

Identification of Material Parameters of Two-Dimensional Anisotropic Bodies Using an Inverse Multi-Loading Boundary Element Technique

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Abstract: An inverse technique, based on the boundary element method (BEM) and elastostatic experiments for identification of elastic constants of orthotropic and general anisotropic 2D bodies is presented. Displacement measurements at several points on the boundary of the body, obtained by a few known load cases are used in the inverse analysis to find the unknown elastic constants of the body. Using data from more than one elastostatic experiment results in a more accurate and stable solution for the identification problem. In the inverse analysis, sensitivities of displacements of only boundary points with respect to the elastic constants are needed. Therefore, the BEM is a very useful and effective method for this purpose. An iterative Tikhonov regularization method is used for the inverse analysis. A method for simple computation of initial guesses for unknown elastic constants and a procedure for appropriate selection of the regularization parameter appearing in the inverse analysis is also proposed. Convergence and accuracy of the presented method with respect to measurement errors and number of load cases are investigated by presenting several examples.

Keywords: inverse method, boundary element method, orthotropic, anisotropic, elastic constants

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1 Introduction

Anisotropic composite materials are widely used in engineering structures. Some natural materials and bio-structures show anisotropy too. Rock mass is generally considered anisotropic because of existence of micro cracks and joint sets [Ohkami, Ichikawa, and Kawamoto (1991)]. Bamboo stalks contain longitudinal fibers, which make bamboo an orthotropic material [Silva, Walters, and Paulino (2006)]. Bone tissue, like most biological materials is anisotropic [Geng, Tan, and Liu (2001); Fan et al. (2002); DeTolla et al. (2000)]. Identification of elastic parameters of man-made or natural anisotropic materials and structures is very important for predicting their behavior.

Techniques for identification of elastic constants are based on inverse methods. In a direct problem, the geometry, boundary conditions, material properties, and applied loads are known and the objective is to find field variables in the domain and over the boundary of the problem. However, in an inverse problem, boundary conditions, material constants, or applied loads are fully or partially unknown, and instead, the magnitude of the field variables at some points in the domain or on the boundary of the problem are given. Input data for inverse problems are usually provided by measurement, and therefore, they include some errors. Inverse problems are ill-posed, i.e. they are very sensitive to small changes in input data [Hadamard (1923); Ling and Atluri (2006)]. This behavior makes the inverse problems much more complicated in comparison with the direct problems. To overcome this difficulty, a regularization method is usually used in the inverse analysis and the regularization parameter appearing in the formulation should be carefully selected [Wang and Xiao (2001); Xie and Zou (2002); Khosravifard and Hematiyan, (2011); Liu and Kuo (2011)].

The most important methods for identification of elastic constants are based on either static or dynamic measurements. Some researchers have presented inverse methods based on the finite element method (FEM) and static measurements; see for example [Wang and Kam (2000); Lecompte et al. (2007)]. Cunha and Piranda (2000) presented a method based on the FEM and dynamic measurements for identification of stiffness properties of composite tubes. Rikards, Chate, and Gailis (2001) presented a method for identification of elastic constants of laminates based on dynamic measurements. They also used the FEM in their formulation.

A few researchers have used the BEM for identification of elastic constants of orthotropic or anisotropic materials. Ohkami, Ichikawa, and Kawamoto (1991) presented an identification method based on static measurements and the BEM for a 2D orthotropic medium. They used the Gauss-Newton method in their formulation. Huang et al. (2004) presented an inverse BEM based on the displacement measure-

ments and the Levenberg-Marquardt method for identification of elastic parameters of 2D orthotropic bodies. Comino and Gallego (2005) presented an inverse method based on the BEM for identification of elastic constants of 2D anisotropic materials. They used static measurements and the Levenberg-Marquardt method in their inverse technique.

In the above-mentioned studies on mechanical property identification of anisotropic materials, the unknowns have been computed using only one static load case. To obtain an acceptable solution using only one static experiment, a complicated load case should be considered. Often, it is impossible to make a sample with a simple standard geometry for performing the required measurements. In such cases, the original body should be used for the measurements. On the other hand, setting up a single experiment with a load case that efficiently includes effects of all elastic constants may be impossible or difficult. Therefore, it is reasonable to carry out several experiments with different simple load cases to find out the unknowns. This is the main idea of the present paper. In this work, an inverse method based on static experiments for identification of elastic constants of orthotropic and general anisotropic 2D bodies is proposed. Displacements at several boundary points, obtained by a few known load cases are considered as measured data. Equations generated from the inverse formulation of all load cases are coupled and solved simultaneously. The BEM is used for sensitivity analysis in the inverse method. The BEM is a very useful and effective method for this purpose, because a sensitivity analysis of displacements at only boundary points with respect to the elastic constants is needed in the inverse analysis.

The iterative Tikhonov regularization method [Tikhonov and Arsenin (1977); Tikhonov and Arsenin (1986)], including a regularization parameter, is used for the inverse analysis. A procedure for appropriate selection of the regularization parameter is also proposed. A set of initial guesses should be considered for starting the inverse analysis. An efficient method for simple computation of initial guesses for unknown elastic constants is also presented. Some examples are presented to demonstrate the rate of convergence and accuracy of the proposed method.

