal element of the Jacobi matrix by some another number. The uniqueness and existence results for solution of the inverse problem are established and an explicit algorithm of reconstruction of the matrix from the two spectra is given.

Keywords: Jacobi matrix, difference equation, eigenvalue, normalizing numbers, inverse spectral problem.

1 Introduction

Let J be an $N \times N$ Jacobi matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{1}$$

where for each n, a_n and b_n are arbitrary complex numbers such that a_n is different from zero:

$$a_n, b_n \in \mathbb{C}, \quad a_n \neq 0. \tag{2}$$

[1] Department of Mathematics, Atilim University, 06836 Incek, Ankara, Turkey. E-mail: guseinov@atilim.edu.tr

A distinguishing feature of the Jacobi matrix (1) from other matrices is that the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$ is equivalent to the second order linear difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{3}$$

$$n \in \{0,1,\ldots,N-1\}, \quad a_{-1} = a_{N-1} = 1,$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0. \tag{4}$$

This allows, using techniques from the theory of three-term linear difference equations Atkinson (1964), to develop a thorough analysis of the eigenvalue problem $Jy = \lambda y$.

Define $\widetilde{J}$ to be the Jacobi matrix where all a_n and b_n are the same as J, except b_0 is replaced by $\widetilde{b}_0 \in \mathbb{C}$, that is,

$$\widetilde{J} = \begin{bmatrix} \widetilde{b}_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}. \tag{5}$$

We shall assume that

$$\widetilde{b}_0 \neq b_0. \tag{6}$$

Denote by $\lambda_1,\ldots,\lambda_p$ all the distinct eigenvalues of the matrix J and by $m_1,\ldots,m_p$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J - \lambda I)$ so that $1 \le p \le N$, $m_1 + \ldots + m_p = N$, and

$$\det(\lambda I - J) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}. \tag{7}$$

Further, denote by $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_q$ all the distinct eigenvalues of the matrix $\widetilde{J}$ and by $n_1,\ldots,n_q$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(\widetilde{J} - \lambda I)$ so that $1 \le q \le N$, $n_1 + \ldots + n_q = N$, and

$$\det(\lambda I - \widetilde{J}) = (\lambda - \widetilde{\lambda}_1)^{n_1} \cdots (\lambda - \widetilde{\lambda}_q)^{n_q}. \tag{8}$$

The collections

$$\{\lambda_k,\ m_k\ (k=1,\ldots,p)\} \quad \text{and} \quad \{\widetilde{\lambda}_i,\ n_i\ (i=1,\ldots,q)\} \tag{9}$$

form the spectra (together with their multiplicities) of the matrices J and $\widetilde{J}$, respectively. We call these collections the *two spectra* of the matrix J.

The inverse problem about two spectra consists in reconstruction of the matrix J by its two spectra. This problem consists of the following parts:

(i) Is the matrix J determined uniquely by its two spectra?

(ii) To indicate an algorithm for the construction of the matrix J from its two spectra.

(iii) To find necessary and sufficient conditions for two collection of numbers in (9) to be the two spectra for some matrix of the form (1) with entries from class (2).

For real finite Jacobi matrices this problem is completely solved in Guseinov (2012). Note that in case of real entries the finite Jacobi matrix is selfadjoint and its eigenvalues are real and distinct. In the complex case the Jacobi matrix is, in general, no longer selfadjoint and its eigenvalues may be complex and multiple.

In the present paper we show, by reducing the inverse problem about two spectra to the inverse problem about spectral data consisting of the eigenvalues and normalizing numbers of the matrix, that the complex Jacobi matrix is determined from two spectra given in (9) uniquely up to signs of the off-diagonal elements of the matrix. We indicate also necessary and sufficient conditions for two collections of numbers of the form given in (9) to be two spectra of a Jacobi matrix J of the form (1) with entries belonging to the class (2).

For given collections in (9), assuming that

$$\sum_{k=1}^{p} m_k\lambda_k - \sum_{i=1}^{q} n_i\widetilde{\lambda}_i =: a \neq 0, \tag{10}$$

we construct the numbers

$$\beta_{kj} = \frac{1}{a(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\frac{\prod_{i=1}^{q}(\lambda-\widetilde{\lambda}_i)^{n_i}}{\prod_{l=1,l\neq k}^{p}(\lambda-\lambda_l)^{m_l}} \tag{11}$$

$$(j = 1,\ldots,m_k;\ k=1,\ldots,p)$$

and then set

$$s_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \quad l = 0, 1, 2, \dots, \tag{12}$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j-1 > l$. Using these numbers we introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \dots. \tag{13}$$

The main result of this paper is the following theorem.

