 CMES, vol.86, no.5, pp.435-468, 2012

A Higher Order Solution of the Elastic Problem for a Homogeneous, Linear-Elastic and Isotropic Half-Space Subjected to a Point-Load Perpendicular to the Surface

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Abstract: A recent experimental programme with the aim of acquiring the strains induced by aircraft traffic in concrete pavements [Ferretti and Bignozzi (2012); Ferretti (2012a)] has provided the opportunity of reviewing the classical solution of Boussinesq's problem for a homogeneous linear-elastic and isotropic half-space subjected to a point-load. In this document, we have proposed a second order solution to Boussinesq's problem, which allows us to account for the new experimental evidence.

Keywords: Boussinesq's problem, potentials properties, second order solution.

1 Introduction

Real soil bodies, composed of discrete particles, can be transformed into a form whereby useful deductions can be made through the exact processes of mathematics, by introducing the abstraction of a continuum, or continuous medium. This basic assumption allows us to perform densities and rates in the neighborhood of a material point. In particular, at each point of a medium that macroscopically acts as elastic (the same load-displacement path is followed during both the loading and the unloading processes), we can define the elastic constants as being the result of the limit process when all the neighborhood dimensions tend to zero. If the elastic constants are the same at all points within a region of a body, that region is said to be homogeneous. Additionally, the medium in the neighborhood of a point is called isotropic if its defining parameters are the same in all directions emanating from that point. Isotropy reduces the number of independent elastic constants at a point to two: E, the longitudinal modulus of elasticity or Young's modulus, and υ, Poisson's ratio [Harr, 1966]. A third elastic constant can be used to describe the behavior of the neighborhood, the shear modulus G, connected to the former two

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by the relationship:

$$G = \frac{E}{2(1+\upsilon)}. \tag{1}$$

Alternatively to E and υ, the two Lamé parameters λ and μ may be used as independent elastic constants within the neighborhood:

$$\lambda = E\frac{\upsilon}{(1+\upsilon)(1-2\upsilon)}, \tag{2}$$

$$\mu = G. \tag{3}$$

A homogeneous body is not necessarily isotropic and an isotropic body may not be also homogeneous. Furthermore, if the loading-unloading path of an elastic body is linear (the displacement is proportional to the applied load), the material that composes that body is said to be linear-elastic. In this case, the constitutive equations that relate σ_i to ε_i, the normal stresses and strains in the $i = x,y,z$ directions, and τ_{ij} to γ_{ij}, the shearing stresses and strains in the $i,j = x,y,z$ directions, with $i \neq j$, are linear. They are expressed by Hooke's well-known generalized laws:

$$\begin{aligned}
\sigma_x &= \lambda I_{1\varepsilon} + 2\mu\varepsilon_x \\
\sigma_y &= \lambda I_{1\varepsilon} + 2\mu\varepsilon_y \\
\sigma_z &= \lambda I_{1\varepsilon} + 2\mu\varepsilon_z \\
\tau_{yz} &= \mu\gamma_{yz} \\
\tau_{xz} &= \mu\gamma_{xz} \\
\tau_{xy} &= \mu\gamma_{xy}
\end{aligned} \tag{4}$$

where the strains are related to the displacements along the directions of the x, y, z axes, u, v, w, respectively, in the assumption of linear relationships between the components of strain and the displacement derivatives of the first order (theory of the first order for small strains):

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z}
\end{aligned}$$

$$
\begin{aligned}
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{aligned}
\tag{5}
$$

and $I_{1\varepsilon} = tr(\varepsilon)$ is the first invariant of strain, giving the bulk (volumetric) strain, $\Delta V/V$, in the first order theory:

$$I_{1\varepsilon} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\Delta V}{V}. \tag{6}$$

Hooke's generalized laws may be inverted to give the components of strain in functions of the components of stress:

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right] \\
\varepsilon_y &= \frac{1}{E}\left[\sigma_y - \upsilon\left(\sigma_z + \sigma_x\right)\right] \\
\varepsilon_z &= \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right] \\
\gamma_{yz} &= \frac{1}{G}\tau_{yz} \\
\gamma_{xz} &= \frac{1}{G}\tau_{xz} \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy}
\end{aligned}
\tag{7}
$$

The conservation of the linear momentum and Hooke's laws give rise to the governing equations in the partial differential forms:

$$
\begin{cases}
(\lambda + \mu)\frac{\partial I_{1\varepsilon}}{\partial x} + \mu\nabla^2 u + f_x = 0 \\
(\lambda + \mu)\frac{\partial I_{1\varepsilon}}{\partial y} + \mu\nabla^2 v + f_y = 0 \\
(\lambda + \mu)\frac{\partial I_{1\varepsilon}}{\partial z} + \mu\nabla^2 w + f_z = 0
\end{cases}
\tag{8}
$$

which express the equations of equilibrium in terms of displacements. In Eqs. (8), ∇^2 is the Laplace operator:

$$\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}, \tag{9}$$

and f_x, f_y, f_z are the body forces per unit volume ΔV, along the x, y, and z axes, respectively:

$$\begin{cases} f_x = \lim\limits_{\Delta V \to 0} \frac{\Delta F_x}{\Delta V} = \frac{dF_x}{dV} \\ f_y = \lim\limits_{\Delta V \to 0} \frac{\Delta F_y}{\Delta V} = \frac{dF_y}{dV} \\ f_z = \lim\limits_{\Delta V \to 0} \frac{\Delta F_z}{\Delta V} = \frac{dF_z}{dV} \end{cases} \tag{10}$$

The boundary conditions are:

$$\begin{cases} \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z = p_x \\ \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z = p_y \\ \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = p_z \end{cases} \tag{11}$$

where p_x, p_y, p_z are the surface forces per unit area ΔS:

$$\begin{cases} p_x = \lim\limits_{\Delta S \to 0} \frac{\Delta P_x}{\Delta S} = \frac{dP_x}{dS} \\ p_y = \lim\limits_{\Delta S \to 0} \frac{\Delta P_y}{\Delta S} = \frac{dP_y}{dS} \\ p_z = \lim\limits_{\Delta S \to 0} \frac{\Delta P_z}{\Delta S} = \frac{dP_z}{dS} \end{cases} \tag{12}$$

Finally, by substituting Eqs. (4) into Eqs. (11) and making use of Eqs. (5), we obtain the boundary conditions in terms of displacements:

$$\begin{cases} p_x = \left(\lambda I_{1\varepsilon} + 2\mu \frac{\partial u}{\partial x}\right) n_x + \mu \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) n_y + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) n_z\right] \\ p_y = \left(\lambda I_{1\varepsilon} + 2\mu \frac{\partial v}{\partial y}\right) n_y + \mu \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) n_x + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) n_z\right] \\ p_z = \left(\lambda I_{1\varepsilon} + 2\mu \frac{\partial w}{\partial z}\right) n_z + \mu \left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) n_x + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) n_y\right] \end{cases} \tag{13}$$

In 1885, Boussinesq used the theory of linear elasticity and the properties of potentials in order to obtain the closed form solution for a homogeneous, linear-elastic and isotropic half-space subjected to a point-load perpendicular to the surface [Boussinesq (1885)]. Assuming that the z direction coincides with the direction of gravity, Boussinesq's vertical stress under a concentrated load F is:

$$\sigma_z = \frac{3}{2} \frac{F}{\pi r^2} \cos^3 \vartheta, \tag{14}$$

where r is the distance between the application and the evaluation points, and ϑ is the angle between the point-load vector and the radial arm connecting the application to the evaluation point (Fig. 1).

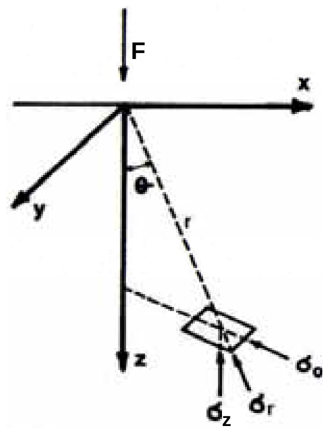

Figure 1: Parameter definition for Eq. (14)

Since $\cos\vartheta$ can be expressed as the ratio (Fig. 1):

$$\cos\vartheta = \frac{z}{r}, \tag{15}$$

it follows that:

$$r = \frac{z}{\cos\vartheta}. \tag{16}$$

By substituting Eq. (16) into Eq. (14), Eq. (14) may be rewritten as:

$$\sigma_z = \frac{3}{2}\frac{F}{\pi z^2}\cos^5\vartheta, \tag{17}$$

while by substituting Eq. (15) into Eq. (14) gives Boussinesq's equation in Cartesian coordinates:

$$\sigma_z = \frac{3}{2}\frac{Fz^3}{\pi r^5}. \tag{18}$$

Further forms of Eq. (14) in Cartesian coordinates are:

$$\sigma_z = \frac{3}{2}\frac{Fz^3}{\pi\left(R^2+z^2\right)^{\frac{5}{2}}}, \tag{19}$$

where R is the horizontal distance from the application to the evaluation point, and:

$$\sigma_z = I_B\frac{F}{z^2}, \tag{20}$$

where I_B, the so-called influence factor, takes the form:

$$I_B = \frac{3}{2\pi}\left[1+\left(\frac{R}{z}\right)^2\right]^{-\frac{5}{2}}. \tag{21}$$

It should be noted in Eq. (14), or, equivalently, in Eq. (17), (18), (19) and (20), that the vertical normal stress (σ_z) is independent of the elastic parameters E and υ. That is, Boussinesq's vertical normal stress spreads in the medium independently of the kind of medium itself.

