

## **Comprehensive Investigation into the Accuracy and Applicability of Monte Carlo Simulations in Stochastic Structural Analysis**

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**Abstract:** Monte Carlo simulation method has been used extensively in probabilistic analyses of engineering systems and its popularity has been growing. While it is widely accepted that the simulation results are asymptotically accurate when the number of samples increases, certain exceptions do exist. The major objectives of this study are to reveal the conditions of the applicability of Monte Carlo method and to provide new insights into the accuracy of the simulation results in stochastic structural analysis. Firstly, a simple problem of a spring with random axial stiffness subject to a deterministic tension is investigated, using normal and lognormal distributions. Analytical solutions for moments of spring elongation are derived through the explicit integration, and numerical solutions by Monte Carlo simulations with different sample sizes are carried out. This study shows analytically that when a normal distribution is assumed, integrals for calculating the moments do not exist and the first moment has a Cauchy principal value, and numerically that Monte Carlo simulation method may fail to yield convergent results for the non-existent moments. Secondly, parallel and series spring systems with normally distributed and correlated axial stiffness values are considered, and the same findings are made as for the single spring problem. Finally, conclusions are made on the importance of checking integrability before Monte Carlo simulations are conducted in the stochastic analysis and the advantages of lognormal distribution for modelling material parameters. Considering that Monte Carlo simulation method has great potential in engineering applications due to the ever-increasing computer power, the findings are crucial for the stochastic analysis in a variety of engineering fields.

**Keywords:** Monte Carlo simulation, stochastic structural analysis, asymptotical

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exactness, normal distribution, lognormal distribution.

## 1 Introduction

Nondeterministic analysis of structures with uncertain properties has attracted more and more attentions recently [Jiang and Han (2007); Moens, Munck, and Vandepitte (2007); Liu, Gao, Song, and Zhang (2011)]. As a well-known probabilistic method, Monte Carlo simulation (MCS) has been widely used to predict the variability of structural responses such as nodal displacements when randomness of parameters of the structure and/or external loads is taken into account [Stefanou (2009)]. In this method, a number of sample models are created according to the given distributions and subsequently analysed to get populations of deterministic responses. Based on these populations, approximate values for the variability of structural responses are calculated using simple relationships of statistics. As one of the early explorations of the method in structural analysis, Astill, Nosseir, and Shinozuka (1972) proposed a MCS method to deal with the stress-wave propagation through a random structure under impact loading. Later, Shinozuka and Leno (1976) developed a probabilistic model for spatial distribution of material properties and applied the MCS to examine the statistical size effect of nonhomogeneous structural systems. Since then, the MCS has been applied to various problems. It has been widely accepted that the accuracy of the results so obtained depends on the number of samples used in the simulation and, with a larger number of samples, the results obtained will be more accurate. In particular, when  $N$  samples are used for one MCS and the simulation is repeated for  $M$  times, the standard deviation of mean values of the  $M$  sets of response samples is inversely proportional to  $\sqrt{N}$ , see, for example, Soong and Grigoriu (1993) and Schuëller (2006). On the other hand, the MCS is easy to implement as no special algorithm other than the one for calculating deterministic responses is required. Because of the asymptotical exactness and the straightforwardness of MCS, the method is often employed to derive reference solutions for stochastic structural analysis problems. Over the years, many researchers have used MCS results to validate new proposed methods, including Liu, Belytchko, and Mani (1986a; 1986b), Ghanem and Spanos (1991a; 1991b), Anders and Hori (1999), and Kim and Inoue (2004).

One of the main tasks of stochastic structural analysis is to determine the statistical features of interested response quantities, such as moments of nodal displacements and member internal force components. From a mathematical point of view, MCS for calculating moments of a structural response may be regarded as a numerical technique of approximating the corresponding integrals [Evans and Swartz (2000)]. Hence, it is necessary to check whether the target integral values exist or not before the simulation is conducted; otherwise, the results will be meaningless if the

integrals do not exist at all. However, this issue has received little attention in the literature so far and it seems that people tend to use the MCS by taking the integrability of integrals, and therefore the meaningfulness of simulation results for granted. There are many examples that the MCS is used and the results taken as reference solutions for assessing the validity of newly proposed methods without questioning the existence of the target values. While in many cases the target values do exist and the MCS provides accurate estimations, there are situations where target values do not exist at all and simulations with relatively small sample sizes may yield seemingly convergent but wrong results, as shown later in this paper. Therefore, there is a need to investigate the applicability of MCS and reveal possible pitfalls.