Theorem 1. *Let two collections of numbers in (9) be given, where $\lambda_1, \dots, \lambda_p$ are distinct complex numbers with $p \in \{1, \dots, N\}$ and $m_1, \dots, m_p$ are positive integers such that $m_1 + \dots + m_p = N$; the $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_q$ are distinct complex numbers with $q \in \{1, \dots, N\}$ and $n_1, \dots, n_q$ are positive integers such that $n_1 + \dots + n_q = N$. In order for these collections to be two spectra for a Jacobi matrix J of the form (1) with entries belonging to the class (2), it is necessary and sufficient that the following two conditions be satisfied:*

(i) $\lambda_k \neq \widetilde{\lambda}_i$ *for all* $k \in \{1, \dots, p\}, i \in \{1, \dots, q\}$, *and (10) holds;*

(ii) $D_n \neq 0$, *for* $n \in \{1, 2, \dots, N-1\}$, *where D_n is the determinant defined by (13), (12), (11).*

Under the conditions (i) and (ii) the entries a_n and b_n of the matrix J for which the collections in (9) are two spectra, are recovered by the formulae

$$a_n = \frac{\pm\sqrt{D_{n-1} D_{n+1}}}{D_n}, \quad n \in \{0, 1, \dots, N-2\}, \ D_{-1} = 1, \tag{14}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0, 1, \dots, N-1\}, \ \Delta_{-1} = 0, \ \Delta_0 = s_1, \tag{15}$$

where D_n is defined by (13), (12), (11), and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \dots, s_{2n+1}$. Further, the element $\widetilde{b}_0$ of the matrix $\widetilde{J}$ corresponding to the matrix J by (5) is determined by the formula

$$\widetilde{b}_0 = b_0 + \sum_{i=1}^{q} n_i \widetilde{\lambda}_i - \sum_{k=1}^{p} m_k \lambda_k.$$

It follows from the above solution of the inverse problem about two spectra that the matrix (1) is not uniquely restored from the two spectra. This is linked with the fact that the a_n are determined from (14) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs $+$ and $-$. Namely, let $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ be a given finite sequence, where for each $n \in \{0, 1, \ldots, N-2\}$ the σ_n is $+$ or $-$. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (14) for $n \in \{0, 1, \ldots, N-2\}$ we can choose the sign σ_n when extracting the square root. In this way we get precisely 2^{N-1} distinct Jacobi matrices possessing the same two spectra. The inverse problem is solved uniquely from the data consisting of the two spectra and a sequence $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ of signs $+$ and $-$. Thus, we can say that the inverse problem with respect to the two spectra is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix.

The paper is organized as follows. Section 2 is auxiliary and presents solution of the inverse spectral problem for complex finite Jacobi matrices in terms of the eigenvalues and normalizing numbers. Last Section 3 presents the solution of the inverse problem for complex finite Jacobi matrices in terms of the two spectra.

2 Auxiliary facts

In this section we follow the auhtor's paper Guseinov (2009). Given a Jacobi matrix J of the form (1) with the entries (2), consider the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$, that is equivalent to the problem (3), (4). Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of Eq. (3) satisfying the initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1; \tag{16}$$

$$Q_{-1}(\lambda) = -1, \quad Q_0(\lambda) = 0. \tag{17}$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n and is called a polynomial of first kind and $Q_n(\lambda)$ is a polynomial of degree $n-1$ and is known as a polynomial of second kind. These polynomials can be found recurrently from Eq. (3) using initial conditions (16) and (17). The leading terms of the polynomials $P_n(\lambda)$ and $Q_n(\lambda)$ have the forms

$$P_n(\lambda) = \frac{\lambda^n}{a_0 a_1 \cdots a_{n-1}} + \ldots,\ n \geq 0; \quad Q_n(\lambda) = \frac{\lambda^{n-1}}{a_0 a_1 \cdots a_{n-1}} + \ldots,\ n \geq 1. \tag{18}$$

The equality

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) \tag{19}$$

holds [see Guseinov (2009)] so that the eigenvalues of the matrix J coincide with the zeros of the polynomial $P_N(\lambda)$.

The Wronskian of the solutions $P_n(\lambda)$ and $Q_n(\lambda)$,

$$a_n[P_n(\lambda)Q_{n+1}(\lambda)-P_{n+1}(\lambda)Q_n(\lambda)],$$

does not depend on $n\in\{-1,0,1,\ldots,N-1\}$. On the other hand, the value of this expression at $n=-1$ is equal to 1 by (16), (17), and $a_{-1}=1$. Therefore

$$a_n[P_n(\lambda)Q_{n+1}(\lambda)-P_{n+1}(\lambda)Q_n(\lambda)]=1 \quad \text{for all} \quad n\in\{-1,0,1,\ldots,N-1\}.$$

Putting, in particular, $n=N-1$, we arrive at

$$P_{N-1}(\lambda)Q_N(\lambda)-P_N(\lambda)Q_{N-1}(\lambda)=1. \tag{20}$$

Let $R(\lambda)=(J-\lambda I)^{-1}$ be the resolvent of the matrix J (by I we denote the identity matrix of needed dimension) and e_0 be the N-dimensional column vector with the components $1,0,\ldots,0$. The rational function

$$w(\lambda)=-\langle R(\lambda)e_0,e_0\rangle=\left\langle(\lambda I-J)^{-1}e_0,e_0\right\rangle, \tag{21}$$

introduced earlier in Hochstadt (1974), we call the *resolvent function* of the matrix J, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{C}^N$. This function is known also as the Weyl-Titchmarsh function of J.

The entries $R_{nm}(\lambda)$ of the matrix $R(\lambda)=(J-\lambda I)^{-1}$ (resolvent of J) are of the form

$$R_{nm}(\lambda)=\begin{cases} P_n(\lambda)[Q_m(\lambda)+M(\lambda)P_m(\lambda)], & 0\le n\le m\le N-1,\\ P_m(\lambda)[Q_n(\lambda)+M(\lambda)P_n(\lambda)], & 0\le m\le n\le N-1,\end{cases} \tag{22}$$

[see Guseinov (2009)] where

$$M(\lambda)=-\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{23}$$

According to (21), (22), (23) and using initial conditions (16), (17), we get

$$w(\lambda)=-R_{00}(\lambda)=-M(\lambda)=\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{24}$$

We will use the following well-known useful lemma. We bring it here for easy reference.