Boussinesq also gives the radial stress σ_r for $\upsilon = 0.5$ (Poisson's ratio not far from reality for most soils):

$$\sigma_r = \sigma_1 = \frac{3}{2}\frac{F}{\pi r^2}\cos\vartheta, \tag{22}$$

where σ_1 is the first principal stress.

For $\upsilon = 0.5$, both the vertical and radial stress contours below a concentrated load take the form of a ball-shaped surface (Fig. 2).

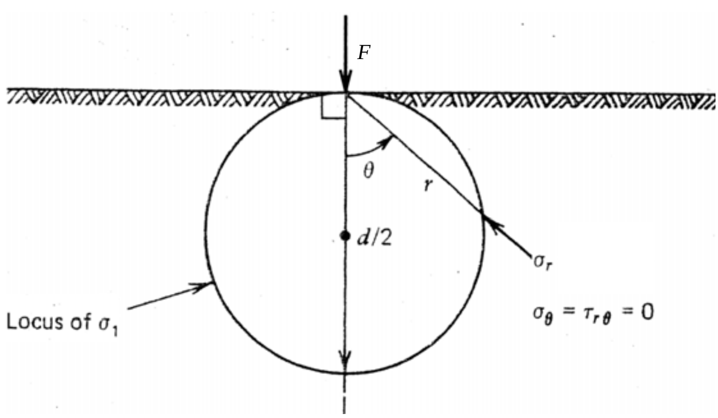

Figure 2: Stress distribution of Boussinesq for $\nu = 0.5$

Some of the most important fields of application for Boussinesq's solution are the design of airfield pavements [Ferretti and Bignozzi (2012); Ferretti (2012a)] and soil compaction modeling in agricultural soils. The US Army is concerned with the stress that a mine will experience as a MDT (Mine Detonation Trailer) rolls over a minefield area [Olmstead and Fischer (2009)]. Also of specific interest to the U.S. Air Force is to understand the difficulties encountered by its intra-theater airlift capabilities (e.g., C-130s and C-17s) when taking off and landing on unpaved and semi-prepared airfields.

As far as the soil compaction resulting from heavy tractor tires is concerned, it was found that agricultural soils distribute stresses differently from the ball-shaped surfaces shown in Fig. 2 [Ayers and Van Riper (1991)]: soil stresses are greater under the load axis and smaller further outside. In 1934, Eq. (14), Boussinesq's point-load equation, was modified by Fröhlich to incorporate concentration factors to account for agricultural soils:

$$\sigma_z = \frac{n}{2}\frac{F}{\pi r^2}\cos^n \vartheta, \tag{23}$$

where n is Fröhlich's stress concentration factor. By substituting Eq. (16) into Eq. (23), we can eliminate the radial distance r, so that the dependence on the depth z becomes explicit:

$$\sigma_z = \frac{n}{2}\frac{F}{\pi z^2}\cos^{(n+2)} \vartheta, \tag{24}$$

and by substituting Eq. (15) into Eq. (23), we obtain Fröhlich's equation in Cartesian coordinates:

$$\sigma_z = \frac{n}{2}\frac{F z^n}{\pi r^{n+2}}, \tag{25}$$

or:

$$\sigma_z = \frac{n}{2}\frac{F z^n}{\pi \left(R^2 + z^2\right)^{\frac{n+2}{2}}}. \tag{26}$$

With a concentration factor $n = 3$, Eqs. (23), (24), (25) and (26) provide Eqs. (14), (17), (18) and (19), respectively. Higher concentration factors increase the depth at which the stresses propagate so that the stress contours protrude deeper into the soil, modifying the ball-shaped surface of Fig. 2 into an ellipsoidal-shaped surface (Fig. 3).

The concentration factor, n, is a parameter that cannot be measured directly. Its evaluation is still an open question. In particular, it is not clear whether it depends upon the soil properties only [Söehne (1953), Söehne (1958), Binger and Wells (1989), Sharifat and Kushwaha (2000), Trautner (2003)] or also upon the applied load [Horn (1990)]. It is also unclear whether n is greater when the soil is harder [Trautner (2003)] or softer [Söehne (1953), Söehne (1958), Binger and

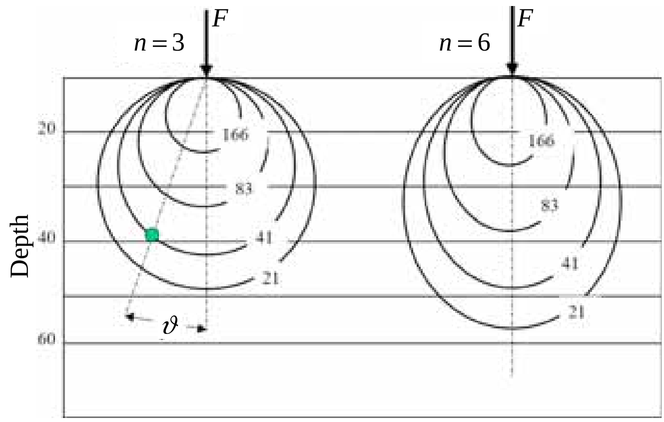

Figure 3: Fröhlich's curves of equal vertical normal stress in the soil

Wells (1989), Horn (1990), Sharifat and Kushwaha (2000)]. For a discussion on the values of n to be considered for hard, medium and soft soil, see Keller (2004) and Ferretti and Bignozzi (2012).

While it seems that experimental stress measurements compare well with predicted stresses at a depth of 20 cm [Ayers and Van Riper (1991)], it must be kept in mind that Fröhlich's equation empirically modifies that of Boussinesq, and is not an analytical solution to the elastic problem. As a consequence, the conditions of equilibrium and compatibility may not be satisfied. Veverka (1973) showed that, in order to satisfy these two conditions of equilibrium and compatibility, Young's modulus must vary with depth, if the concentration factor differs from 3, that is, if the equation of Fröhlich does not coincide with the equation of Boussinesq:

$$E = Cz^{\frac{n-1}{2}}, \tag{27}$$

where C is a constant.

Koolen and Kuipers (1983) also found that soil strength influences stress propagation. They justified the impossibility for Boussinesq's solution to account for soil strength on the fact that soil is not the homogeneous, isotropic, ideal elastic medium it should be to match the assumptions on which Boussinesq's solution is based.

Lastly, in Ferretti (2012a), it was found that a weak tensile state of strain appears in front of the wheel of an aircraft (section B-B in Fig. 4) that is taxiing over a concrete pavement. Since the concrete behavior is linear-elastic for operating loads, we can use Hooke's laws for deriving stresses from strains, finding that a weak tensile state of vertical stress appears in front of the wheel. Depending on the velocity of the aircraft, a second positive peak can appear behind the wheel. This second peak is always smaller than the peak in the front of the wheel.

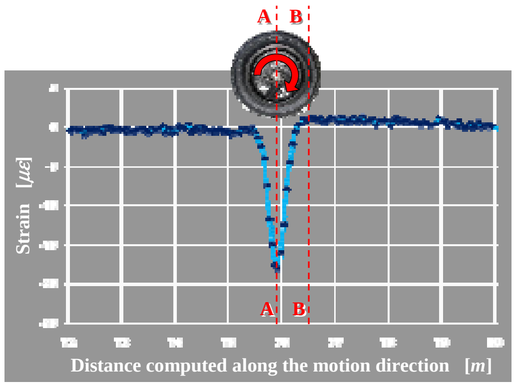

Figure 4: Vertical strains in function of distance when a wheel passes over a strain-gauge

The acquired positive strains do not seem to be caused by friction forces developing at the interface between the pavement and the wheel, since they appear even when the truck speed is very low, that is, in quasi-static conditions. In the latter case, the vertical strain profile of Fig. 4 becomes symmetric with respect to the wheel and exhibits two positive peaks of equal intensity, one before and one after the wheel.

Boussinesq's closed elastic solution for a homogeneous, linear-elastic and isotropic half-space subjected to a point-load perpendicular to the surface does not provide any tensile state of stress near the pavement surface. Nevertheless, several researchers [Spangler (1935); Hossain, Muqtadir and Hoque (1997); Darestani, Thambiratnam, Baweja and Nataatmadja (2006)] have assumed, in the past, that a tensile state of stress does arise in pavements, in order to explain some of the main mechanisms of pavement distress that cannot be justified based upon Boussinesq's theory. To the knowledge of the Author, the actual presence of a tensile state of stress was experimentally verified in Ferretti and Bignozzi (2012) for the first time.

2 The first order elastic solution

The original treatment of Boussinesq [Boussinesq (1885)] is briefly summarized and discussed in the present paragraph in order to clarify the manipulations leading to the higher order solution.

Let us assume the surface of the soil to be the x/y plane of a Cartesian coordinate system and the direction of gravity to be the z axis. With this position, the outgoing normal versor, n, is opposite to the z axis:

$$n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}. \tag{28}$$

By substituting Eq. (28) into Eq. (13), we obtain the boundary conditions in terms of displacements for the problem of Boussinesq:

$$\begin{cases} p_x = -\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ p_y = -\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ p_z = -\lambda I_{1\varepsilon} - 2\mu\frac{\partial w}{\partial z} \end{cases} \tag{29}$$

where positive and negative values of p_z indicate that p_z is concordant and discordant to the z axis, respectively.