2 The BEM for elastostatic analysis of 2D anisotropic bodies

Before dealing with the core issue of the inverse analysis, it is useful to review some fundamental equations in two-dimensional anisotropic elasticity and the corresponding formulation in the BEM. For a homogeneous generally anisotropic elastic material in plane stress, the constitutive relations can be expressed in matrix

form as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{16} \\ c_{12} & c_{22} & c_{26} \\ c_{16} & c_{26} & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}, \quad (1)$$

where σ_{ij} and ε_{ij} ($i, j=1, 2$) represent the stresses and strains, respectively, and the coefficients c_{mn} and a_{mn} are the elastic stiffness and compliance constants of the material, respectively. These compliances may be given in terms of engineering constants as follows [Lekhnitskii (1968)]:

$$\begin{aligned} a_{11} &= 1/E_1, & a_{22} &= 1/E_2, & a_{12} &= -\nu_{12}/E_1 = -\nu_{21}/E_2, \\ a_{16} &= \eta_{12,1}/E_1 = \eta_{1,12}/G_{12}, & a_{26} &= \eta_{12,2}/E_2 = \eta_{2,12}/G_{12}, & a_{66} &= 1/G_{12}, \end{aligned} \quad (2)$$

where E_k is the Young's modulus in the direction of the x_k -axis and G_{12} is the shear modulus on the $x_1 - x_2$ plane; ν_{ij} is the Poisson's ratio, and $\eta_{i,jl}$, $\eta_{ij,l}$ are the coefficients of mutual influence of the first and second kind, respectively. Equation (2) is also applicable to the case of plane strain, provided b_{jk} is substituted for a_{jk} by

$$b_{jk} = a_{jk} - a_{j3}a_{k3}/a_{33}, \quad (j, k = 1, 2), \quad (3)$$

where, with the index 3 referring to the x_3 -axis, a_{m3} are given by

$$a_{j3} = -\nu_{j3}/E_j = -\nu_{3j}/E_3, \quad a_{33} = 1/E_3, \quad a_{63} = \eta_{12,3}/E_3 = \eta_{3,12}/G_{12}. \quad (4)$$

By introducing Airy's stress functions, Lekhnitskii (1968) has shown that the characteristic equation for an anisotropic material in stable equilibrium is

$$a_{11}\mu^4 - a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - a_{26}\mu + a_{22} = 0. \quad (5)$$

It has further been shown that the roots of this characteristic equation must be complex, and are given by two distinct pairs of complex conjugates. These roots are denoted by

$$\mu_j = \alpha_j + i\beta_j, \quad (j = 1, 2), \quad (6)$$

where $i = \sqrt{-1}$ and β_j must be positive from thermodynamic considerations. By following the above notation for material properties, the position of a general field point at (x_1, x_2) can be described by

$$z_j = x_1 + \mu_j x_2, \quad (j = 1, 2). \quad (7)$$

The analytical basis of the BEM is the boundary integral equation (BIE). Two key requirements are necessary for its derivation, namely, (a) the fundamental solution to the governing differential equations, and (b) a reciprocal theorem relating the displacements and the tractions on the elastic body. These are provided, respectively, by the unit load solutions for an infinite body, and the Betti-Rayleigh's reciprocal work theorem. Carrying out the usual limiting process results in the BIE, relating the displacements u_i and the tractions t_i on the boundary S of the domain Ω , as follows:

$$C_{ij}u_i(P) = \int_S t_i(Q)U_{ij}(P,Q)dS - \int_S u_i(Q)T_{ij}(P,Q)dS, \quad (8)$$

in which P and Q represent the source point and the field point on S , respectively, and C_{ij} are coefficients associated with the boundary geometry at P . In the boundary integral equation, $U_{ij}(P,Q)$ and $T_{ij}(P,Q)$ are the fundamental solutions for displacements and tractions at Q in the x_i -direction, respectively, when a unit load is applied at P in the x_j -direction. Their explicit forms are given by [Cruse (1988)]:

$$U_{ij}(P,q) = 2Re\{r_{i1}A_{j1} \log z_1 + r_{i2}A_{j2} \log z_2\}, \quad (9a)$$

$$T_{1j}(P,Q) = 2n_1Re\{\mu_1^2 A_{j1}/z_1 + \mu_2^2 A_{j2}/z_2\} - 2n_2Re\{\mu_1 A_{j1}/z_1 + \mu_2 A_{j2}/z_2\}, \quad (9b)$$

$$T_{2j}(P,Q) = -2n_1Re\{\mu_1 A_{j1}/z_1 + \mu_2 A_{j2}/z_2\} + 2n_2Re\{A_{j1}/z_1 + A_{j2}/z_2\}, \quad (9c)$$

where r_{ij} and A_{ji} are complex quantities associated with the material properties [Cruse (1988)], $Re\{\}$ is the operator which takes the real part of complex quantities, and z_i is a generalized complex variable. This variable for the field point Q at (x_1, x_2) , with reference to the source point P at (x_{p1}, x_{p2}) , is defined as follows:

$$z = (x_1 - x_{p1}) + \mu_i(x_2 - x_{p2}) = \zeta_1 + \mu_i\zeta_2, \quad (10)$$

where ζ_i represent the local coordinates with the origin located at P . In Eqs. (9b) and (9c), n_i are components of the unit outward normal vector at Q . The BIE, Eq. (8), can generally be solved only by numerical means. It involves, discretizing the boundary into a mesh with, say, M elements and N distinct nodes. When using quadratic isoparametric elements, the geometry and all the primary solution variables are assumed to vary in a quadratic manner over each element. With the use of interpolation by shape functions $N^c(\zeta)$ expressed in terms of the intrinsic coordinate ζ ($-1 \leq \zeta \leq 1$), the coordinates and solution variables at a general field point can then be expressed as

$$x_i(\zeta) = \sum_{c=1}^3 N^c(\zeta)x_i^c, \quad \mu_i(\zeta) = \sum_{c=1}^3 N^c(\zeta)\mu_i^c, \quad t_i(\zeta) = \sum_{c=1}^3 N^c(\zeta)t_i^c, \quad (11)$$

where $N^c(\zeta)$ are given by

$$N^1(\zeta) = -\zeta(1-\zeta)/2, \quad N^2(\zeta) = (1-\zeta^2), \quad N^3(\zeta) = \zeta(1+\zeta)/2. \quad (12)$$

By substituting Eqs. (11) and (12) into the BIE, Eq. (8), the discretized form of the boundary integral equation is obtained, as follows:

$$\begin{aligned} C_{ij}(P^a)u_{ij}(P^a) &= \sum_{b=1}^M \sum_{c=1}^3 b^c_{t_i} \int_S U_{ij}(P^a, Q) N^c(\zeta) J(\zeta) dS \\ &\quad - \sum_{b=1}^M \sum_{c=1}^3 b^c_{u_i} \int_S T_{ij}(P^a, Q) N^c(\zeta) J(\zeta) dS \end{aligned} \quad (13)$$

where P^a stands for the a -th node of the mesh ($P^a=1 \sim N$) and the superscripts b and c represent the b -th element and the c -th node of each element, respectively. $J(\zeta)$ is the Jacobian of coordinate transformation. Equation (13) represents a set of $2N$ linear algebraic equations for the unknown displacements/tractions at the boundary nodes. It can be solved by, for example, the Gaussian elimination method.