This paper critically examines the applicability of MCS to stochastic structural analysis considering random material properties. To start with, a simple model of a spring with random axial stiffness subject to a deterministic unit tension is taken as the sample problem. The normal and lognormal distributions are assumed for the axial stiffness of the spring and moments of the spring axial elongation are considered. It is shown analytically that when the spring axial stiffness is normally distributed, the moments of the spring elongation do not exist. Numerical studies with MCS are also carried out for different variations and sample sizes, showing that the MCS is effective only when the integral exists, otherwise the simulation results will become unstable as the sample size increases. Following the discussions on the simple model with a single spring, parallel and series systems of multiple springs are also investigated, and it is shown that when the spring axial stiffness is modelled with normal distribution, the moments of the free-end displacement do not exist either. Finally, concluding remarks are made about the applicability of normal distribution and MCS in stochastic structural analysis.

## 2 Problem description

As the simplest structure containing random material properties, a spring with random axial stiffness  $K$  is considered. The spring is fixed at one end and loaded by a deterministic unit tension at the other end (see Fig. 1). Let the probability density function (PDF) of the random variable  $K$  be  $f_K(k)$ , and the moments of the axial elongation of the spring are to be determined.

By definition, the  $i$ -th moment of a function of the random variable  $K$ ,  $Y = g(k)$ , denoted by  $I_i$ , is given by the following integral:

$$I_i = E[Y^i] = E[g^i(K)] = \int_{R_K} [g(k)]^i f_K(k) dk \quad (1)$$

where  $E[\cdot]$  is the mean value operator, and  $R_K$  is the distribution domain of random



Figure 1: A spring subject to unit tension

variable  $K$ .

When a deterministic unit tension is applied, the spring elongation can be expressed as a function of spring axial stiffness only

$$g(K) = \frac{1}{K} \tag{2}$$

and moments of the axial elongation are given by

$$I_i = \int_{R_K} \left(\frac{1}{k}\right)^i f_K(k) dk, \quad (i = 1, 2, \dots) \tag{3}$$

which are improper when either normal or lognormal distribution is used, because the integration domain  $R_K$  is now unbounded and may contain the singular point  $k = 0$ . It is known that the existence of these integrations depends on the behaviour of the integrand near its singularities and the tail behaviour of the integrand [Evans and Swartz (2000)]. Therefore, the integrals in Eq. (3) may or may not exist, depending on the behaviours of the PDF  $f_K(k)$ . In the following discussion, both the normal distribution and the lognormal distribution will be considered and the existence of these integrals investigated.

### 3 Analytical solutions

In this section, moments of the spring axial elongation given in Eq. (3) will be evaluated analytically for normal and lognormal distributions and major differences between the two random variable models are summarised.

#### 3.1 Normal distribution

The normal distribution [Ang and Tang (1975)] is the best-known and maybe the most widely used probability distribution. In this section, it will be shown analytically that with the normally distributed axial stiffness, moments of the spring axial elongation do not exist.

Suppose the axial stiffness  $K$  is a normal variable with mean value  $\mu$  ( $\mu > 0$ ) and standard deviation  $\sigma$  ( $\sigma > 0$ ), the PDF  $f_K(k)$  will be

$$f_K(k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \quad k \in \mathbf{R}_K = (-\infty, +\infty) \quad (4)$$

Introducing a non-dimensional random variable  $T = K/\mu$ , the PDF  $f_T(t)$  will be

$$f_T(t) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(t-1)^2}{2\delta^2}} \quad t \in \mathbf{R}_T = (-\infty, +\infty) \quad (5)$$

where  $\delta = \sigma/\mu$  is the coefficient of variation (CV) of the axial stiffness  $K$ .