In order to obtain the normal and shear stresses, for the depth z, on each horizontal plane (parallel to x/y) with the normal versor opposite to the z axis, the third among Eqs. (29) must be changed in sign:

$$\begin{cases} \tau_{xz} = -\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \tau_{yz} = -\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \sigma_z = \lambda I_{1\varepsilon} + 2\mu\frac{\partial w}{\partial z} \end{cases} \tag{30}$$

where positive values of σ_z indicate normal stresses of traction and negative values of σ_z indicate normal stresses of compression.

The assumption of linear-elastic behavior makes it possible to apply the superposition principle. This means that we can separately analyze and superpose the stress field induced in the soil by the point-load and the stress field induced in the soil by the weight of the soil itself. Concentrating on the first case, we can write Eqs. (8), the equilibrium equations, in the form:

$$\begin{cases} (\lambda+\mu)\frac{\partial I_{1\varepsilon}}{\partial x} + \mu\nabla^2 u = 0 \\ (\lambda+\mu)\frac{\partial I_{1\varepsilon}}{\partial y} + \mu\nabla^2 v = 0 \\ (\lambda+\mu)\frac{\partial I_{1\varepsilon}}{\partial z} + \mu\nabla^2 w = 0 \end{cases}, \tag{31}$$

where the body forces have been set equal to zero since gravity has been neglected:

$$f_x = f_y = f_z = 0. \tag{32}$$

2.1 First integral of the equilibrium problem

The first solution of Boussinesq is based on the similarity between the system of Eqs. (31) and the system:

$$\begin{cases} \frac{\partial}{\partial x}\left(2\frac{\partial P}{\partial z}\right) - \nabla^2 \frac{\partial (zP)}{\partial x} = 0 \\ \frac{\partial}{\partial y}\left(2\frac{\partial P}{\partial z}\right) - \nabla^2 \frac{\partial (zP)}{\partial y} = 0 \\ \frac{\partial}{\partial z}\left(2\frac{\partial P}{\partial z}\right) - \nabla^2 \frac{\partial (zP)}{\partial z} = 0 \end{cases}, \tag{33}$$

that is satisfied by any potential function P for which the Laplacian is equal to zero:

$$\nabla^2 P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0, \tag{34}$$

where P is referred to a fictive surface of finite dimensions, lying on the plane x/y, with the mass M of the fictive surface:

$$M = \int dm, \tag{35}$$

depending on the load conditions. For the purposes of the present paper, from here on, the fictive surface will be called load surface.

Since the points Q of the load surface lie on the plane x/y, their third coordinate is equal to zero. Denoting the further two coordinates of Q with x_1 and y_1, we have:

$$Q \equiv (x_1, y_1, 0). \tag{36}$$

Let $\rho(x_1, y_1)$ be the mass density for unit surface at the point Q. With this position, we can write Eq. (35) in the form:

$$\int dm = \int \rho(x_1, y_1)\, dx_1 dy_1, \tag{37}$$

where $\rho(x_1, y_1)$ is an arbitrary continuous function that is defined only inside the loaded area.

Boussinesq derived Eqs. (33) by performing the three partial derivatives of the function $\nabla^2 zP$, using Eq. (34) to simplify the result:

$$\nabla^2 zP = z\nabla^2 P + 2\frac{\partial P}{\partial z} = 2\frac{\partial P}{\partial z}. \tag{38}$$

By assuming:

$$u = -\frac{\partial}{\partial x}(zP), \tag{39}$$

$$v = -\frac{\partial}{\partial y}(zP), \tag{40}$$

$$w = -\frac{\partial}{\partial z}(zP) + KP, \tag{41}$$

$$\frac{\lambda+\mu}{\mu} I_{1\varepsilon} = 2\frac{\partial P}{\partial z}, \tag{42}$$

where K is a constant coefficient, and making use of Eq. (34), the three Eqs. (33) give the three Eqs. (31) divided by μ.

The expression for P chosen by Boussinesq is the logarithmic potential Ψ of the loaded surface for the prefixed point (x,y,z) of the semi-space under the surface, at the distance r from Q:

$$P = \Psi = \int \log(z+r)\,dm, \tag{43}$$

with:

$$r = \sqrt{(x-x_1)^2 + (y-y_1)^2 + z^2}, \tag{44}$$

$$\nabla^2\Psi = 0. \tag{45}$$

In Eqs. (43) and (44), x_1 and y_1 are variable within the boundaries of the loaded surface, while x, y and z are prefixed. With the position in Eq. (43), from Eqs. (39) – (42) we can find:

$$u = -\frac{\partial}{\partial x}(z\Psi), \tag{46}$$

$$v = -\frac{\partial}{\partial y}(z\Psi), \tag{47}$$

$$w = -\frac{\partial}{\partial z}(z\Psi) + K\Psi, \tag{48}$$

$$\frac{\lambda+\mu}{\mu} I_{1\varepsilon} = 2\frac{\partial \Psi}{\partial z}, \tag{49}$$

By differentiating and substituting Eqs. (46), (47) and (48) into Eq. (6), with $I_{1\varepsilon}$ obtained from Eq. (49), we have the value of K:

$$K = 2\frac{\lambda+2\mu}{\lambda+\mu}. \tag{50}$$

Finally, by substituting Eqs. (46), (47) and (48) into Eq. (30), with the value of K provided by Eq. (50), we find the three components of stress acting on a plane element at the depth z:

$$\begin{cases} \tau_{xz} = 2\mu \frac{\partial}{\partial x}\left(z\frac{\partial \Psi}{\partial z} - \frac{\mu}{\lambda+\mu}\Psi\right) \\ \tau_{yz} = 2\mu \frac{\partial}{\partial y}\left(z\frac{\partial \Psi}{\partial z} - \frac{\mu}{\lambda+\mu}\Psi\right) \\ \sigma_z = -2\mu\left(z\frac{\partial^2 \Psi}{\partial z^2} - \frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial \Psi}{\partial z}\right) \end{cases} \tag{51}$$

Eqs. (46), (47) and (48) satisfy the equations of equilibrium in terms of displacements but do not provide the right displacement field, since they do not vanish at infinity as fast as the function $1/r$ (see Boussinesq (1885) for details). In order to match this further requirement, Boussinesq used the first derivative $\partial\Psi/\partial z$ of Eq. (43) instead of the logarithmic potential Ψ for the potential P in Eqs. (33), which is allowable since $\partial\Psi/\partial z$ still gives a Laplacian equal to zero (as do all the derivatives of Ψ):

$$P = \frac{\partial \Psi}{\partial z} = \int \frac{dm}{r}, \tag{52}$$

with:

$$\nabla^2 \frac{\partial \Psi}{\partial z} = 0. \tag{53}$$

Note that the position in Eq. (52) is equivalent to substituting the logarithmic potential Ψ by the inverse potential U, defined by Lamé just as:

$$U = \int \frac{dm}{r}. \tag{54}$$

From Eqs. (39) – (42) and the position in Eq. (52), it follows that:

$$\begin{cases} u = -\frac{\partial^2}{\partial x \partial z}\int r dm = -z\frac{\partial}{\partial x}\int \frac{dm}{r} \\ v = -\frac{\partial^2}{\partial y \partial z}\int r dm = -z\frac{\partial}{\partial y}\int \frac{dm}{r} \\ w = -\frac{\partial^2}{\partial z^2}\int r dm + 2\frac{\lambda+2\mu}{\lambda+\mu}\int \frac{dm}{r} = \frac{\lambda+3\mu}{\lambda+\mu}\int \frac{dm}{r} + z\int \frac{z dm}{r^3} \end{cases} \tag{55}$$

$$I_{1\varepsilon} = \frac{2\mu}{\lambda+\mu}\frac{\partial}{\partial z}\int \frac{dm}{r} = -\frac{2\mu}{\lambda+\mu}\int \frac{z dm}{r^3}, \tag{56}$$

where the following identities have been used:

$$z\frac{\partial \Psi}{\partial z} = \int \frac{z}{r} dm = \frac{\partial}{\partial z}\int r dm. \tag{57}$$

Eqs. (55) can be put in the condensed form:

$$\begin{cases} (u,v) = -z\frac{\partial}{\partial(x,y)}\frac{\partial\Psi}{\partial z} \\ w = -z\frac{\partial^2\Psi}{\partial z^2} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial\Psi}{\partial z} \end{cases} \tag{58}$$

The stress components follow from Eqs. (51) by substituting Ψ with $\partial\Psi/\partial z$ and making use of the second equality in Eq. (52):

$$\begin{cases} \tau_{xz} = -2\mu\frac{\partial}{\partial x}\left(\frac{\mu}{\lambda+\mu}\int\frac{dm}{r} + z\int\frac{zdm}{r^3}\right) \\ \tau_{yz} = -2\mu\frac{\partial}{\partial y}\left(\frac{\mu}{\lambda+\mu}\int\frac{dm}{r} + z\int\frac{zdm}{r^3}\right) \\ \sigma_z = -2\mu\int\left(\frac{\mu}{\lambda+\mu}\frac{z}{r^3} + 3\frac{z^3}{r^5}\right)dm \end{cases} \tag{59}$$