3 Inverse analysis

An anisotropic elastic problem under stable equilibrium in which the material properties, displacement boundary conditions, and applied loads are known is a well-posed problem, i.e. it possesses a unique solution. However, an inverse problem with unknown elastic constants is an ill-posed problem. In the inverse problem, the unknowns are to be found by using some additional information obtained by measurements. In an inverse analysis, an optimization method including a sensitivity analysis and a regularization technique should be employed.

3.1 The inverse problem statement and formulation

A general 2D anisotropic body with unknown elastic constants is considered. The vector of elastic constants which contains these unknowns is defined as follows

$$\mathbf{C} = [C_1 \quad C_2 \quad \cdots \quad C_6]^T \quad (14)$$

where

$$C_1 = c_{11}, \quad C_2 = c_{22}, \quad C_3 = c_{66}, \quad C_4 = c_{12}, \quad C_5 = c_{16}, \quad C_6 = c_{26} \quad (15)$$

To find these material constants, a few elastostatic experiments with different load conditions and constraints are performed. The number of experiments may be two, three, or more. We consider three load cases to formulate the problem. The

formulation with a different number of load cases is similarly accomplished. An anisotropic body under three different load and constraint conditions is shown in Fig. 1.

In each load case, the displacements at some selected boundary points are measured. The location of measurements can be different in each load case. Assume that there are N_1 , N_2 , and N_3 measurement data in the load cases 1, 2, and 3, respectively.

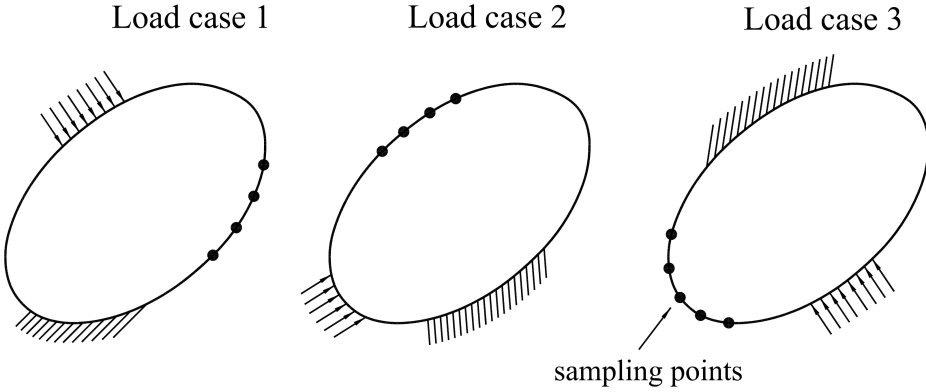


Figure 1: A body subjected to three different load cases

The vectors of measured data are represented by $\mathbf{Y}^{(1)}$, $\mathbf{Y}^{(2)}$, and $\mathbf{Y}^{(3)}$, where, for example, $\mathbf{Y}^{(1)}$ contains measured data obtained from the load case 1 and can be expressed as follows:

$$\mathbf{Y}^{(1)} = \left[Y_1^{(1)} \quad Y_2^{(1)} \quad \dots \quad Y_{N_1}^{(1)} \right]^T \quad (16)$$

The vector of displacements at the sampling points in the load case 1 computed by considering a set of elastic constants is represented by $\mathbf{U}^{(1)}$, which can be expressed as follows:

$$\mathbf{U}^{(1)} = \left[U_1^{(1)} \quad U_2^{(1)} \quad \dots \quad U_{N_1}^{(1)} \right]^T \quad (17)$$

Vectors $\mathbf{U}^{(2)}$ and $\mathbf{U}^{(3)}$ are similarly defined. To find the vector of elastic constants, the Tikhonov regularization method is used. In this method, the following objective function S is formed [Tikhonov and Arsenin (1977)]:

$$S = (\mathbf{Y} - \mathbf{U})^T (\mathbf{Y} - \mathbf{U}) + \mu \mathbf{C}^T \mathbf{C} \quad (18)$$

where the vectors \mathbf{Y} and \mathbf{U} are expressed as follows

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}^{(1)} \\ \mathbf{Y}^{(2)} \\ \mathbf{Y}^{(3)} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} \mathbf{U}^{(1)} \\ \mathbf{U}^{(2)} \\ \mathbf{U}^{(3)} \end{bmatrix} \quad (19)$$

The unknown vector \mathbf{C} is found by minimizing the summation S . In Eq. (18), μ is a regularization parameter. The first term in Eq. (18) is used to make sure that the difference between the vectors \mathbf{Y} and \mathbf{U} is small. The second term in Eq. (18) is used to prevent the elastic constants vector having a large norm. Small values of μ lead to oscillatory solutions in some cases. Increasing the value of the regularization parameter damps the oscillations; however, the difference between the measured and computed values of the displacements at the sampling points increases. Minimization of S with respect to the vector \mathbf{C} leads to

$$\frac{\partial S}{\partial \mathbf{C}} = -2\mathbf{X}^T(\mathbf{Y} - \mathbf{U}) + 2\mu\mathbf{C} = 0 \quad (20)$$