Lemma 1. *Let $A(\lambda)$ and $B(\lambda)$ be polynomials with complex coefficients and* $\deg A<\deg B=N$. *Next, suppose that $B(\lambda)=b(\lambda-z_1)^{m_1}\cdots(\lambda-z_p)^{m_p}$, where*

$z_1,\dots,z_p$ are distinct complex numbers, b is a nonzero complex number, and $m_1,\dots,m_p$ are positive integers such that $m_1+\dots+m_p=N$. Then there exist uniquely determined complex numbers a_{kj} $(j=1,\dots,m_k;\ k=1,\dots,p)$ such that

$$\frac{A(\lambda)}{B(\lambda)}=\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{a_{kj}}{(\lambda-z_k)^j} \tag{25}$$

for all values of λ different from $z_1,\dots,z_p$. The numbers a_{kj} are given by the equation

$$a_{kj}=\frac{1}{(m_k-j)!}\lim_{\lambda\to z_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left[(\lambda-z_k)^{m_k}\frac{A(\lambda)}{B(\lambda)}\right], \tag{26}$$

$j=1,\dots,m_k;\ k=1,\dots,p.$

Proof. For each $k\in\{1,\dots,p\}$ we have

$$\frac{A(\lambda)}{B(\lambda)}=\frac{C_k(\lambda)}{(\lambda-z_k)^{m_k}}, \tag{27}$$

where the function

$$C_k(\lambda)=(\lambda-z_k)^{m_k}\frac{A(\lambda)}{B(\lambda)}$$

$$=\frac{A(\lambda)}{b(\lambda-z_1)^{m_1}\cdots(\lambda-z_{k-1})^{m_{k-1}}(\lambda-z_{k+1})^{m_{k+1}}\cdots(\lambda-z_p)^{m_p}}$$

is regular (analytic) at z_k. We can expand $C_k(\lambda)$ into a Taylor series about the point z_k,

$$C_k(\lambda)=\sum_{s=0}^{\infty}d_{ks}(\lambda-z_k)^s, \tag{28}$$

where

$$d_{ks}=\frac{C_k^{(s)}(z_k)}{s!},\quad s=0,1,2,\dots\ .$$

Substituting (28) in (27) we get that near z_k,

$$\frac{A(\lambda)}{B(\lambda)}=\sum_{s=0}^{m_k-1}\frac{d_{ks}}{(\lambda-z_k)^{m_k-s}}+(\text{a Taylor series about } z_k).$$

Consider the function

$$\Phi(\lambda)=\frac{A(\lambda)}{B(\lambda)}-\sum_{k=1}^{p}\sum_{s=0}^{m_k-1}\frac{d_{ks}}{(\lambda-z_k)^{m_k-s}}.$$

This function is analytic everywhere, that is, $\Phi(\lambda)$ is an entire function. Next, since $\deg A<\deg B$,

$$\Phi(\lambda)\to 0 \quad \text{as} \quad |\lambda|\to\infty.$$

Thus the entire function $\Phi(\lambda)$ is bounded and tends to zero as $|\lambda|\to\infty$. By the well-known Liouville theorem, we conclude that $\Phi(\lambda)\equiv 0$. Thus we have

$$\frac{A(\lambda)}{B(\lambda)}=\sum_{k=1}^{p}\sum_{s=0}^{m_k-1}\frac{d_{ks}}{(\lambda-z_k)^{m_k-s}}=\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{d_{k,m_k-j}}{(\lambda-z_k)^{j}}$$

and

$$d_{k,m_k-j}=\frac{C_k^{(m_k-j)}(z_k)}{(m_k-j)!}=\frac{1}{(m_k-j)!}\lim_{\lambda\to z_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left[(\lambda-z_k)^{m_k}\frac{A(\lambda)}{B(\lambda)}\right].$$

These prove (25) and (26). Note that decomposition (25) is unique as for the a_{kj} in this decomposition Eq. (26) necessarily holds. □

By (19) and (7) we have

$$P_N(\lambda)=c(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p},$$

where c is a nonzero constant. Therefore we can decompose the rational function $w(\lambda)$ expressed by (24) into partial fractions (Lemma 1) to get

$$w(\lambda)=\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(\lambda-\lambda_k)^{j}}, \tag{29}$$

where

$$\beta_{kj}=\frac{1}{(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left[(\lambda-\lambda_k)^{m_k}\frac{Q_N(\lambda)}{P_N(\lambda)}\right] \tag{30}$$

are called the *normalizing numbers* of the matrix J.

The collection of the eigenvalues and normalizing numbers

$$\{\lambda_k,\ \beta_{kj}\ (j=1,\dots,m_k;\ k=1,\dots,p)\}, \tag{31}$$

of the matrix J of the form (1), (2) is called the *spectral data* of this matrix.

Determination of the spectral data of a given Jacobi matrix is called the *direct spectral problem* for this matrix.

Thus, the spectral data consist of the eigenvalues and associated normalizing numbers derived by decomposing the resolvent function (Weyl-Titchmarsh function) $w(\lambda)$ into partial fractions using the eigenvalues.