The integrals in Eq. (3) can then be expressed as

$$I_i = \int_{\mathbf{R}_T} \frac{1}{(\mu t)^i} f_T(t) dt = S \int_{-\infty}^{+\infty} \frac{1}{t^i} e^{-\frac{(t^2-2t)}{2\delta^2}} dt, \quad (i = 1, 2, \dots) \quad (6)$$

where  $S$  is a finite quantity defined as

$$S = \frac{1}{\sqrt{2\pi}\delta\mu^i} e^{-\frac{1}{2\delta^2}} \quad (7)$$

To examine the integrability of  $I_i$  in Eq. (6), we split the integral into three parts (see, e.g. [Neri (1971)])

$$I_i = S \int_{-\infty}^{-c} \frac{1}{t^i} e^{-\frac{(t^2-2t)}{2\delta^2}} dt + S \int_{-c}^c \frac{1}{t^i} e^{-\frac{(t^2-2t)}{2\delta^2}} dt + S \int_c^{+\infty} \frac{1}{t^i} e^{-\frac{(t^2-2t)}{2\delta^2}} dt \quad (8)$$

where  $c$  is a small positive value.

The exponential term in integral  $I_i$  can be approximated with the following power series near the singularity point,  $t = 0$ ,

$$e^{-\frac{(t^2-2t)}{2\delta^2}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{t^2-2t}{2\delta^2} \right)^j \quad (9)$$

which is convergent when the following inequality stands:

$$\left| \frac{t^2-2t}{2\delta^2} \right| \leq 1 \quad (10)$$

If the CV is small to moderate, say,  $\delta \leq 1/\sqrt{2} \approx 0.707$ , the solutions of inequality (10) are

$$1 - \alpha \leq t \leq 1 - \beta \quad \text{and} \quad 1 + \beta \leq t \leq 1 + \alpha \quad (11)$$

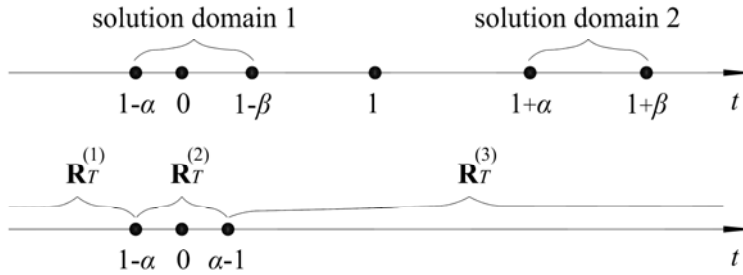


Figure 2: Solutions of inequality and integration sub-domains

where  $\alpha = \sqrt{1 + 2\delta^2}$  and  $\beta = \sqrt{1 - 2\delta^2}$  (see Fig. 2).

It can be easily shown that  $0 < \alpha - 1 < 1 - \beta$ , and therefore, the power series in Eq. (9) is convergent over the domain  $[1 - \alpha, \alpha - 1]$ . Thus, setting  $c = \alpha - 1$ , Eq. (8) becomes

$$I_i = I_i^{(1)} + I_i^{(2)} + I_i^{(3)} = S \int_{\mathbf{R}_T^{(1)}} \frac{1}{t^i} e^{-\frac{(t^2-2t)}{2\delta^2}} dt + S \int_{\mathbf{R}_T^{(2)}} \frac{1}{t^i} e^{-\frac{(t^2-2t)}{2\delta^2}} dt + S \int_{\mathbf{R}_T^{(3)}} \frac{1}{t^i} e^{-\frac{(t^2-2t)}{2\delta^2}} dt \tag{12}$$

with the sub-domains defined as (see Fig. 2)

$$\mathbf{R}_T^{(1)} = (-\infty, 1 - \alpha] \quad \mathbf{R}_T^{(2)} = [1 - \alpha, \alpha - 1] \quad \mathbf{R}_T^{(3)} = [\alpha - 1, +\infty) \tag{13}$$

Thus, integral  $I_i$  can be examined by considering each of the three sub-integrals  $I_i^{(1)}$ ,  $I_i^{(2)}$  and  $I_i^{(3)}$  in Eq. (12) separately.