The matrix \mathbf{X} in Eq. (20) is the sensitivity matrix of all load cases, which can be expressed as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \\ \mathbf{X}^{(3)} \end{bmatrix} \quad (21)$$

where $\mathbf{X}^{(L)}$ is the sensitivity matrix of the load case L and is expressed as

$$\mathbf{X}^{(L)} = \begin{bmatrix} X_{11}^{(L)} & X_{12}^{(L)} & \cdots & X_{16}^{(L)} \\ X_{21}^{(L)} & X_{22}^{(L)} & \cdots & X_{26}^{(L)} \\ \vdots & \vdots & \vdots & \vdots \\ X_{N_L1}^{(L)} & X_{N_L2}^{(L)} & \cdots & X_{N_L6}^{(L)} \end{bmatrix} \quad L = 1, 2, \text{ and } 3 \quad (22)$$

The components of the sensitivity matrix $\mathbf{X}^{(L)}$ can be expressed as follows

$$X_{ij}^{(L)} = \frac{\partial U_i^{(L)}}{\partial C_j} \quad (23)$$

In order to compute the components of the sensitivity matrix given in Eq. (22), the derivative of boundary displacements with respect to the elastic constants must be obtained. Two main approaches are usually used for this purpose. One is the finite difference scheme, and the second is by direct differentiation of the integral or

matrix equations, representing the problem. The latter is much more complicated; in this work, the finite difference method is used.

The unknown vector \mathbf{C} can be found using Eq. (20) by an iterative procedure. Suppose that the vector $\tilde{\mathbf{C}}$ is an estimate for the vector of elastic constants, and $\tilde{\mathbf{U}}^{(1)}$, $\tilde{\mathbf{U}}^{(2)}$, and $\tilde{\mathbf{U}}^{(3)}$ are the corresponding displacement vectors for the load cases 1, 2, and 3, respectively. The displacement vector can be approximated as follows

$$\mathbf{U} = \tilde{\mathbf{U}} + \mathbf{X}(\mathbf{C} - \tilde{\mathbf{C}}) \quad (24)$$

where

$$\tilde{\mathbf{U}} = \begin{bmatrix} \tilde{\mathbf{U}}^{(1)} \\ \tilde{\mathbf{U}}^{(2)} \\ \tilde{\mathbf{U}}^{(3)} \end{bmatrix} \quad (25)$$

Substituting Eq. (24) into Eq. (20) and after some mathematical manipulations, the following relationship is obtained

$$\mathbf{C} = [\mathbf{X}^T \mathbf{X} + \mu \mathbf{I}]^{-1} [\mathbf{X}^T (\mathbf{Y} - \tilde{\mathbf{U}}) + \mathbf{X}^T \mathbf{X} \tilde{\mathbf{C}}]. \quad (26)$$

Eq. (26) is to be used in an iterative procedure, therefore, it is appropriately written in the following form

$$\mathbf{C}^{k+1} = [(\mathbf{X}^k)^T \mathbf{X}^k + \mu^k \mathbf{I}]^{-1} [(\mathbf{X}^k)^T (\mathbf{Y} - \mathbf{U}^k) + (\mathbf{X}^k)^T \mathbf{X}^k \mathbf{C}^k] \quad (27)$$

where k and $k+1$ represent iteration numbers. The convergence criterion is then defined as

$$\|\mathbf{C}^{k+1} - \mathbf{C}^k\| \leq e \quad (28)$$

where e is a specified tolerance.

The existence and uniqueness of the solution for the inverse problem may be assured by physical reasoning. If a sufficient number of measured data (greater than the number of unknowns) are used in the inverse analysis, a solution very close to the exact solution can be obtained even in case of noisy input data.

3.2 Selection of the regularization parameter

The regularization parameter μ^k in Eq. (27) should be carefully selected at each iteration. The L-curve method [Hansen (1998)] is a well-known method for finding the optimum value of the regularization parameter. However, an alternative

method based on measurement errors is used here instead. Suppose that the measured data have a Gaussian noise distribution, and each measured displacement can be expressed as follows:

$$Y_i = Y_i^{exact} + e_i Y_i^{exact} \quad i = 1, 2, \dots, N_t \quad (29)$$

where N_t is the total number of measured data and e_i is the relative error. e_i is a random number from a Gaussian distribution with a zero mean and the standard deviation $\hat{\sigma}$. We usually have sufficient information about the measurement error and there exists a suitable estimation for the standard deviation. Assuming the vector \mathbf{E} contains errors of computed displacements with respect to measured data, and using Eq. (24), we can write:

$$\mathbf{E} = \mathbf{U}(\mathbf{C}^{k+1}) - \mathbf{Y} = \mathbf{U}^k + \mathbf{X}^k(\mathbf{C}^{k+1} - \mathbf{C}^k) - \mathbf{Y} \quad (30)$$

When we use a small number of measurement data, for example, equal to the number of unknowns, with $\mu^k = 0$, the computed vector \mathbf{C}^{k+1} will be very noisy; however, the value of $\|\mathbf{E}\|$ will be very small and

$$\text{STD}(\mathbf{E}) < \hat{\sigma} \quad (31)$$

where STD stands for the standard deviation. In this case, selecting a positive value for μ^k will result in a much better solution for \mathbf{C}^{k+1} with a larger $\text{STD}(\mathbf{E})$. The regularization parameter μ^k is selected in a such way that

$$\text{STD}(\mathbf{E}) \approx \hat{\sigma} \quad (32)$$

When a sufficient number of measurement data (considerably more than the number of unknowns) is used, $\text{STD}(\mathbf{E}) > \hat{\sigma}$ even with $\mu^k = 0$. In this case, satisfactory results can be obtained with $\mu^k = 0$