It follows from (24) by (18) that $\lambda w(\lambda)$ tends to 1 as $\lambda \to \infty$. Therefore multiplying (29) by λ and passing then to the limit as $\lambda \to \infty$, we find that

$$\sum_{k=1}^{p} \beta_{k1} = 1. \tag{32}$$

The *inverse spectral problem* consists in reconstruction of the matrix J by its spectral data. This problem was solved by the author in Guseinov (2009) and we will present here the final result.

Let us set

$$s_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \quad l = 0, 1, 2, \dots , \tag{33}$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j-1 > l$. Next, using these numbers s_l we introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \dots . \tag{34}$$

Let us bring two important properties of the determinants D_n in the form of two lemmas.

Lemma 2. *Given any collection (31), for the determinants D_n defined by (34), (33), we have $D_n = 0$ for $n \geq N$, where $N = m_1 + \ldots + m_p$.*

Proof. Given a collection (31), define a linear functional Ω on the linear space of all polynomials in λ with complex coefficients as follows: if $G(\lambda)$ is a polynomial then the value $\langle \Omega, G(\lambda) \rangle$ of the functional Ω on the element (polynomial) G is

$$\langle \Omega, G(\lambda) \rangle = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \beta_{kj} \frac{G^{(j-1)}(\lambda_k)}{(j-1)!}, \tag{35}$$

where $G^{(n)}(\lambda)$ denotes the n-th order derivative of $G(\lambda)$ with respect to λ. Let $m \geq 0$ be a fixed integer and set

$$T(\lambda) = \lambda^m(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}$$

$$= t_m\lambda^m + t_{m+1}\lambda^{m+1} + \ldots + t_{m+N-1}\lambda^{m+N-1} + \lambda^{m+N}. \quad (36)$$

Then, according to (35),

$$\left\langle \Omega, \lambda^l T(\lambda) \right\rangle = 0, \quad l = 0,1,2,\ldots\ . \quad (37)$$

Consider (37) for $l = 0,1,2,\ldots,N+m$, and substitute (36) in it for $T(\lambda)$. Taking into account that

$$\left\langle \Omega, \lambda^l \right\rangle = s_l, \quad l = 0,1,2,\ldots\ , \quad (38)$$

where s_l is defined by (33), we get

$$t_m s_{l+m} + t_{m+1}s_{l+m+1} + \ldots + t_{m+N-1}s_{l+m+N-1} + s_{l+m+N} = 0,$$

$$l = 0,1,2,\ldots,N+m.$$

Therefore $(0,\ldots,0,t_m,t_{m+1},\ldots,t_{m+N-1},1)$ is a nontrivial solution of the homogeneous system of linear algebraic equations

$$x_0 s_l + x_1 s_{l+1} + \ldots + x_m s_{l+m} + x_{m+1}s_{l+m+1} + \ldots + x_{m+N-1}s_{l+m+N-1}$$

$$+ x_{m+N}s_{l+m+N} = 0, \quad l = 0,1,2,\ldots,N+m,$$

with the unknowns $x_0,x_1,\ldots,x_m,x_{m+1},\ldots,x_{m+N-1},x_{m+N}$. Therefore the determinant of this system, which coincides with D_{N+m}, must be zero. □

Lemma 3. *If collection (31) is the spectral data of the matrix J of the form (1) with entries belonging to the class (2), then for the determinants D_n defined by (34), (33) we have $D_n \neq 0$ for $n \in \{0,1,\ldots,N-1\}$.*

Proof. We have

$$D_0 = s_0 = \sum_{k=1}^{p} \beta_{k1} = 1 \neq 0$$

by (32). Consider now D_n for $n \in \{1,\ldots,N-1\}$. For any $n \in \{1,\ldots,N-1\}$ let us consider the homogeneous system of linear algebraic equations

$$\sum_{k=0}^{n} g_k s_{k+m} = 0, \quad m = 0,1,\ldots,n, \quad (39)$$

with unknowns $g_0, g_1, \ldots, g_n$. The determinant of system (39) coincides with the D_n. Therefore to prove $D_n \neq 0$, it is sufficient to show that system (39) has only a trivial solution. Assume the contrary: let (39) have a nontrivial solution $\{g_0, g_1, \ldots, g_n\}$. For each $m \in \{0, 1, \ldots, n\}$ take an arbitrary complex number h_m. Multiply both sides of (39) by h_m and sum the resulting equation over $m \in \{0, 1, \ldots, n\}$ to get

$$\sum_{m=0}^{n} \sum_{k=0}^{n} h_m g_k s_{k+m} = 0.$$

Substituting expression (38) for s_{k+m} in this equation and denoting

$$G(\lambda) = \sum_{k=0}^{n} g_k \lambda^k, \quad H(\lambda) = \sum_{m=0}^{n} h_m \lambda^m,$$

we obtain

$$\langle \Omega, G(\lambda) H(\lambda) \rangle = 0. \tag{40}$$

Since $\deg G(\lambda) \leq n$, $\deg H(\lambda) \leq n$ and the plynomials $P_0(\lambda), P_1(\lambda), \ldots, P_n(\lambda)$ form a basis (their degrees are different) of the linear space of polynomials of degree $\leq n$, we have expansions

$$G(\lambda) = \sum_{k=0}^{n} c_k P_k(\lambda), \quad H(\lambda) = \sum_{k=0}^{n} d_k P_k(\lambda).$$