### 3.1.1 The existence of $I_i^{(2)}$

Substituting Eq. (9) into the sub-integral  $I_i^{(2)}$  defined in Eq. (12) yields

$$I_i^{(2)} = S \int_{1-\sqrt{1+2\delta^2}}^{\sqrt{1+2\delta^2}-1} \sum_{j=0}^{\infty} \frac{1}{\delta^{2j} j!} \left[ t^{j-i} \left( 1 - \frac{t}{2} \right)^j \right] dt \tag{14}$$

Then the sub-integral for the first moment can be computed as

$$I_1^{(2)} = S \int_{1-\sqrt{1+2\delta^2}}^{\sqrt{1+2\delta^2}-1} \frac{1}{t} dt + S \int_0^{\sqrt{1+2\delta^2}-1} \sum_{j=1}^{\infty} \frac{1}{\delta^{2j} j!} \left[ t^{j-1} \left( 1 - \frac{t}{2} \right)^j \right] dt + S \int_0^{\sqrt{1+2\delta^2}-1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{\delta^{2j} j!} \left[ t^{j-1} \left( 1 + \frac{t}{2} \right)^j \right] dt \tag{15}$$

The first term on the right-hand side of Eq. (15), being an integral of an odd function over a symmetric domain, has a Cauchy principal value (CPV) of zero, but the integral itself does not exist (see, e.g. Neri (1971) and Kreyszig (1993)); the second term as an integration of a uniformly convergent series is finite, and its value can be determined by the following convergent term-wise integration (see Kreyszig (1993))

$$S \int_0^{\sqrt{1+2\delta^2}-1} \sum_{j=1}^{\infty} \frac{1}{\delta^{2j} j!} \left[ t^{j-1} \left(1 - \frac{t}{2}\right)^j \right] dt = S \sum_{j=1}^{\infty} \frac{1}{\delta^{2j} j!} \left[ \int_0^{\sqrt{1+2\delta^2}-1} t^{j-1} \left(1 - \frac{t}{2}\right)^j dt \right] \quad (16)$$

in which each integration term can be numerically computed by using Gauss-Legendre quadrature [Press (1992)]; and the third term as an integration of an absolutely convergent series has a finite value, as shown below.

The third term on the right-hand side of Eq. (15) can be rewritten as the following term-wise integration

$$S \int_0^{\sqrt{1+2\delta^2}-1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{\delta^{2j} j!} \left[ t^{j-1} \left(1 + \frac{t}{2}\right)^j \right] dt = S \sum_{j=1}^{\infty} Q_j \quad (17)$$

in which

$$Q_j = \frac{1}{\delta^{2j}} \frac{(-1)^{j-1}}{j!} \left[ \int_0^{\sqrt{1+2\delta^2}-1} t^{j-1} \left(1 + \frac{t}{2}\right)^j dt \right] \quad (18)$$

Consequently,

$$Q_{j+1} = \frac{1}{\delta^{2(j+1)}} \frac{(-1)^j}{(j+1)!} \left[ \int_0^{\sqrt{1+2\delta^2}-1} t^j \left(1 + \frac{t}{2}\right)^{j+1} dt \right] = (-1)^j \left( \frac{1}{\delta^2} \frac{1}{j+1} \right) \left( \frac{1}{\delta^{2j}} \frac{1}{j!} \right) \left\{ \int_0^{\sqrt{1+2\delta^2}-1} \left[ t \left(1 + \frac{t}{2}\right) \right] \left[ t^{j-1} \left(1 + \frac{t}{2}\right)^j \right] dt \right\} \quad (19)$$

Functions,  $t(1+t/2)$  and  $t^{j-1}(1+t/2)^j$ , are both continuous and positive over the integration domain  $[0, \sqrt{1+2\delta^2}-1]$ , so

$$|Q_{j+1}| \leq \left( \frac{1}{\delta^2} \frac{1}{j+1} \right) \left( \frac{1}{\delta^{2j}} \frac{1}{j!} \right) \left\{ G \int_0^{\sqrt{1+2\delta^2}-1} \left[ t^{j-1} \left(1 + \frac{t}{2}\right)^j \right] dt \right\} = \left( \frac{G}{\delta^2} \frac{1}{j+1} \right) |Q_j| \quad (20)$$

where

$$G = \max \left\{ t \left( 1 + \frac{t}{2} \right), t \in \left[ 0, \sqrt{1 + 2\delta^2} - 1 \right] \right\} \tag{21}$$

Because both  $G$  and  $\delta$  are of fixed values for a given distribution, there exists a positive integer,  $L$ , such that the following relationship holds

$$|Q_{j+1}| \leq \left( \frac{G}{\delta^2} \frac{1}{j+1} \right) |Q_j| < |Q_j| \tag{22}$$

when  $j \geq L$ .

Hence,  $\sum_{j=1}^{\infty} Q_j$  converges to a finite value.