3.3 Sensitivity analysis of boundary displacements with respect to the elastic constants

In order to compute the components of the sensitivity matrix given in Eq. (23), the derivative of boundary displacements with respect to the elastic constants is required. Two main approaches are usually used for this purpose. One is the use of finite differences [Ohkami (1991)], and the second is carried out by direct differentiation of the integral or matrix equations, representing the problem [Huang et al (2004); Comino and Gallego (2005); Gallego Comino, and Ruiz-Cabello (2005)]. The computational cost of the second approach is less than the first one; however, the second approach is much more complicated. In this work, the approach based

on the finite difference method is used. The sensitivity coefficient given in Eq. (23) can be approximated by a finite difference as follows:

$$X_{ij}^{(L)} = \frac{\partial U_i^{(L)}}{\partial C_j} = \frac{U_i^{(L)}|_{C_j+\varepsilon C_j} - U_i^{(L)}|_{C_j}}{\varepsilon C_j} \quad (33)$$

where ε is a small value. The value of $\varepsilon = 0.001$ is used in the present work.

3.4 Initial guesses for the elastic constants

Using suitable initial guesses for the elastic constants results in a smaller number of total iterations in the inverse analysis. Often, there exist some information about the material constants, and it is possible to suggest suitable initial guesses for the unknown constants. For cases where there is no information about the values of the elastic constants, a method for generating initial guesses should be proposed. In this part of the paper, methods for computing initial guesses for orthotropic as well as general anisotropic materials are presented.

3.4.1 Initial guesses for orthotropic materials

We use elastic constants of an isotropic material as initial guesses for unknown elastic constants of the orthotropic material. Elastic constants of an isotropic material in terms of the elastic modulus E and the Poisson's ratio ν can be expressed as

$$\begin{aligned} C_1 &= \frac{E}{1-\nu^2}, & C_2 &= \frac{E}{1-\nu^2}, & C_3 &= \frac{E}{2(1+\nu)}, \\ C_4 &= \frac{E\nu}{1-\nu^2}, & C_5 &= 0, & C_6 &= 0 \end{aligned} \quad (34)$$

for plane stress conditions and

$$\begin{aligned} C_1 &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, & C_2 &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, & C_3 &= \frac{E}{2(1+\nu)}, \\ C_4 &= \frac{E\nu}{(1+\nu)(1-2\nu)}, & C_5 &= 0, & C_6 &= 0 \end{aligned} \quad (35)$$

for plane strain conditions.

Now we attempt to find the elastic modulus of an isotropic material with $\nu = 0.3$, which approximately gives measured data in the load case 1. Suppose that the vector of displacements at sampling points in the load case 1 obtained by considering an isotropic material with $E = 1$ and

$\nu = 0.3$ is denoted by $\bar{\mathbf{U}}_1$, i.e.

$$\bar{\mathbf{U}}_1 = \mathbf{U}_1(E = 1, \nu = 0.3) \quad (36)$$

We also define $\bar{\mathbf{U}}_E$ as follows:

$$\bar{\mathbf{U}}_E = \mathbf{U}_1(E, \nu = 0.3) \quad (37)$$

Due to the linearity of displacement with respect to $1/E$, it is possible to write

$$\bar{\mathbf{U}}_E = a\bar{\mathbf{U}}_1 \quad (38)$$

where $a = 1/E$

To find a suitable value of a we can minimize the following function

$$F = (\mathbf{Y}_1 - a\bar{\mathbf{U}}_1)^T(\mathbf{Y}_1 - a\bar{\mathbf{U}}_1) \quad (39)$$

which leads to

$$\frac{\partial F}{\partial a} = -2\bar{\mathbf{U}}_1^T(\mathbf{Y}_1 - a\bar{\mathbf{U}}_1) = 0 \quad (40)$$

Therefore, the following relationship is obtained:

$$E = \frac{1}{a} = \frac{\bar{\mathbf{U}}_1^T \bar{\mathbf{U}}_1}{\bar{\mathbf{U}}_1^T \mathbf{Y}_1} \quad (41)$$

We can calculate E by Eq. (41) without any iteration and regularization. The initial guesses for the unknown elastic constants can be found by substituting the computed value of E and $\nu = 0.3$ into Eq. (34) or Eq. (35). The numerical examples presented in Section 4 show that the proposed method for computation of initial guesses for the elastic constants is very actually effective indeed

3.4.2 Initial guesses for general anisotropic materials

Similar to the case of orthotropic materials, we use elastic constants of an isotropic material as initial guesses for C_1 to C_4 . Non-zero initial guesses should be selected for C_5 and C_6 too. The constants C_5 and C_6 may be positive or negative. We can use $\pm 0.1a^{-1}$ as initial guesses for C_5 and C_6 . The example in Section 4 shows that using these initial guesses results in a very good convergence of the inverse analysis.

4 Numerical examples

In this section, the proposed inverse technique is applied for identification of elastic constants of bodies. Two different cases are considered. In the first case, the body is made of an orthotropic material, while it is made of an anisotropic material in the second case. In both cases, it is considered that no preliminary knowledge of

the elastic constants is available. The method described in Sec. 3.4 is first used to obtain reasonable initial guesses for the inverse method. Then, the initial guesses are iteratively updated until the algorithm converges. In each case, a direct analysis is performed and the displacements at several boundary points obtained using this direct analysis, are used in place of experimental measurements. To account for the inherent experimental errors, a vector of errors with Gaussian distribution is added to the results of the direct analysis. The effect of the standard deviation of the errors on the identified elastic constants is also studied.

The shape of the body for which the elastic constants are sought is shown in Fig. 2. For the sensitivity analysis by the BEM, the boundary of the body is discretized into 34 quadratic boundary elements.