Substituting these in (40) and using the orthogonality relations [see Guseinov (2009)]

$$\langle \Omega, P_m(\lambda) P_n(\lambda) \rangle = \delta_{mn}, \quad m, n \in \{0, 1, \ldots, N-1\},$$

where δ_{mn} is the Kronecker delta (at this place we use the condition that collection (31) is the spectral data for a matrix J of the form (1), (2)), we get

$$\sum_{k=0}^{n} c_k d_k = 0.$$

Since the polynomial $H(\lambda)$ is arbitrary, we can take $d_k = \overline{c_k}$ in the last equality and get that $c_0 = c_1 = \ldots = c_n = 0$, that is, $G(\lambda) \equiv 0$. But this is a contradiction and the proof is complete. □

The solution of the above inverse problem is given by the following theorem [see Guseinov (2009)].

Theorem 2. *Let an arbitrary collection (31) of numbers be given, where* $1 \leq p \leq N$, $m_1, \ldots, m_p$ *are positive integers with* $m_1 + \ldots + m_p = N$, $\lambda_1, \ldots, \lambda_p$ *are distinct* *complex numbers. In order for this collection to be the spectral data for a Jacobi matrix J of the form (1) with entries belonging to the class (2), it is necessary and sufficient that the following two conditions be satisfied:*

(i) $\sum_{k=1}^{p} \beta_{k1} = 1$;

(ii) $D_n \neq 0$, *for* $n \in \{1,2,\ldots,N-1\}$, *where* D_n *is the determinant defined by (34), (33).*

Under the conditions (i) and (ii) the entries a_n *and* b_n *of the matrix J for which the collection (31) is spectral data, are recovered by the formulae*

$$a_n = \frac{\pm\sqrt{D_{n-1}D_{n+1}}}{D_n}, \quad n \in \{0,1,\ldots,N-2\},\ D_{-1} = 1, \tag{41}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0,1,\ldots,N-1\},\ \Delta_{-1} = 0,\ \Delta_0 = s_1, \tag{42}$$

where D_n *is defined by (34), (33), and* Δ_n *is the determinant obtained from the determinant* D_n *by replacing in* D_n *the last column by the column with the components* $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.

It follows from the above solution of the inverse problem that the matrix (1) is not uniquely restored from the spectral data. This is linked with the fact that the a_n are determined from (41) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs $+$ and $-$. Namely, let $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ be a given finite sequence, where for each $n \in \{0,1,\ldots,N-2\}$ the σ_n is $+$ or $-$. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (41) for $n \in \{0,1,\ldots,N-2\}$ we can choose the sign σ_n when extracting the square root. In this way we get precisely 2^{N-1} distinct Jacobi matrices possessing the same spectral data. The inverse problem is solved uniquely from the data consisting of the spectral data and a sequence $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ of signs $+$ and $-$. Thus, we can say that the inverse problem with respect to the spectral data is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix.

Note that in the case of arbitrary real distinct numbers $\lambda_1, \ldots, \lambda_N$ and positive numbers $\beta_1, \ldots, \beta_N$ the condition (ii) of Theorem 2 is satisfied automatically and in this case we have $D_n > 0$, for $n \in \{1,2,\ldots,N-1\}$; see Guseinov (2009). However, in the complex case the condition (ii) of Theorem 2 need not be satisfied automatically. Indeed, let $N = 3$ and as the collection (31) we take

$$\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \beta_1,\ \beta_2,\ \beta_3\},$$

where $\lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3$ are arbitrary complex numbers such that

$$\lambda_1 \neq \lambda_2, \quad \lambda_1 \neq \lambda_3, \quad \lambda_2 \neq \lambda_3,$$

$\beta_1 \neq 0, \quad \beta_2 \neq 0, \quad \beta_3 \neq 0, \quad \beta_1 + \beta_2 + \beta_3 = 1.$

We have

$$s_l = \beta_1 \lambda_1^l + \beta_2 \lambda_2^l + \beta_3 \lambda_3^l, \quad l = 0, 1, 2, \ldots,$$

and it is not difficult to show that

$$D_1 = \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}$$

$$= \beta_1\beta_2(\lambda_1 - \lambda_2)^2 + \beta_1\beta_3(\lambda_1 - \lambda_3)^2 + \beta_2\beta_3(\lambda_2 - \lambda_3)^2,$$

$$D_2 = \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \beta_1\beta_2\beta_3(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2,$$

$$\Delta_0 = s_1 = \beta_1\lambda_1 + \beta_2\lambda_2 + \beta_3\lambda_3,$$

$$\Delta_1 = \begin{vmatrix} s_0 & s_2 \\ s_1 & s_3 \end{vmatrix} = \beta_1\beta_2(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2$$

$$+\beta_1\beta_3(\lambda_1 + \lambda_3)(\lambda_1 - \lambda_3)^2 + \beta_2\beta_3(\lambda_2 + \lambda_3)(\lambda_2 - \lambda_3)^2,$$

$$\Delta_2 = \begin{vmatrix} s_0 & s_1 & s_3 \\ s_1 & s_2 & s_4 \\ s_2 & s_3 & s_5 \end{vmatrix} = \beta_1\beta_2\beta_3 \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{vmatrix}.$$

We see that the condition $D_1 \neq 0$ is not satisfied automatically and therefore one must require $D_1 \neq 0$ as a condition. For example, if take

$$\beta_1 = \beta_2 = \beta_3 = \frac{1}{3}, \quad \lambda_1 = \frac{1 \pm i\sqrt{3}}{2}, \quad \lambda_2 = 1, \quad \lambda_3 = 0,$$

then we get $D_1 = 0$.