For the points of the surface, Boussinesq provides the results:

$$\begin{cases} u = 0 \\ v = 0 \\ w = \frac{\lambda+3\mu}{\lambda+\mu}\int\frac{dm}{r} \end{cases} \tag{60}$$

$$I_{1\varepsilon} = -4\pi\frac{\mu}{\lambda+\mu}\rho\left(x,y\right), \tag{61}$$

$$\begin{cases} p_x = \lim\limits_{z\to 0} -2\frac{\mu^2}{\lambda+\mu}\frac{\partial}{\partial x}\int\frac{dm}{r} \\ p_y = \lim\limits_{z\to 0} -2\frac{\mu^2}{\lambda+\mu}\frac{\partial}{\partial y}\int\frac{dm}{r} \\ p_z = \lim\limits_{z\to 0} -2\mu\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial}{\partial z}\int\frac{dm}{r} = 4\pi\mu\frac{\lambda+2\mu}{\lambda+\mu}\rho\left(x,y\right) \end{cases} \tag{62}$$

obtained by the conversion from rectangular coordinates (x_1,y_1,z) to polar coordinates (R,ω,z) in Fig. 5:

$$x_1 = x + R\cos\omega, \tag{63}$$

$$y_1 = y + R\sin\omega. \tag{64}$$

Eqs. (60) – (62) have been obtained from Eqs. (55), (56) and (59) by performing the limit process for $z\to 0$. In the opinion of the Author, the way this limit process has been performed is questionable, since the finite dimensions of the load surface are not take into account correctly. In order to clarify this statement, the solution given by Boussinesq will now be discussed: the expressions in Eqs. (60) – (62) come from the preventive evaluation in polar coordinates (Fig. 5) of the three limits:

$$\lim_{z\to 0}\int\frac{dm}{r}, \tag{65}$$

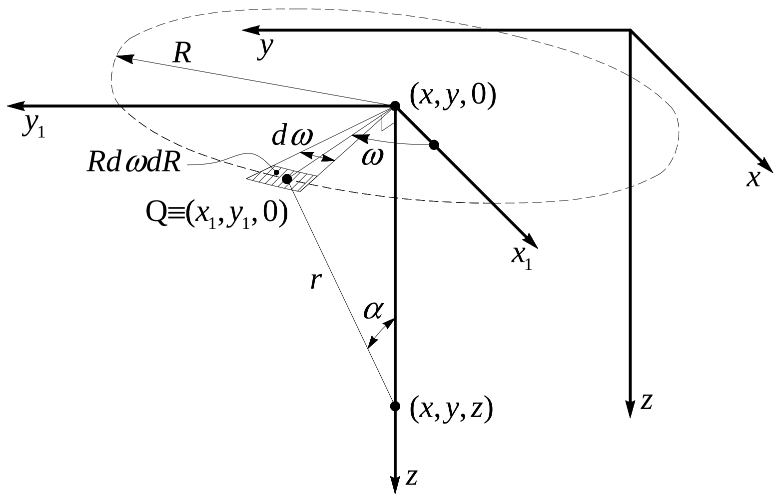

Figure 5: Relationship between rectangular and polar coordinates

$$\lim_{z\to 0}\int \frac{zdm}{r^3}, \tag{66}$$

$$\lim_{z\to 0}\int \frac{z^3dm}{r^5}, \tag{67}$$

where $dm = \rho\,(x_1, y_1)\,dx_1dy_1$ takes the form:

$$dm = \rho\,(x + R\cos\omega, y + R\sin\omega)\,Rd\omega dR, \tag{68}$$

and r, the distance between Q and the point (x,y,z), is equal to:

$$r = \sqrt{z^2 + R^2}. \tag{69}$$

The first limit provides:

$$\lim_{z\to 0}\int \frac{dm}{r} == \lim_{z\to 0}\int_0^{2\pi} d\omega \int_0^{\infty} \frac{\rho\,(x + R\cos\omega, y + R\sin\omega)\,R}{\sqrt{z^2 + R^2}}dR. \tag{70}$$

As far as Eq. (70) is concerned, it may be argued that the limits of the integration variable R should be 0 and the actual dimension of the load surface, and not 0 and ∞. This means that the form of the load surface has to be specified and the distance from the boundaries of the load surface has to be related to the anomaly ω. A mass density ρ defined on the whole x/y plane may be used if we admit that ρ could be set equal to zero outside the load surface. Indeed, the form of the load surface must be specified even in this case, and ρ may no longer be a continuous function. Consequently, the equivalence expressed by Eq. (70), provided by Boussinesq, is only valid for the case of a load surface that has infinite dimensions along both the x and y axes, contradicting Boussinesq's assumption itself. Note also that, for an indefinite load surface, the limit in Eq. (70) could not provide a finite value. However, this does not have any relevant implication on the result since the double integral in Eq. (70) is not solved by Boussinesq and its value is not used in the following manipulations.

Similar observations may be carried out for the other two limits, which Boussinesq puts in the form:

$$\lim_{z\to 0}\int \frac{zdm}{r^3} = \lim_{z\to 0}\int_0^{2\pi} d\omega \int_0^{\infty} \frac{\rho\left(x+R\cos\omega, y+R\sin\omega\right) zR}{\sqrt{z^2+R^2}^3} dR, \tag{71}$$

$$\lim_{z\to 0}\int \frac{z^3dm}{r^5} = \lim_{z\to 0}\int_0^{2\pi} d\omega \int_0^{\infty} \frac{\rho\left(x+R\cos\omega, y+R\sin\omega\right) z^3R}{\sqrt{z^2+R^2}^5} dR. \tag{72}$$

This second time, the solutions of the double integrals are found by performing a change of variable from R to $q=\tan\alpha$ (Fig. 5):

$$R = zq. \tag{73}$$

In polar coordinates (q,ω,z), we have:

$$x_1 = x + zq\cos\omega, \tag{74}$$

$$y_1 = y + zq\sin\omega, \tag{75}$$

$$dR = zdq, \tag{76}$$

$$\lim_{z\to 0}\int \frac{zdm}{r^3} = \lim_{z\to 0}\int_0^{2\pi} d\omega \int_0^{\infty} \frac{\rho\left(x+zq\cos\omega, y+zq\sin\omega\right) q}{\sqrt{1+q^2}^3} dq, \tag{77}$$

$$\lim_{z\to 0}\int \frac{z^3dm}{r^5} = \lim_{z\to 0}\int_0^{2\pi} d\omega \int_0^{\infty} \frac{\rho\left(x+zq\cos\omega, y+zq\sin\omega\right) q}{\sqrt{1+q^2}^5} dq. \tag{78}$$

Then, Boussinesq computes the two integrals:

$$\int_0^{\infty} \frac{1}{\sqrt{1+q^2}^3} dq = -\left[\frac{1}{\sqrt{1+q^2}}\right]_0^{\infty} = 1, \tag{79}$$

$$\int_0^\infty \frac{1}{\sqrt{1+q^2}^5}dq = -\frac{1}{3}\left[\frac{1}{\sqrt{1+q^2}^3}\right]_0^\infty = \frac{1}{3}, \tag{80}$$

and, assuming that ρ depends only on x and y for each assigned point (x,y,z):

$$\rho\left(x+R\cos\omega, y+R\sin\omega\right) = \rho\left(x,y\right), \tag{81}$$

which is equal to assuming that ρ is equal to its average value at the point $(x,y,0)$, the normal projection of (x,y,z) on the plane x/y:

$$\rho\left(x+R\cos\omega, y+R\sin\omega\right) = \bar{\rho}\left(x,y,0\right), \tag{82}$$

puts ρ out of the integrals in Eqs. (77) and (78), finding:

$$\lim_{z\to 0}\int\frac{zdm}{r^3} = 2\pi\rho\left(x,y\right), \tag{83}$$

$$\lim_{z\to 0}\int\frac{z^3dm}{r^5} = \frac{2}{3}\pi\rho\left(x,y\right). \tag{84}$$

Eqs. (60) – (62) follow from substituting Eqs. (83) and (84) into Eqs. (55), (56) and (59).

This second time, we may argue that the limits for q are correct in Eqs. (77) and (78), since:

$$\lim_{z\to 0}\alpha|_{x_1\neq x\wedge y_1\neq y} = \frac{\pi}{2}, \tag{85}$$

$$\lim_{z\to 0}\tan\alpha|_{x_1\neq x\wedge y_1\neq y} \to +\infty, \tag{86}$$

while not the limits for ω. Actually, since the load surface has finite dimensions, it is always possible to find a circumference of centre $(x,y,0)$ which intersects the load surface. The arches of circumferences which are outside the load surface are lucus of points $(x_1,y_1,0)$ that do not belong to the load surface. These points must not be considered in the calculus.

In order to avoid any intersection and taking into account only the arches of circumferences lying inside the load surface, the lower and upper bounds of ω must be modified in function of the values assumed by x and y. Once again, the solution provided by Boussinesq is valid for infinite load surfaces only. The question is not irrelevant, since the value provided by the third of Eqs. (62) is used to build the coefficients in the linear combination giving the point-load solution (§2.4).

Likewise, it seems unnecessary to perform integrals on the whole load surface, since the aim of the treatment is to find the solution for a single point-load, and not for a distributed load. In effect, after obtaining the general solution, Boussinesq gives the point-load solution by substituting the integrals with their integrands, that is, by causing the dimensions of the load surface in the x/y plane to vanish. It therefore seems possible, besides being simpler, to build the point-load solution directly, by defining the potentials for the infinitesimal superficial neighborhood of the point $(x_1, y_1, 0)$, rather than for a finite load surface to vanish. In other words, the load surface may be allowed to coincide with the infinitesimal neighborhood of mass dm of the point $(x_1, y_1, 0)$. With this in mind, here we propose to define the logarithmic potential of the infinitesimal neighborhood of mass dm as:

$$\psi = \log(z+r)\,dm, \tag{87}$$

which satisfies the condition:

$$\nabla^2\psi = 0, \tag{88}$$

and assume:

$$P = \frac{\partial\psi}{\partial z} = \frac{1}{r}dm, \tag{89}$$

in Eqs. (33).