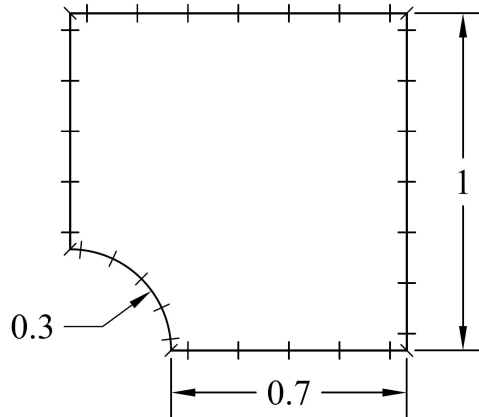


Figure 2: The geometry of the problem along with 34 boundary elements

Fig. 3 shows the three load cases used for identification of the material constants. In this figure, the value of the applied traction q , for each case is equal to 10^4 N/m. 20 sampling points are selected on the boundary of the body in each load case. The dots in the figure represent the sampling points where the measurements are made. The elastic constants of the body are obtained in three different situations. First, the measurements of only the first load case are used for identification of the material constants. Then, the measurements of the first and the second load cases are used together, and another set of constants are identified. Finally, the measurements of all the three load cases are used together to obtain material constants. The results of these three situations are compared to a case in which all loadings are applied to the body simultaneously (see Fig. 4).

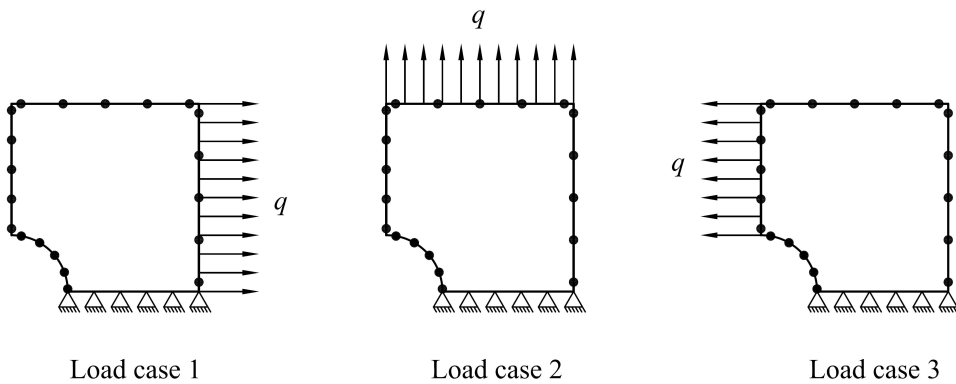


Figure 3: The three load cases used for determination of material constants

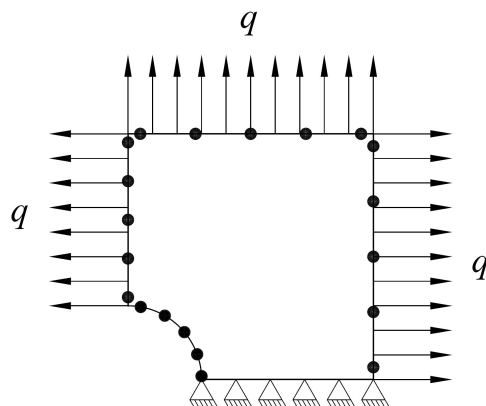


Figure 4: The load case with simultaneous application of various loadings

4.1 Case I: Identification of the elastic constants of an orthotropic body

In this case, it is assumed that the body shown in Fig. 2 is made of an orthotropic material for which the vector of elastic constants is:

$$\mathbf{C} = [10 \quad 5 \quad 3.5 \quad 1.5]^T \quad \text{GPa} \quad (42)$$

The relative error of measured data is assumed 10%, i.e. a relative error with a Gaussian distribution and the standard deviation of $\hat{\sigma} = 0.033$ is assumed. Using the measured data and employing the method described in Sec. 3.4, the following initial guess for the vector \mathbf{C} is obtained:

$$\mathbf{C}^0 = [6.5 \quad 6.5 \quad 2.3 \quad 1.9]^T \quad \text{GPa} \quad (43)$$

It is clear that the proposed method results in a vector of elastic constants whose values are relatively close to the exact values. In this way, the number of iterations needed for the convergence of the algorithm is efficiently reduced. Table 1, gives the values of elastic constants obtained by load cases of Fig. 3. This table also lists the values of the elastic constant obtained by application of all the loads together in one experiment (see Fig. 4).

Table 1: The identified elastic constants (in GPa) of the orthotropic material with various load cases (with 10% measurement error)

	c_{11} (error)	c_{22} (error)	c_{66} (error)	c_{12} (error)
Exact value	10	5	3.5	1.5
1-test	10.41 (4.1%)	4.95 (1.0%)	3.53 (0.9%)	1.17 (22%)
2-test	11.32 (13%)	5.02 (0.3%)	3.61 (3.1%)	1.69 (12%)
3-test	10.38 (3.8%)	5.02 (0.3%)	3.43 (2.0%)	1.41 (5.8%)
1-test (all-in-one loading)	10.06 (0.6%)	5.13 (2.5%)	2.99 (15%)	1.42 (5.6%)

Table 1 suggests that the most reliable results are obtained in the case that three different experiments are performed and the results of all three experiments are used together for identification of the elastic constants. Another benefit of using more than one test is that relatively simple tests can be performed in each case for identification of the constants. If only one test is to be used, the applied loads will need to be much more complicated.

The effect of measurement errors on the identified elastic constants is investigated. The standard deviation of the error is so chosen as to result in 3, 5, 10, and 20

Table 2: Effect of measurement error on the identified elastic constants of the orthotropic body

	c_{11} (error)	c_{22} (error)	c_{66} (error)	c_{12} (error)
Exact value (GPa)	10	5	3.5	1.5
3% measurement error	10.11 (1.1%)	5.00 (0.1%)	3.48 (0.6%)	1.48 (1.6%)
5% measurement error	10.19 (1.9%)	5.01 (0.2%)	3.45 (1.0%)	1.46 (2.8%)
10% measurement error	10.38 (3.8%)	5.02 (0.3%)	3.43 (2.0%)	1.41 (5.8%)
20% measurement error	10.80 (8.0%)	5.03 (0.7%)	3.36 (3.9%)	1.32 (13%)

percent tolerance of measurements error. Table 2, shows the elastic constants predicted for each case. The results of this table are obtained using the three load cases shown in Fig. 3.