3 Construction of a complex Jacobi matrix from two of its spectra

Let J be an $N \times N$ Jacobi matrix of the form (1) with entries satisfying (2). Define $\widetilde{J}$ to be the Jacobi matrix given by (5), where the number $\widetilde{b}_0$ satisfies (6). We denote all the distinct eigenvalues of the matrices J and $\widetilde{J}$ by $\lambda_1, \ldots, \lambda_p$ and $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_q$, respectively. Let m_k be the multiplicity of λ_k as the root of the characteristic polynomial $\det(\lambda I - J)$ and n_i the multiplicity of $\widetilde{\lambda}_i$ as the root of the characteristic polynomial $\det(\lambda I - \widetilde{J})$. We call the collections $\{\lambda_k,\ m_k\ (k = 1, \ldots, p)\}$ and $\{\widetilde{\lambda}_i,\ n_i\ (i = 1, \ldots, q)\}$ the *two spectra* of the matrix J. Note that $m_1 + \ldots + m_p = N$ and $n_1 + \ldots + n_q = N$.

The inverse problem for two spectra consists in the reconstruction of the matrix J by two of its spectra.

We will reduce the inverse problem for two spectra to the inverse problem for eigenvalues and normalizing numbers solved above in Section 2.

First let us study some necessary properties of the two spectra of the Jacobi matrix J.

Let $P_n(\lambda)$ and $Q_n(\lambda)$ be the polynomials of the first and second kind for the matrix J. The similar polynomials for the matrix $\widetilde{J}$ we denote by $\widetilde{P}_n(\lambda)$ and $\widetilde{Q}_n(\lambda)$. By (19) we have

$$\det(J-\lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda), \tag{43}$$

$$\det\left(\widetilde{J}-\lambda I\right) = (-1)^N a_0 a_1 \cdots a_{N-2} \widetilde{P}_N(\lambda), \tag{44}$$

so that the eigenvalues $\lambda_1,\ldots,\lambda_p$ and $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_q$ of the matrices J and $\widetilde{J}$ and their multiplicities coincide with the zeros and their multiplicities of the polynomials $P_N(\lambda)$ and $\widetilde{P}_N(\lambda)$, respectively.

The $P_n(\lambda)$ and $\widetilde{P}_n(\lambda)$ satisfy the same equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \{1,\ldots,N-1\}, \quad a_{N-1} = 1, \tag{45}$$

subject to the initial conditions

$$P_0(\lambda) = 1, \quad P_1(\lambda) = \frac{\lambda - b_0}{a_0}; \tag{46}$$

$$\widetilde{P}_0(\lambda) = 1, \quad \widetilde{P}_1(\lambda) = \frac{\lambda - \widetilde{b}_0}{a_0}. \tag{47}$$

The $Q_n(\lambda)$ also satisfies Eq. (45); besides

$$Q_0(\lambda) = 0, \quad Q_1(\lambda) = \frac{1}{a_0}. \tag{48}$$

Since $P_n(\lambda)$ and $\widetilde{P}_n(\lambda)$ form, for $b_0 \neq \widetilde{b}_0$, linearly independent solutions of Eq. (45), the solution $Q_n(\lambda)$ will be a linear combination of the solutions $P_n(\lambda)$ and $\widetilde{P}_n(\lambda)$. Using initial conditions (46), (47), and (48) we find that

$$Q_n(\lambda) = \frac{1}{\widetilde{b}_0 - b_0}\left[P_n(\lambda) - \widetilde{P}_n(\lambda)\right], \quad n \in \{0,1,\ldots N\}. \tag{49}$$

Replacing $Q_N(\lambda)$ and $Q_{N-1}(\lambda)$ in (20) by their expressions from (49), we get

$$P_{N-1}(\lambda)\widetilde{P}_N(\lambda) - P_N(\lambda)\widetilde{P}_{N-1}(\lambda) = b_0 - \widetilde{b}_0. \tag{50}$$

Lemma 4. *The matrices J and $\widetilde{J}$ have no common eigenvalues, that is, $\lambda_k \neq \widetilde{\lambda}_i$ for all values of k and i.*

Proof. Suppose that λ is an eigenvalue of the matrices J and $\widetilde{J}$. Then by (43) and (44) we have $P_N(\lambda) = \widetilde{P}_N(\lambda) = 0$. But this is impossible by (50) and the condition $\widetilde{b}_0 \neq b_0$. □

The following lemma allows us to calculate the difference $\widetilde{b}_0 - b_0$ in terms of the two spectra.

Lemma 5. *The equality*

$$\sum_{k=1}^{p} m_k \lambda_k - \sum_{i=1}^{q} n_i \widetilde{\lambda}_i = b_0 - \widetilde{b}_0 \tag{51}$$

holds.