The solution following by the position in Eq. (89) is:

$$\begin{cases} u = -\frac{\partial^2 r}{\partial x \partial z}dm = -z\frac{\partial}{\partial x}\left(\frac{1}{r}\right)dm = \frac{z(x-x_1)}{r^3}dm \\ v = -\frac{\partial^2 r}{\partial y \partial z}dm = -z\frac{\partial}{\partial y}\left(\frac{1}{r}\right)dm = \frac{z(y-y_1)}{r^3}dm \\ w = -\frac{\partial^2 r}{\partial z^2}dm + 2\frac{\lambda+2\mu}{\lambda+\mu}\frac{dm}{r} = \frac{\lambda+3\mu}{\lambda+\mu}\frac{dm}{r} + \frac{z^2 dm}{r^3} = \frac{r^2(\lambda+3\mu)+z^2(\lambda+\mu)}{r^3(\lambda+\mu)}dm \end{cases} \tag{90}$$

$$I_{1\varepsilon} = \frac{2\mu}{\lambda+\mu}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)dm = -\frac{2\mu}{\lambda+\mu}\frac{z}{r^3}dm, \tag{91}$$

$$\begin{cases} \tau_{xz} = -2\mu\frac{\partial}{\partial x}\left(\frac{\mu}{\lambda+\mu}\frac{1}{r} + \frac{z^2}{r^3}\right)dm = 2\mu\,(x-x_1)\,\frac{\mu r^2+3(\lambda+\mu)z^2}{(\lambda+\mu)r^5}dm \\ \tau_{yz} = -2\mu\frac{\partial}{\partial y}\left(\frac{\mu}{\lambda+\mu}\frac{1}{r} + \frac{z^2}{r^3}\right)dm = 2\mu\,(y-y_1)\,\frac{\mu r^2+3(\lambda+\mu)z^2}{(\lambda+\mu)r^5}dm \\ \sigma_z = -2\mu z\left(\frac{\mu}{\lambda+\mu}\frac{1}{r^3} + 3\frac{z^2}{r^5}\right)dm = -2\mu z\frac{\mu r^2+3(\lambda+\mu)z^2}{(\lambda+\mu)r^5}dm \end{cases} \tag{92}$$

For $z \to 0$, we find:

$$\begin{cases} u = 0 \\ v = 0 \\ w = \frac{\lambda+3\mu}{r(\lambda+\mu)}dm \end{cases} \tag{93}$$

$$I_{1\varepsilon} = \begin{cases} 0 & \text{for } r > 0 \\ -\frac{2\mu}{\lambda+\mu}\rho\left(x_1, y_1\right) & \text{for } r \to 0 \end{cases} \tag{94}$$

$$\begin{cases} p_x = \frac{2\mu^2}{\lambda+\mu}\frac{x-x_1}{r^3}dm \\ p_y = \frac{2\mu^2}{\lambda+\mu}\frac{y-y_1}{r^3}dm \\ p_z = \lim\limits_{z\to 0} -2\mu\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial^2\psi}{\partial z^2} = \begin{cases} 0, & r > 0 \\ 2\mu\frac{\lambda+2\mu}{\lambda+\mu}\rho\left(x_1, y_1\right), & r \to 0 \end{cases} \end{cases} \tag{95}$$

Note how Eqs. (93) and the first two of Eqs. (95) may be obtained directly from the integrands in Eqs. (60) and the first two Eqs. (62), respectively, by means of the position in Eq. (89).

Each time the normal component of the external load is assigned in function of x_1 and y_1, from the third of Eqs. (62) we find:

$$\rho\left(x, y\right) = \frac{\lambda+\mu}{4\pi\mu\left(\lambda+2\mu\right)}p_z, \tag{96}$$

while, when the logarithmic potential given in Eq. (87) is used instead of Ψ, the relationship between ρ and the external pressure p_z is found for $r \to 0$ from the third of Eqs. (95):

$$\rho\left(x_1, y_1\right) = \frac{\lambda+\mu}{2\mu\left(\lambda+2\mu\right)}p_z. \tag{97}$$

Eq. (96) or Eq. (97), together with the first two of Eqs. (60), tell us that we have obtained the solution for the case in which the boundary conditions consist in giving the normal component of the external load and assuming the horizontal displacements on the surface to be equal to zero.

2.2 Second integral of the equilibrium problem

The second solution of the equilibrium problem follows from the position:

$$\begin{cases} u = \frac{\partial P}{\partial x} \\ v = \frac{\partial P}{\partial y} \\ w = \frac{\partial P}{\partial z} \end{cases} \tag{98}$$

Due to Eq. (34), in this second case, the bulk strain is equal to zero:

$$I_{1\varepsilon} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla^2 P = 0. \tag{99}$$

and the equilibrium equations expressed by Eqs. (31) are identically satisfied.

Substituting Eqs. (98) into Eqs. (30), we derive the condensed form of the stresses for the general case:

$$\begin{cases} (\tau_{xz}, \tau_{yz}) = -2\mu \frac{\partial}{\partial(x,y)} \frac{\partial P}{\partial z} \\ \sigma_z = 2\mu \frac{\partial^2 P}{\partial z^2} \end{cases} \tag{100}$$

and for the special case in which P is the logarithmic potential Ψ (Eq. (43)):

$$\begin{cases} (\tau_{xz}, \tau_{yz}) = -2\mu \frac{\partial}{\partial(x,y)} \int \frac{dm}{r} \\ \sigma_z = 2\mu \frac{\partial}{\partial z} \int \frac{dm}{r} \end{cases} \tag{101}$$

For the points of the surface, by performing the limit process for $z \to 0$ of Eqs. (101), we find the following boundary conditions:

$$\begin{cases} p_x = \lim\limits_{z \to 0} -2\mu \frac{\partial}{\partial x} \int \frac{dm}{r} \\ p_y = \lim\limits_{z \to 0} -2\mu \frac{\partial}{\partial y} \int \frac{dm}{r} \\ p_z = 4\pi\mu\rho\,(x,y) \end{cases} \tag{102}$$

where Eq. (83) has been used to perform the third limit. As previously discussed (§2.1), even here the integration limits in the limit process do not seem to be adequate. Moreover, with the aim of finding a point-load solution, it seems reasonable to use the logarithmic potential given in Eq. (87) instead of Ψ.

With the position:

$$P = \psi = \log(z+r)\,dm, \tag{103}$$

the following may be found:

$$\begin{cases} u = \frac{\partial \psi}{\partial x} = \frac{x-x_1}{r(z+r)} dm \\ v = \frac{\partial \psi}{\partial y} = \frac{y-y_1}{r(z+r)} dm \\ w = \frac{\partial \psi}{\partial z} = \frac{1}{r} dm \end{cases} \tag{104}$$

$$I_{1\varepsilon} = \nabla^2 \psi = 0, \tag{105}$$

$$\begin{cases} (\tau_{xz}, \tau_{yz}) = -2\mu \frac{\partial}{\partial(x,y)} \left(\frac{1}{r}\right) dm \\ \sigma_z = 2\mu \frac{\partial}{\partial z} \left(\frac{1}{r}\right) dm = -2\mu \frac{z}{r^3} dm \end{cases} \tag{106}$$

and, for the points of the surface:

$$\begin{cases} u = \frac{\partial \psi}{\partial x} = \frac{x-x_1}{r^2} dm \\ v = \frac{\partial \psi}{\partial y} = \frac{y-y_1}{r^2} dm \\ w = \frac{\partial \psi}{\partial z} = \frac{1}{r} dm \end{cases} \tag{107}$$

$$I_{1\varepsilon} = 0, \tag{108}$$

$$\begin{cases} p_x = 2\mu \frac{x-x_1}{r^3} dm \\ p_y = 2\mu \frac{y-y_1}{r^3} dm \\ p_z = \lim_{z\to 0} -2\mu \frac{\partial^2 \psi}{\partial z^2} = \begin{cases} 0 & \text{for } r > 0 \\ 2\mu\rho\,(x_1, y_1) & \text{for } r \to 0 \end{cases} \end{cases} \tag{109}$$

From the third of Eqs. (102), we find the value to give to ρ for Boussinesq:

$$\rho\,(x,y) = \frac{1}{4\pi\mu} p_z. \tag{110}$$

The value of $\rho\,(x_1, y_1)$ proposed here for $r \to 0$ can be taken from the last of Eqs. (109):

$$\rho\,(x_1, y_1) = \frac{1}{2\mu} p_z. \tag{111}$$

Eq. (110) or Eq. (111), together with Eqs. (102), tell us that we have obtained the solution for the case in which the normal component of the external load is given and the shear components of the external load depend upon the values assumed by the normal component in the points of the surface. The displacements satisfy the integration condition:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{112}$$

A similar integration condition is satisfied by the shear stresses on the planes parallel to x/y:

$$\frac{\partial p_x}{\partial y} = \frac{\partial p_y}{\partial x}. \tag{113}$$

2.3 Third integral of the equilibrium problem

The third solution of the equilibrium problem follows from the position:

$$\begin{cases} u = -\frac{\partial P}{\partial y} \\ v = \frac{\partial P}{\partial x} \\ w = 0 \end{cases} \tag{114}$$

that is, from the assumption of plane strain.

As for the second solution, in this case also, the bulk strain is equal to zero:

$$I_{1\varepsilon} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{115}$$

and the equilibrium equations are identically satisfied, due to Eq. (34). Since the bulk strain is equal to zero (Eq. (115)), the condition of plane strain implies plane stress in each point of the body.