Fig. 5 and Fig. 6 depict the values of the elastic constants versus the iteration number for the case with three tests and with 10% and 20% measurement error, respectively. These figures clearly show that the convergence of the method is very fast. The reason for this high rate of convergence is the reasonable initial guess that is selected here

4.2 Case II: Identification of the elastic constants of an anisotropic body

In this case, it is assumed that the body shown in Fig. 2 is made of an anisotropic material for which the vector of elastic constants is as follows:

$$\mathbf{C} = [544.8 \quad 531.1 \quad 243.5 \quad 153.6 \quad -81.2 \quad 89.7]^T \text{ GPa} \quad (44)$$

These values correspond to those for alumina crystal, the three principal axes of which have been rotated clockwise by 30° , 45° and 60° , respectively as reported in a paper by Tan, Shiah, and Lin (2009). Following the procedure described in Sections 3.4.1 and 3.4.2, the following four vectors of initial guesses are obtained.

$$\begin{aligned} \mathbf{C}^0 &= [486.6 \quad 486.6 \quad 170.3 \quad 146.0 \quad 48.7 \quad 48.7]^T \text{ GPa} \\ \mathbf{C}^0 &= [486.6 \quad 486.6 \quad 170.3 \quad 146.0 \quad -48.7 \quad 48.7]^T \text{ GPa} \\ \mathbf{C}^0 &= [486.6 \quad 486.6 \quad 170.3 \quad 146.0 \quad 48.7 \quad -48.7]^T \text{ GPa} \\ \mathbf{C}^0 &= [486.6 \quad 486.6 \quad 170.3 \quad 146.0 \quad -48.7 \quad -48.7]^T \text{ GPa} \end{aligned} \quad (45)$$

The results for this example are obtained by utilizing the first vector of initial guesses given in Eq. (45), mentioning that other vectors give converged solutions.

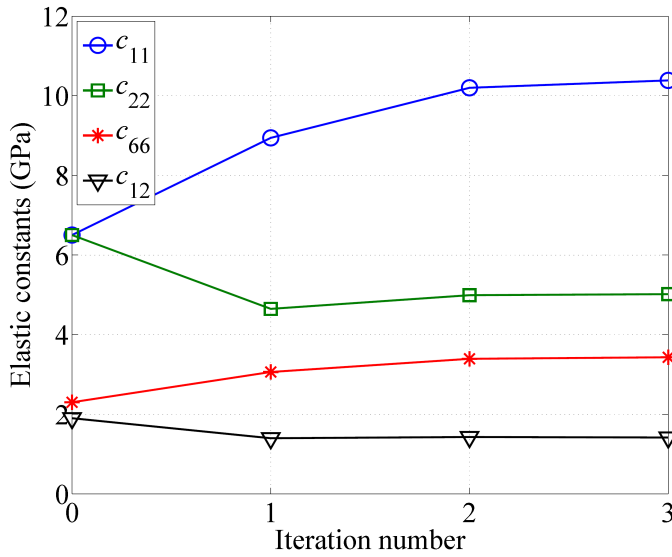


Figure 5: Convergence of the proposed method for identification of the elastic constants of the orthotropic body (with 10% measurement error)

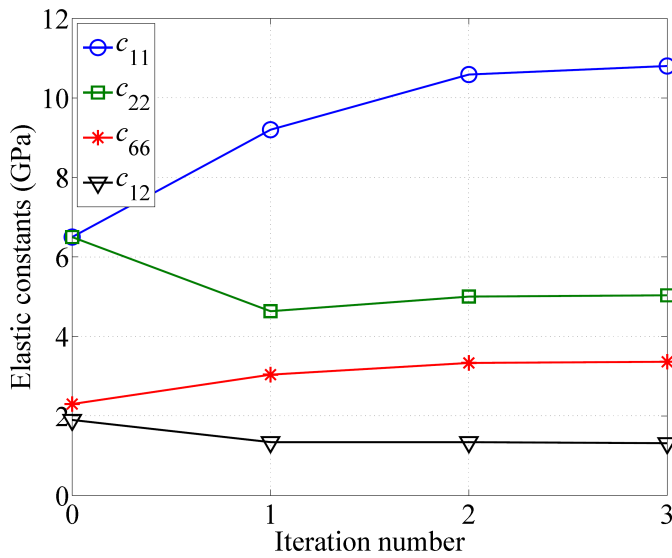


Figure 6: Convergence of the proposed method for identification of the elastic constants of the orthotropic body (with 20% measurement error)

Table 3 lists the identified elastic constants based on the load cases of Fig. 3. The results are also compared with those obtained by the load case of Fig. 4. A close review of the table suggests that the most reliable results are obtained in the case that three tests with the three load cases are conducted and the measurements are used together. The values reported in Table 3, are obtained when the tolerance of measurement errors is considered to be 10%.

In order to have a good appreciation of the usefulness of the obtained vector of initial guesses, a vector with seemingly appropriate initial guesses is selected as follows:

$$\mathbf{C}^0 = [300 \quad 250 \quad 150 \quad 150 \quad -100 \quad 100]^T \text{ GPa} \quad (46)$$

Although the values of the initial guesses given in Eq. (46) are close to the exact values and even the signs of the constants c_{16} and c_{26} are consistent with the actual values, the inverse algorithm would not converge with this vector.

To investigate the effect of measurement errors on the identified elastic constants, a vector of errors with Gaussian distribution is generated. The standard deviation of the errors for the same vector is so chosen as to result in 3, 5, 10, and 20 percent tolerance of measurement error. Table 4, reports the elastic constants predicted for each case. The results of this table are obtained with the three loading cases shown in Fig. 3.