Therefore, sub-integral  $I_1^{(2)}$  given by Eq. (15) does not exist but has a finite CPV.

The sub-integrals for the second moment can be expressed as

$$\begin{aligned} I_2^{(2)} = & S \int_{1-\sqrt{1+2\delta^2}}^{\sqrt{1+2\delta^2}-1} \frac{1}{t^2} dt + \frac{S}{\delta^2} \int_{1-\sqrt{1+2\delta^2}}^{\sqrt{1+2\delta^2}-1} \frac{1}{t} dt - \frac{S}{\delta^2} \int_{1-\sqrt{1+2\delta^2}}^{\sqrt{1+2\delta^2}-1} \frac{1}{2} dt \\ & + S \int_0^{\sqrt{1+2\delta^2}-1} \sum_{j=2}^{\infty} \frac{1}{\delta^{2j} j!} \left[ t^{j-2} \left( 1 - \frac{t}{2} \right)^j \right] dt \\ & + S \int_0^{\sqrt{1+2\delta^2}-1} \sum_{j=2}^{\infty} \frac{(-1)^{j-2}}{\delta^{2j} j!} \left[ t^{j-2} \left( 1 + \frac{t}{2} \right)^j \right] dt \end{aligned} \tag{23}$$

On the right-hand side of Eq. (23), the first two integrals do not exist; the other integrals can be shown to be finite. Therefore, sub-integral  $I_2^{(2)}$  does not exist.

Following a similar procedure as for  $I_2^{(2)}$ , it can be shown that all the higher-order moments,  $I_i^{(2)}$  ( $i \geq 3$ ), do not exist.

### 3.1.2 The existence of $I_i^{(1)}$

The  $I_i^{(1)}$  defined in Eq. (12) can be rewritten as the following Riemann-Stieltjes integral [Stroock (1994)]

$$I_i^{(1)} = \frac{1}{\mu^i} \int_{\sqrt{1+2\delta^2}-1}^{+\infty} \frac{1}{t^i} dF_T(t) \tag{24}$$

where  $F_T(t)$  is the cumulative distribution function (CDF) of random variable  $T$ .

Because  $F_T(t)$  is a monotonic function and  $1/t^i$  is continuous and bounded over the integration domain  $\left[ \sqrt{1 + 2\delta^2} - 1, +\infty \right)$ ,  $I_i^{(1)}$  is integrable to a finite value (see, e.g. Stroock (1994)).



3.1.3 The existence of  $I_i^{(3)}$

Sub-integral  $I_i^{(3)}$  defined in Eq. (12) can be rewritten as the following Riemann-Stieltjes integral

$$I_i^{(3)} = \frac{1}{\mu^i} \int_{-\infty}^{1-\sqrt{1+2\delta^2}} \frac{1}{t^i} dF_T(t) \tag{25}$$

which, like  $I_i^{(1)}$  in Eq. (24), can also be proven to be finite.

As  $I_i^{(2)}$  does not exist and both  $I_i^{(1)}$  and  $I_i^{(3)}$  are finite,  $I_i$  does not exist. From the above investigations, it is known that  $I_1$  has a CPV. For the calculation of the CPV of  $I_1$ , the sub-integrals  $I_1^{(1)}$  and  $I_1^{(3)}$  can be numerically evaluated by using Gauss-Laguerre quadrature [Press (1992)], and  $I_1^{(2)}$  by the convergent power series defined in Eqs. (16) and (17).

3.2 Lognormal distribution

The use of normal distribution for describing random material properties is questionable, as outcomes of normal random variables can be negative, whereas the material properties are positive in nature. Lognormal distribution as another option can guarantee a positive value and thus is more suitable for describing random material properties [Sudret and Der Kiureghian (2000)].

Suppose the logarithm of the stiffness  $K$  be a normal distribution  $N(\mu_{\ln K}, \sigma_{\ln K})$ , the PDF  $f_K(k)$  will be [Nowak and Collins (2000)]

$$f_K(k) = \frac{1}{\sqrt{2\pi\sigma_{\ln K}k}} e^{-\frac{(\ln k - \mu_{\ln K})^2}{2\sigma_{\ln K}^2}} \quad k \in \mathbf{R}_K = (0, +\infty) \tag{26}$$

The mean value and standard deviation of  $\ln K$ ,  $\mu_{\ln K}$  and  $\sigma_{\ln K}$ , can be expressed as

$$\mu_{\ln K} = \ln \mu - \frac{1}{2} \ln(1 + \delta^2) \tag{27}$$

$$\sigma_{\ln K}^2 = \ln(1 + \delta^2) \tag{28}$$

in which  $\mu$  and  $\sigma$  are the mean value and standard deviation of  $K$ , respectively, and  $\delta$  is the CV.