Substituting Eqs. (114) into Eqs. (30), we obtain the stresses:

$$\begin{cases} \tau_{xz} = \mu \frac{\partial^2 P}{\partial y \partial z} \\ \tau_{yz} = -\mu \frac{\partial^2 P}{\partial x \partial z} \\ \sigma_z = 0 \end{cases} \tag{116}$$

with the first two stress components of Eqs. (116) satisfying the relationship:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0. \tag{117}$$

With the position in Eq. (103), the third solution of Boussinesq, where $P = \Psi$, is substituted by:

$$\begin{cases} \tau_{xz} = \mu \frac{\partial^2 \psi}{\partial y \partial z} \\ \tau_{yz} = -\mu \frac{\partial^2 \psi}{\partial x \partial z} \\ \sigma_z = 0 \end{cases} \tag{118}$$

which follow from the assumption:

$$\begin{cases} u = -\frac{\partial \psi}{\partial y} = -\frac{y-y_1}{r(z+r)} dm \\ v = \frac{\partial \psi}{\partial x} = \frac{x-x_1}{r^2} dm \\ w = 0 \end{cases} \tag{119}$$

In this third case, we have found the solution for the case in which the normal component of the external load is set equal to zero and the two shear components stand in the relationship represented by Eq. (117). The displacements take place horizontally. Moreover, since the third of Eqs. (114) and Eq. (115) provide:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{120}$$

the specific variation of area over the planes parallel to x/y (plane problem) is equal to zero:

$$\frac{\Delta S}{S}=0. \tag{121}$$

2.4 Elastic solution for a point-load perpendicular to the surface

Due to the superposition principle, it is always possible to find further solutions to the equilibrium problem by combining the former three solutions with each other. Along these lines, in order to obtain simplified expressions of the stresses acting on the planes parallel to x/y, Boussinesq formed two linear combinations of Eqs. (58) and (98), in which $P=\Psi$. The linear combination giving the solution of the point-load perpendicular to the surface is obtained by multiplying Eqs. (58) by the inverse of $4\pi\mu$ and Eqs. (98), with $P=\Psi$, by the inverse of $-4\pi(\lambda+\mu)$. Due to the superposition principle, the same multiplying factors can be taken to build a second linear combination of Eqs. (59) and (101), providing the solution in terms of stresses. The solution found by Boussinesq is:

$$(\tau_{xz},\tau_{yz})=\frac{3}{2}\frac{z^2}{\pi}\int\frac{1}{r^4}\frac{\partial r}{\partial(x,y)}dm, \tag{122}$$

$$\sigma_z=-\frac{3}{2}\frac{z^2}{\pi}\int\frac{1}{r^4}\frac{\partial r}{\partial z}dm, \tag{123}$$

for a point inside the soil, and

$$\begin{cases} p_x=0\\ p_y=0\\ p_z=\lim\limits_{z\to 0}-\frac{1}{2\pi}\frac{\partial^2\Psi}{\partial z^2}=\rho(x,y)\end{cases} \tag{124}$$

for a point of the surface.

The multiplying factors of the linear combination have been chosen specifically to find the equality between p_z and $\rho(x,y)$ shown by the third of Eqs. (124), since it allows us to put dm and the external load in the simple relationship given in Eq. (127). In effect, if dF is the external load applied to the infinitesimal element dx_1dy_1 of the surface, given by:

$$dF = p_z dx_1 dy_1, \tag{125}$$

from the equality between p_z and $\rho(x,y)$, it follows that:

$$dF = \rho(x,y)\, dx_1 dy_1. \tag{126}$$

It must be recalled that $\rho(x,y)$ is defined at the point $(x,y,0)$, which is the projection of (x,y,z) on the plane x/y, and not at the point $(x_1,y_1,0)$. Thus, in all those cases where the external load in not uniformly distributed on the load surface, it is not admissible to confuse the function $\rho(x,y)$ with the function $\rho(x_1,y_1)$. Consequently, the equality imposed by Boussinesq at this point:

$$dF = \rho(x_1,y_1)\, dx_1 dy_1 = dm, \tag{127}$$

has not a general validity. On the basis of the former discussion, we can also argue that the equality between dF and dm in Eq. (127) cannot be established in all cases, since the multiplying factors of the linear combination derive from values (provided by the third of Eqs. (62) and the third of Eqs. (102)) obtained by performing integrals between limits that do not seem to be properly chosen. This puts in discussion the substitution of dm by dF into Eqs. (122), which, in the case of infinitesimal load surface, give the well-known solutions of Boussinesq for a point-load perpendicular to the surface:

$$\begin{cases} \tau_{xz} = \frac{3}{2\pi}\frac{z^2}{r^5}(x-x_1)\, dF \\ \tau_{yz} = \frac{3}{2\pi}\frac{z^2}{r^5}(y-y_1)\, dF \\ \sigma_z = -\frac{3}{2\pi}\frac{z^3}{r^5}dF \end{cases} \tag{128}$$

where the integrals of Eqs. (122) have been substituted by their integrands due to the infinitesimal dimensions of the load surface. For a finite load F, the third of Eqs. (128) gives Eq. (18). In conclusion, it seems more correct to derive the elastic solution for a point-load perpendicular to the surface as linear combination of Eqs. (90) and (104), for displacements, and of Eqs. (92) and (106), for stresses. Even here, the multiplying factors of the linear combination will be chosen in such a way that it is possible to establish equality between p_z and ρ, which, in this case, is given as $\rho(x_1,y_1)$. To this purpose, Eqs. (90) and (92) will be multiplied by the inverse of 2μ and Eqs. (104) and (106) will be multiplied by the inverse of $-2(\lambda+\mu)$. The result is:

$$\begin{cases} u = \frac{x-x_1}{2\mu}\left[\frac{z}{r^3} - \frac{\mu}{\lambda+\mu}\frac{1}{r(z+r)}\right]dF \\ v = \frac{y-y_1}{2\mu}\left[\frac{z}{r^3} - \frac{\mu}{\lambda+\mu}\frac{1}{r(z+r)}\right]dF \\ w = \frac{1}{2\mu r}\left[\frac{\lambda+2\mu}{\lambda+\mu} + \frac{z^2}{r^2}\right]dF \end{cases} \tag{129}$$

$$I_{1\varepsilon} = \frac{1}{\lambda+\mu}\frac{d}{dz}\left(\frac{1}{r}\right)dm = -\frac{1}{\lambda+\mu}\frac{z}{r^3}dF, \tag{130}$$

$$\begin{cases} \tau_{xz} = 3\frac{z^2}{r^5}(x-x_1)\,dF \\ \tau_{yz} = 3\frac{z^2}{r^5}(y-y_1)\,dF \\ \sigma_z = -3\frac{z^3}{r^5}dF \end{cases} \tag{131}$$

for a point inside the soil, and

$$\begin{cases} u = -\frac{1}{2(\lambda+\mu)}\frac{x-x_1}{r^2}dP \\ v = -\frac{1}{2(\lambda+\mu)}\frac{y-y_1}{r^2}dP \\ w = \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\frac{1}{r}dP \end{cases} \tag{132}$$

$$I_{1\varepsilon} = \begin{cases} 0 & \text{for } r > 0 \\ -\frac{1}{\lambda+\mu}\rho(x_1,y_1) & \text{for } r \to 0 \end{cases} \tag{133}$$

$$\begin{cases} p_x = 0 \\ p_y = 0 \\ p_z = \begin{cases} 0 & \text{for } r > 0 \\ \lim\limits_{z\to 0} -\frac{\partial^2\psi}{\partial z^2} = \rho(x_1,y_1) & \text{for } r \to 0 \end{cases} \end{cases} \tag{134}$$

for the points of the surface.

As for the solution given by Boussinesq, also in Eqs. (131) the stresses are independent of the elastic coefficients of the medium in which they are calculated.

It is worth noting how the second of Eqs. (133) is the same relationship of direct proportionality between the bulk strain and the load for unit area established by Lamé for the points of the surface. This improves the relationship found by Boussinesq, which is established between $I_{1\varepsilon}$ and $\rho(x,y)$, instead of between $I_{1\varepsilon}$ and $\rho(x_1,y_1)$:

$$I_{1\varepsilon} = -\frac{1}{\lambda+\mu}\rho(x,y). \tag{135}$$

We have already discussed the opportunity of not confusing $\rho(x,y)$ with $\rho(x_1,y_1)$.

3 The higher order elastic solution

Following the spirit of the superposition principle and noting that the partial derivatives of any arbitrary order of the function ψ, defined in Eq. (82), have a zero Laplacian (i.e. they satisfy the condition $\nabla^2 = 0$), it is possible to refine the elastic solution of Boussinesq by adding to it a further solution of Eqs. (33), obtained by substituting ψ with one of its derivatives of the second order. This observation will be used here in order to find a further form of the first integral, which, combined to the former form and the second integral, could provide a stress solution to the vertical point-load problem that also depends on the elastic constants of the soil.