Proof. For any matrix $A = [a_{jk}]_{j,k=1}^{N}$ the spectral trace of A coincides with the matrix trace of A: if $\mu_1, \ldots, \mu_p$ are the distinct eigenvalues of A of multiplicities $m_1, \ldots, m_p$ as the roots of the characteristic polynomial $\det(J - \lambda I)$, then

$$\sum_{k=1}^{p} m_k \mu_k = \sum_{k=1}^{N} a_{kk}.$$

Indeed, this follows from

$$\det(\lambda I - A) = (\lambda - \mu_1)^{m_1} \ldots (\lambda - \mu_p)^{m_p}$$

by comparison of the coefficients of λ^{N-1} on the two sides. Therefore we can write

$$\sum_{k=1}^{p} m_k \lambda_k = b_0 + b_1 + \ldots + b_{N-1} \quad \text{and} \quad \sum_{i=1}^{q} n_i \widetilde{\lambda}_i = \widetilde{b}_0 + b_1 + \ldots + b_{N-1}.$$

Subtracting the last two equalities side by side we arrive at (51). □

Corollary 1. *It follows from (51) that, under the conditon (6),*

$$\sum_{k=1}^{p} m_k \lambda_k - \sum_{i=1}^{q} n_i \widetilde{\lambda}_i \neq 0.$$

The following lemma (together with Lemma 5) gives a formula for calculating the normalizing numbers β_{kj} $(j = 1, \ldots, m_k;\ k = 1, \ldots, p)$ in terms of the two spectra.

Lemma 6. *For each $k \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, m_k\}$ the formula*

$$\beta_{kj} = \frac{1}{(b_0 - \widetilde{b}_0)(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \frac{\prod_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i}}{\prod_{l=1, l \neq k}^{p} (\lambda - \lambda_l)^{m_l}} \tag{52}$$

holds.

Proof. Replacing $Q_N(\lambda)$ in (24) by its expression from (49), we get

$$w(\lambda)=\frac{1}{\widetilde{b}_0-b_0}\left[1-\frac{\widetilde{P}_N(\lambda)}{P_N(\lambda)}\right]. \tag{53}$$

Substituting (29) in the left-hand side we can write

$$\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(\lambda-\lambda_k)^j}=\frac{1}{\widetilde{b}_0-b_0}\left[1-\frac{\widetilde{P}_N(\lambda)}{P_N(\lambda)}\right].$$

Hence, we get, taking into account (43) and (44),

$$\beta_{kj}=\frac{1}{(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left\{\frac{(\lambda-\lambda_k)^{m_k}}{\widetilde{b}_0-b_0}\left[1-\frac{\widetilde{P}_N(\lambda)}{P_N(\lambda)}\right]\right\}$$

$$=\frac{1}{(b_0-\widetilde{b}_0)(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left[(\lambda-\lambda_k)^{m_k}\frac{\widetilde{P}_N(\lambda)}{P_N(\lambda)}\right]$$

$$=\frac{1}{(b_0-\widetilde{b}_0)(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left[(\lambda-\lambda_k)^{m_k}\frac{\det\left(\lambda I-\widetilde{J}\right)}{\det(\lambda I-J)}\right].$$

Substituting here (7) and (8) we arrive at (52). The lemma is proved. □

Theorem 3 (Uniqueness result)**.** *The two spectra in (9) determine the Jacobi matrix J of the form (1) in the class (2) and the number $\widetilde{b}_0\in\mathbb{C}$ in the matrix $\widetilde{J}$ defined by (5) uniquely up to signs of the off-diagonal elements of J.*

Proof. Given the two spectra in (9) of the matrix J we determine uniquely the number (difference) $b_0-\widetilde{b}_0$ by (51) and then the normalizing numbers β_{kj} ($j=1,\dots,m_k$; $k=1,\dots,p$) of the matrix J by (52). Since the collection of the eigenvalues and normalizing numbers of the matrix J determines J uniquely up to signs of the off-diagonal elements of J and the number $\widetilde{b}_0$ is determined uniquely by the equation (Lemma 5)

$$\widetilde{b}_0=b_0+\sum_{i=1}^{q}n_i\widetilde{\lambda}_i-\sum_{k=1}^{p}m_k\lambda_k,$$

the proof is complete. □

Let us now prove Theorem 1 stated above in the Introduction.

The necessity of the conditions of Theorem 1 has been proved above. To prove sufficiency suppose that two collections of numbers in (9) are given which satisfy the conditions of Theorem 1. We construct β_{kj} $(j = 1,\dots,m_k;\ k = 1,\dots,p)$ according to Eqs. (11) and (10). Let us show that

$$\sum_{k=1}^{p} \beta_{k1} = 1. \tag{54}$$

Indeed, we have, for sufficiently large positive number R such that $\lambda_1,\dots,\lambda_p$ are inside the sircle $\{\lambda \in \mathbb{C} : |\lambda| = R\}$,

$$\sum_{k=1}^{p} \beta_{k1} = \sum_{k=1}^{p} \frac{1}{a(m_k-1)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-1}}{d\lambda^{m_k-1}} \frac{\prod_{i=1}^{q}(\lambda-\widetilde{\lambda}_i)^{n_i}}{\prod_{l=1,l\neq k}^{p}(\lambda-\lambda_l)^{m_l}}$$

$$= \frac{1}{a}\sum_{k=1}^{p} \operatorname{Res}_{\lambda=\lambda_k} \frac{(\lambda-\widetilde{\lambda}_1)^{n_1}\cdots(\lambda-\widetilde{\lambda}_q)^{n_q}}{(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}}$$

$$= \frac{1}{2\pi i a} \oint_{|\lambda|=R} \frac{(\lambda-\widetilde{\lambda}_1)^{n_1}\cdots(\lambda-\widetilde{\lambda}_q)^{n_q}}{(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}} d\lambda$$