To illustrate graphically the convergence characteristics of the proposed inverse technique, the values of elastic constants are plotted against the iteration number in Fig. 7 and Fig. 8. The results presented in these figures correspond to the case with 10% and 20% tolerance of measurement error, respectively.

5 Conclusions

An inverse method for identification of the elastic constants of 2D orthotropic and anisotropic materials has been presented. The proposed method is based on the BEM and static measurements. This method uses measured data from more than one experiment. In the numerical examples, it was observed that using two or three experiments instead of one experiment results in solutions that are more accurate. Since the number of measured data obtained from several experiments is considerably larger than the number of unknowns, the inverse analysis can be carried out simply even without any regularization. Initial guesses have an important effect on the convergence of the inverse method. A simple, yet effective, method for computing suitable initial guesses has also been presented. It was observed that this method can compute appropriate initial guesses for a problem without any prior information about the elastic constants.

Table 3: The identified elastic constants (in GPa) of the anisotropic body with various load cases (with 10% measurement error)

	c_{11} (error)	c_{22} (error)	c_{66} (error)	c_{12} (error)	c_{16} (error)	c_{26} (error)
Exact value (GPa)	544.8	531.1	243.5	153.6	-81.2	89.7
1-test	386.4 (29.1%)	524.0 (1.3%)	231.8 (4.8%)	85.4 (44.4%)	-20.5 (74.7%)	122.7 (36.8%)
2-test	542.3 (0.5%)	533.3 (0.4%)	236.3 (3.0%)	158.9 (3.4%)	-64.4 (20.7%)	93.3 (4.0%)
3-test	527.1 (3.2%)	526.3 (0.9%)	245.0 (0.6%)	140.0 (8.9%)	-82.7 (1.8%)	92.5 (3.1%)
1-test (all-in-one loading)	559.6 (2.7%)	533.7 (0.4%)	287.5 (18.1%)	136.1 (11.4%)	-110.0 (35.4%)	111.7 (24.5%)

Table 4: Effect of measurement error on the identified elastic constants of the anisotropic body

	c_{11} (error)	c_{22} (error)	c_{66} (error)	c_{12} (error)	c_{16} (error)	c_{26} (error)
Exact value (GPa)	544.8	531.1	243.5	153.6	-81.2	89.7
3% measurement error	539.3 (1.0%)	529.6 (0.3%)	243.9 (0.2%)	149.4 (2.7%)	-81.6 (0.5%)	90.5 (0.9%)
5% measurement error	535.8 (1.7%)	528.6 (0.5%)	244.2 (0.3%)	146.7 (4.5%)	-81.9 (0.9%)	91.1 (1.5%)
10% measurement error	527.1 (3.2%)	526.3 (0.9%)	245.0 (0.6%)	140.0 (8.9%)	-82.7 (1.8%)	92.5 (3.1%)
20% measurement error	510.9 (6.2%)	522.01 (1.7%)	246.5 (1.3%)	127.1 (17.3%)	-84.3 (3.8%)	95.3 (6.2%)

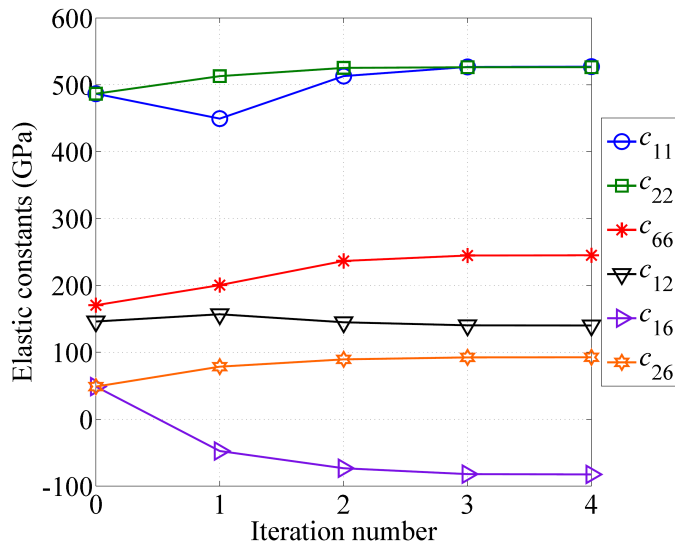


Figure 7: Convergence of the proposed method for identification of the elastic constants of the anisotropic body (with 10% measurement error)

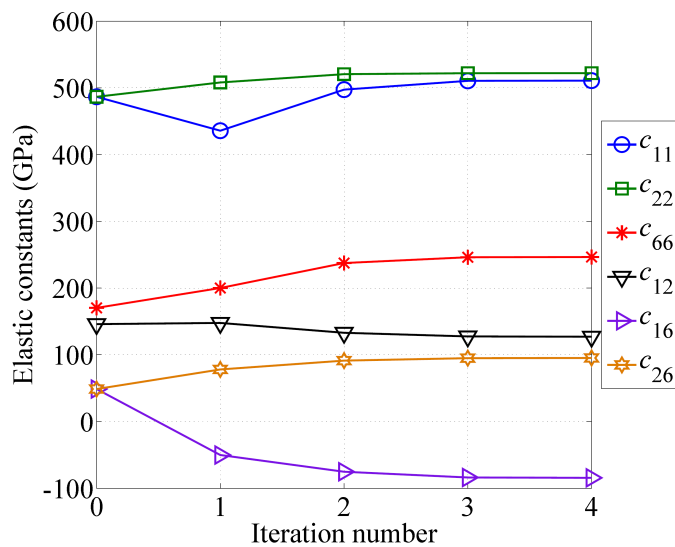


Figure 8: Convergence of the proposed method for identification of the elastic constants of the anisotropic body (with 20% measurement error)

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