3.1 A second order solution of the first integral

Assuming:

$$P = \frac{\partial^2 \psi}{\partial z^2} = -\frac{z}{r^3} dm, \tag{136}$$

we find:

$$\begin{cases} u = -\frac{\partial}{\partial x}\left(z\frac{\partial^2\psi}{\partial z^2}\right) = -z\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial z^2}\right) = -3\frac{z^2(x-x_1)}{r^5}dm \\ v = -\frac{\partial}{\partial y}\left(z\frac{\partial^2\psi}{\partial z^2}\right) = -z\frac{\partial}{\partial y}\left(\frac{\partial^2\psi}{\partial z^2}\right) = -3\frac{z^2(y-y_1)}{r^5}dm \\ w = -\frac{\partial}{\partial z}\left(z\frac{\partial^2\psi}{\partial z^2}\right) + 2\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial^2\psi}{\partial z^2} = -\frac{z}{r^3}\left(3\frac{z^2}{r^2} + \frac{2\mu}{\lambda+\mu}\right)dm \end{cases} \tag{137}$$

$$I_{1\varepsilon} = \frac{2\mu}{\lambda+\mu}\frac{\partial}{\partial z}\left(\frac{\partial^2\psi}{\partial z^2}\right) = \frac{2\mu}{\lambda+\mu}\left(3\frac{z^2}{r^5} - \frac{1}{r^3}\right)dm, \tag{138}$$

$$\begin{cases} \tau_{xz} = 2\mu\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial z}\frac{\partial^2\psi}{\partial z^2} - \frac{\mu}{\lambda+\mu}\frac{\partial^2\psi}{\partial z^2}\right)dm = \frac{6\mu}{\lambda+\mu}\frac{(x-x_1)z}{r^5}\left[\lambda - 5(\lambda+\mu)\frac{z^2}{r^2}\right]dm \\ \tau_{yz} = 2\mu\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial z}\frac{\partial^2\psi}{\partial z^2} - \frac{\mu}{\lambda+\mu}\frac{\partial^2\psi}{\partial z^2}\right)dm = \frac{6\mu}{\lambda+\mu}\frac{(y-y_1)z}{r^5}\left[\lambda - 5(\lambda+\mu)\frac{z^2}{r^2}\right]dm \\ \sigma_z = -2\mu\left(z\frac{\partial^2}{\partial z^2}\frac{\partial^2\psi}{\partial z^2} - \frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial}{\partial z}\frac{\partial^2\psi}{\partial z^2}\right)dm = -2\mu\left(\frac{\lambda+2\mu}{\lambda+\mu}\frac{1}{r^3} + \frac{6\lambda+3\mu}{\lambda+\mu}\frac{z^2}{r^5} - 15\frac{z^4}{r^7}\right)dm \end{cases} \tag{139}$$

inside the soil, and:

$$\begin{cases} u = 0 \\ v = 0 \\ w = 0 \end{cases} \tag{140}$$

$$I_{1\varepsilon} = \begin{cases} -\frac{2\mu}{\lambda+\mu}\frac{1}{r^3}dm & \text{for } r > 0 \\ -\infty & \text{for } r \to 0 \end{cases}, \tag{141}$$

$$\begin{cases} p_x = 0 \\ p_y = 0 \\ p_z = \begin{cases} 2\mu \frac{\lambda+2\mu}{\lambda+\mu} \frac{1}{r^3} dm & \text{for } r > 0 \\ \infty & \text{for } r \to 0 \end{cases} \end{cases} \tag{142}$$

for $z \to 0$.

Since p_x and p_y are equal to zero, Eqs. (139) may be combined with Eqs. (131) without changing the nature of the solved problem, which still is a vertical point-load problem. Moreover, due to the infinite value achieved by p_z for $z, r \to 0$, the second form of the first integral seems to be useful for building the combined solution in all the points of the soil apart from the one of load application.

3.2 Combined solution

The combined solution proposed here is built by multiplying Eqs. (139) for $-C$ and adding the result to Eqs. (131). The resulting stress field is now in relationship with the elastic constants of the soil:

$$\begin{cases} \tau_{xz} = 3\frac{(x-x_1)z}{r^5} \left[z + 2C\mu \left(5\frac{z^2}{r^2} - \frac{\lambda}{\lambda+\mu} \right) \right] dm \\ \tau_{yz} = 3\frac{(y-y_1)z}{r^5} \left[z + 2C\mu \left(5\frac{z^2}{r^2} - \frac{\lambda}{\lambda+\mu} \right) \right] dm \\ \sigma_z = \frac{-1}{r^3} \left[3\frac{z^3}{r^2} + 2C\mu \left(15\frac{z^4}{r^4} - 3\frac{2\lambda+\mu}{\lambda+\mu}\frac{z^2}{r^2} - \frac{\lambda+2\mu}{\lambda+\mu} \right) \right] dm \end{cases} \tag{143}$$

where:

$$r \neq 0. \tag{144}$$

As far as the third of Eqs. (143) is concerned, we may easily verify that the new terms significantly modify the normal stress when approaching the surface, while they are negligible at great depths. Indeed, for $z \to 0$:

$$-p_z = \lim_{z \to 0} \sigma_z = 2C\mu \frac{\lambda + 2\mu}{\lambda + \mu} \frac{1}{r^3} dm, \tag{145}$$

while, for $z \to \infty$, the third of Eqs. (143) gives the combined solution of §2.4.

$$\lim_{z \to \infty} \sigma_z = -3\frac{z^3}{r^5} dm, \tag{146}$$

which, for the position in Eq. (127), is equal to the third of Eqs. (131).

From the comparison between Eqs. (145) and (146), it is clear that, for $C > 0$, the normal stress for $z \to 0$ is opposite in sign to the normal stress for $z \to \infty$:

$$\operatorname{sign}\left(\lim_{z\to 0}\sigma_z\right) = -\operatorname{sign}\left(\lim_{z\to\infty}\sigma_z\right). \tag{147}$$

Therefore, near to the surface, the compressed soil is subjected to a normal stress of traction. This is a result not accounted for in the solution of Boussinesq and, together with the dependence of σ_z on the elastic constants, represents the most important novelty of the new combined solution.

From Eq. (147) we can also argue that, as σ_z is a continuous function, there exists a finite value of depth for which the normal stress is equal to zero. Setting $z_0 = z_0(r, C, \lambda, \mu)$, the function giving the depth for which $\sigma_z = 0$, from Eqs. (143) we find the relationship:

$$z_0^2\left(10C\mu\frac{z_0^2}{r^4} + \frac{z_0}{r^2} - 2C\mu\frac{2\lambda+\mu}{\lambda+\mu}\frac{1}{r^2}\right) = \frac{2}{3}C\mu\frac{\lambda+2\mu}{\lambda+\mu}, \tag{148}$$

in which the banal solution:

$$\sigma_z = 0 \text{ for } r \to \infty, \tag{149}$$

has been eliminated.

As can be easily verified, for $z \to \infty$ the combined solution proposed here is equal to the solution of Boussinesq even for the displacement field.

4 Numerical results and discussion

In the aim of performing a parametric analysis on the two elastic constants E and υ, the value of the calibration constant C that appears in the combined solution of second order (the third of Eqs. (143)) will be set equal to 1.

The plots of the vertical stress of the second order solution for a plane near to the surface are given in Fig. 6 for a prefixed υ and variable values of E, and Fig. 7 for a prefixed E and variable values of υ. The vertical stress contours of the second order solution are given in Fig. 8 for a prefixed υ and variable values of E, and Fig. 9 for a prefixed E and variable values of υ.

As can be easily appreciated in both Figs. 6 and 7, the numerical solution of the second order shows two positive peaks of vertical stress in the proximity of the application point of compression load, in total agreement with the experimental data for vehicular loading (Fig. 4). This result gives a numerical proof that a tensile state of stress actually arises on the surface of soils and pavements when subjected

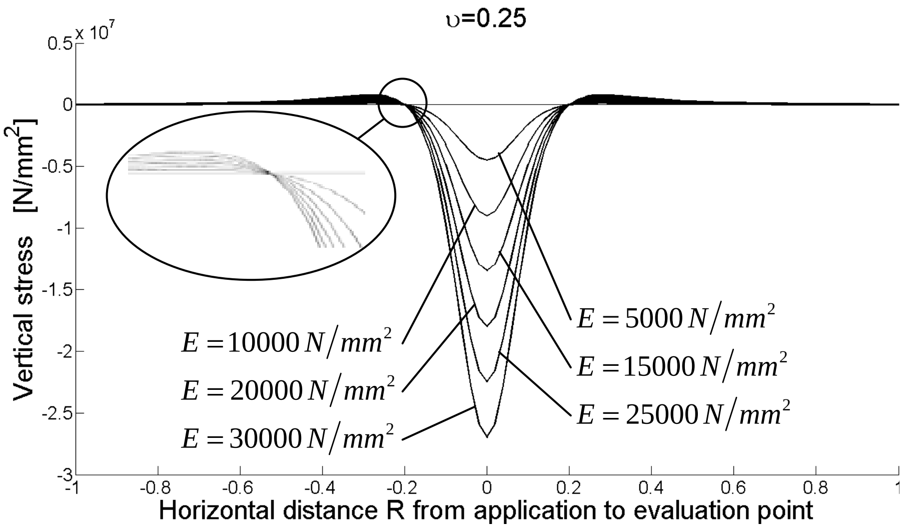

Figure 6: Parametric analysis on E for the vertical stress over the plane $z = 0.2mm$