$$= \frac{1}{2\pi i a} \oint_{|\lambda|=R} \frac{\lambda^N - (n_1\widetilde{\lambda}_1+\ldots+n_q\widetilde{\lambda}_q)\lambda^{N-1}+\ldots}{\lambda^N - (m_1\lambda_1+\ldots+m_p\lambda_p)\lambda^{N-1}+\ldots} d\lambda$$

$$= \frac{1}{2\pi i a} \oint_{|\lambda|=R} \left[1 + \frac{1}{\lambda}\left(\sum_{k=1}^{p} m_k\lambda_k - \sum_{i=1}^{q} n_i\widetilde{\lambda}_i\right) + O\left(\frac{1}{|\lambda|^2}\right)\right] d\lambda$$

$$= \frac{1}{a}\left(\sum_{k=1}^{p} m_k\lambda_k - \sum_{i=1}^{q} n_i\widetilde{\lambda}_i\right) + \frac{1}{2\pi i a} \oint_{|\lambda|=R} O\left(\frac{1}{|\lambda|^2}\right) d\lambda.$$

Passing here to the limit as $R\to\infty$ and noting the definition (10) of a and that

$$\lim_{R\to\infty} \oint_{|\lambda|=R} O\left(\frac{1}{|\lambda|^2}\right) d\lambda = 0,$$

we arrive at (54).

Thus, the collection $\{\lambda_k,\ \beta_{kj}\ (j = 1,\dots,m_k;\ k = 1,\dots,p)\}$ satisfies all the conditions of Theorem 2 and hence there exist a Jacobi matrix J of the form (1) with entries from the class (2) such that λ_k are the eigenvalues of the multiplicity m_k

and β_{kj} are the corresponding normalizing numbers for J. Having the matrix J, in particular, its entry b_0, we construct the number $\widetilde{b}_0$ by

$$\widetilde{b}_0 = b_0 + \sum_{i=1}^{q} n_i \widetilde{\lambda}_i - \sum_{k=1}^{p} m_k \lambda_k \tag{55}$$

and then the matrix $\widetilde{J}$ by (5) according to the matrix J and (55). It remains to show that $\widetilde{\lambda}_i$ are the eigenvalues of $\widetilde{J}$ of multiplicity n_i. To do this we denote the eigenvalues of $\widetilde{J}$ by $\widetilde{\mu}_1, \dots, \widetilde{\mu}_s$ and their multiplicities by $\widetilde{n}_1, \dots, \widetilde{n}_s$. We have to show that $s = q$, $\widetilde{\mu}_i = \widetilde{\lambda}_i$, $\widetilde{n}_i = n_i$ $(i = 1, \dots, q)$. Let us set

$$f(\lambda) = \prod_{l=1}^{p} (\lambda - \lambda_l)^{m_l}, \quad g(\lambda) = \prod_{i=1}^{q} (\lambda - \widetilde{\lambda}_i)^{n_i}, \quad h(\lambda) = \prod_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i}.$$

Then

$$1 - \frac{g(\lambda)}{f(\lambda)} = \frac{g_1(\lambda)}{f(\lambda)}, \quad 1 - \frac{h(\lambda)}{f(\lambda)} = \frac{h_1(\lambda)}{f(\lambda)},$$

where

$$g_1(\lambda) = f(\lambda) - g(\lambda), \quad h_1(\lambda) = f(\lambda) - h(\lambda)$$

are polynomials and $\deg g_1 < \deg f$, $\deg h_1 < \deg f$. Next, by the direct problem we have (Lemma 6)

$$\beta_{kj} = \frac{1}{(b_0 - \widetilde{b}_0)(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \frac{\prod_{i=1}^{s} (\lambda - \widetilde{\mu}_i)^{\widetilde{n}_i}}{\prod_{l=1, l \neq k}^{p} (\lambda - \lambda_l)^{m_l}}$$

$$= -\frac{1}{(b_0 - \widetilde{b}_0)(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left\{ (\lambda - \lambda_k)^{m_k} \left[1 - \frac{h(\lambda)}{f(\lambda)} \right] \right\}$$

$$= -\frac{1}{(b_0 - \widetilde{b}_0)(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - \lambda_k)^{m_k} \frac{h_1(\lambda)}{f(\lambda)} \right] \tag{56}$$

On the other hand, by our construction of β_{kj} we have (11) which can be written in the form

$$\beta_{kj} = -\frac{1}{a(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left\{ (\lambda - \lambda_k)^{m_k} \left[1 - \frac{g(\lambda)}{f(\lambda)} \right] \right\}$$

$$= -\frac{1}{a(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - \lambda_k)^{m_k} \frac{g_1(\lambda)}{f(\lambda)} \right]. \tag{57}$$

Therefore, by Lemma 1 it follows from (56) and (57), taking into account $a = b_0 - \widetilde{b}_0$, that

$$\frac{h_1(\lambda)}{f(\lambda)} = \frac{g_1(\lambda)}{f(\lambda)}.$$

Hence $h_1(\lambda) \equiv g_1(\lambda)$, that is, $h(\lambda) \equiv g(\lambda)$ and consequently $s = q$, $\widetilde{\mu}_i = \widetilde{\lambda}_i$, $\widetilde{n}_i = n_i$ $(i = 1, \ldots, q)$. The proof is complete.

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