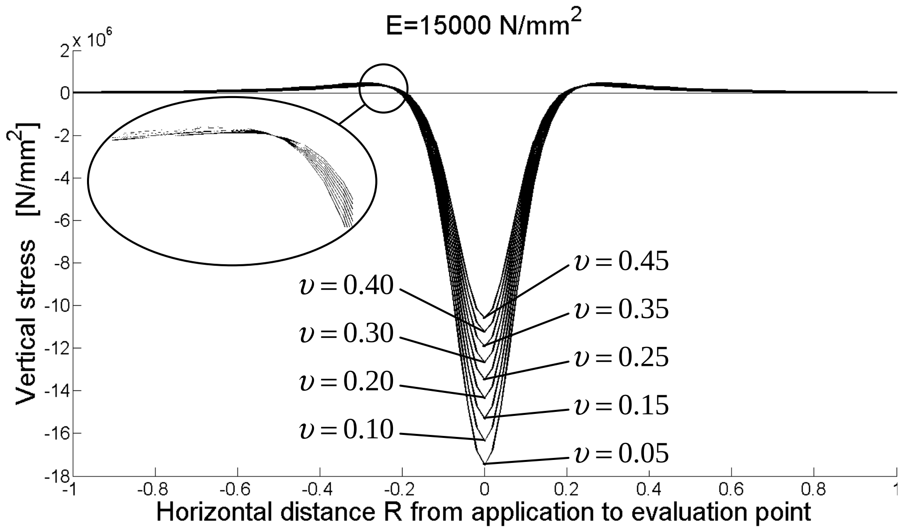

Figure 7: Parametric analysis on ν for the vertical stress over the plane $z = 0.2mm$

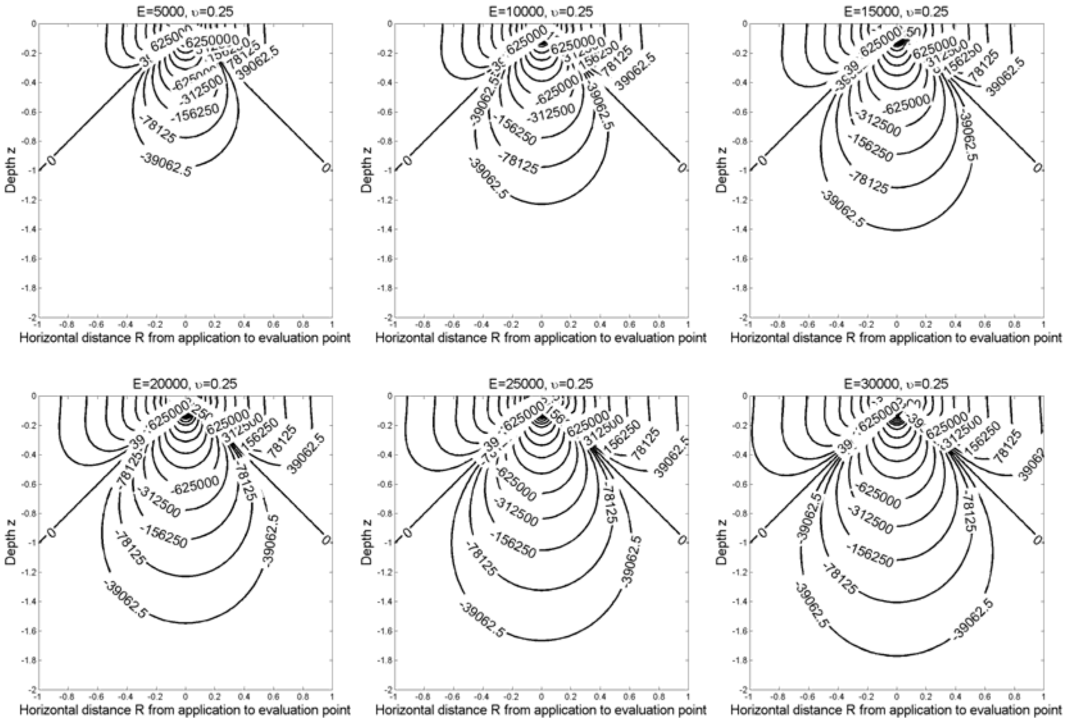

Figure 8: Parametric analysis on E for the vertical stress contours (all distances in *mm*

to compression loads. Since both soil and concrete are assumed as not being resistant to traction – to be on the safe side – the tensile state of stress must be considered with particular attention in these materials.

The parametric analyses also show what the effect is of the elastic constants E and υ on the vertical stress: greater values of E increase the vertical stresses at each depth without modifying the shape of the iso-lines of stress, which change in size homothetically (Fig. 8); greater values of υ modify both the shape of the iso-lines of stress (Fig. 9) and the values of vertical stresses at a given depth. In this latter case, greater values of υ decrease the vertical stresses at each depth. Thus, the effect of higher E modules on the vertical stress is opposite to the effect of higher υ modules.

In Fig. 6, we can see that the point in which the vertical stress change in sign is also a point in which the vertical stress does not depend upon the value of E. Lastly, in Fig. 7, we can find a second point, different to the previous one, in which the

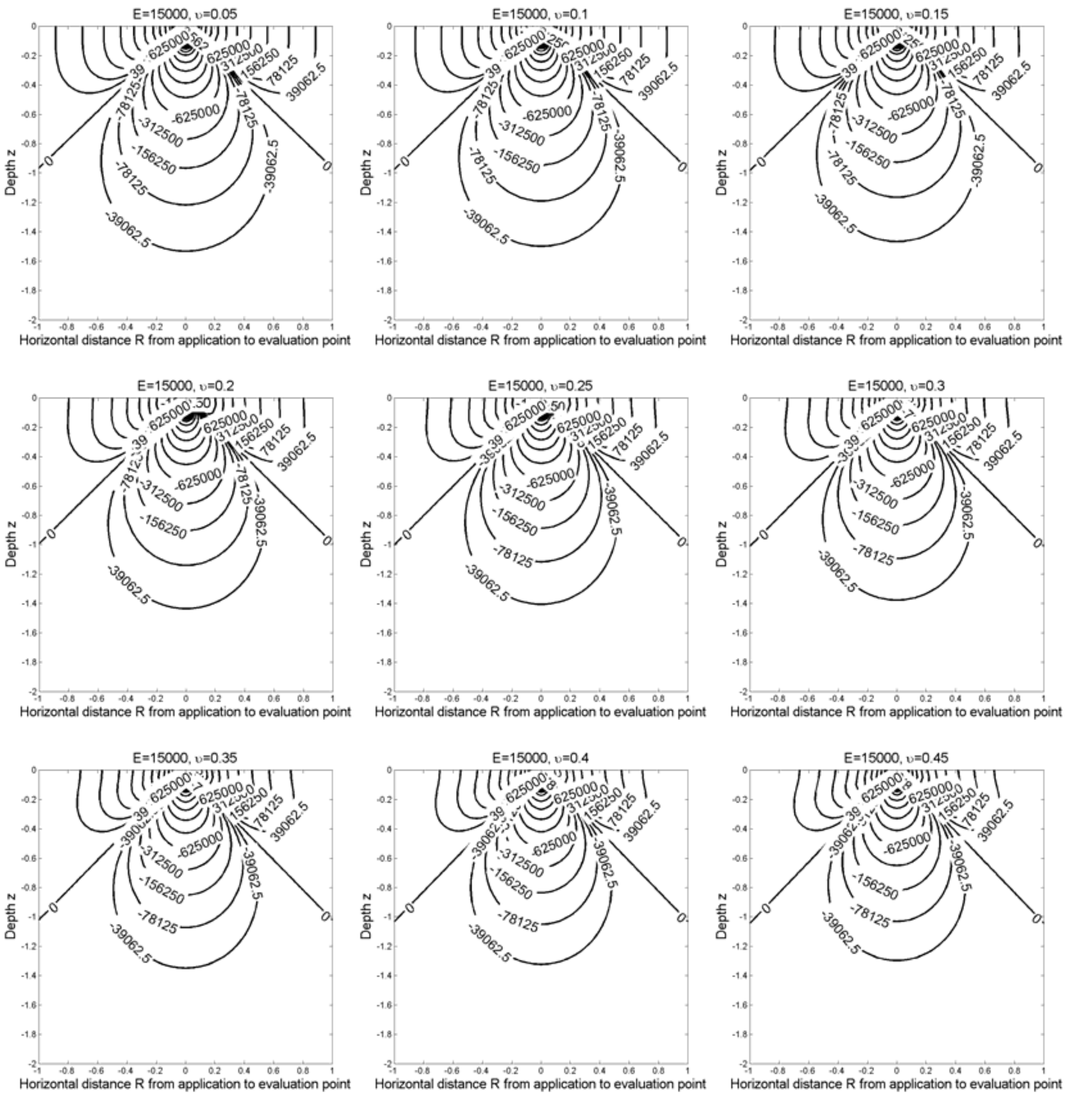

Figure 9: Parametric analysis on ν for the vertical stress contours (all distances in *mm*)

vertical stress does not depend upon the value of υ. Since the two points do not coincide, we can conclude that, in all the points of the half-space, the vertical stress depends on one elastic constant at least.

For the case of vertical stress given by the second order solution for a contact area greater than zero, see Ferretti (2012b), where circular, rectangular and elliptic contact areas are examined together with uniform and a parabolic laws of external pressure distribution.

5 Conclusions

In this paper, we have discussed Boussinesq's solution in the light of both known and new experimental findings on the stress distribution in a half-space subjected to point-loads. The original work carried out by Boussinesq has been reviewed and extended to provide a second order solution.

The plot of the vertical stress at a given depth, given by the second order solution, has been compared with the experimental acquisitions, with a good match found between numerical and experimental data. In particular, the second order solution has shown that a compression point-load always generates a tensile state of stress at the surface, in the proximity of the application point. The existence of a tensile state of stress, not accounted for in Boussinesq's solution, could explain the several observed mechanisms of premature damage that affect concrete pavements subjected to vehicular loading.

The second order solution also allows us to evaluate the effect of the elastic constants E and υ on the stress field, which, in this second case also, is an improvement to Boussinesq's solution.

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