Note that, for the PDF defined in Eq. (26), the parameter given in Eq. (27),  $\mu_{\ln K}$ , has the unit of  $\ln K$ . To eliminate potential problems caused by this inconsistency and facilitate the following discussions, the same non-dimensional random variable  $T = K/\mu$  introduced earlier is used and the PDF for  $T$  is

$$f_T(t) = \frac{1}{\sqrt{2\pi\sigma_{\ln T}t}} e^{-\frac{(\ln t - \mu_{\ln T})^2}{2\sigma_{\ln T}^2}} \quad t \in \mathbf{R}_T = (0, +\infty) \tag{29}$$

where  $\mu_{\ln T}$  and  $\sigma_{\ln T}$  are defined as

$$\mu_{\ln T} = -\frac{1}{2} \ln(1 + \delta^2) \tag{30}$$

$$\sigma_{\ln T}^2 = \ln(1 + \delta^2) \tag{31}$$

Moments of axial elongation then can be expressed as

$$I_i = \int_0^{+\infty} \frac{1}{(\mu t)^i} f_T(t) dt = \int_0^{+\infty} \frac{1}{(\mu t)^i} \frac{1}{\sqrt{2\pi}\sigma_{\ln T} t} e^{-\frac{(\ln t - \mu_{\ln T})^2}{2\sigma_{\ln T}^2}} dt, \quad (i = 1, 2, \dots) \tag{32}$$

By letting  $t = e^s$ , integral  $I_i$  can then be obtained as

$$\begin{aligned} I_i &= \frac{1}{\sqrt{2\pi}\sigma_{\ln T}\mu^i} \int_{-\infty}^{+\infty} e^{-\frac{(s - \mu_{\ln T})^2}{2\sigma_{\ln T}^2} - is} ds = \frac{1}{\mu^i} e^{\frac{i}{2}(i\sigma_{\ln T}^2 - 2\mu_{\ln T})} \\ &= \frac{1}{\mu^i} (1 + \delta^2)^{\frac{i(i+1)}{2}}, \quad (i = 1, 2, \dots) \tag{33} \end{aligned}$$

From Eq. (33) it can be seen that, if the spring axial stiffness is a lognormal random variable, all the moments of the elongation will be finite.

### 3.3 Discussions

It has been a known fact that the use of normal distribution for describing random material properties is problematic because the possible occurrence of negative values contradicts with the physical reality. In this regard, the lognormal distribution is more reasonable considering that all the possible values are positive and therefore physically meaningful.

The above investigation of the simple spring problem reveals another problem with the normal distribution model that the improper integrals for the moments of the spring elongation do not exist at all if it is used to represent the random axial stiffness. This implies that, for any given mean value and standard deviation of the random axial stiffness, the mean value and standard deviation of the spring elongation do not exist. Clearly, this is in conflict with what one may expect. In contrast, with the lognormal distribution, all the moments of the spring elongation are finite functions of the mean value and standard deviation of the axial stiffness. This finding provides another reason for using the lognormal distribution for modelling random material parameters in preference to the normal distribution.

While a lack of integrability of an integral may reveal itself in instability of the approximations as the sample size of the MCS increases, it may take a very large number of samples before the instability can be detected [Evans and Swartz (2000)].

Over the years, the normal distribution has been used extensively in the stochastic structural analysis, often with the MCS. However, to the best knowledge of the present authors, so far there has no study reported on the numerical instability problems associated with normal distribution model yet. A possible explanation for this is that a large number of repeated analyses of even a very small structure may take a substantial amount of computation, and it may take too long for the results to become unstable. For the simple problem considered, it has been shown analytically that the moments of the spring elongation do not exist when the stiffness is modelled as a normal variable. From the above discussion, it is expected that the MCS will produce unstable results when the sample size is large enough. In the following section, numerical results will be presented to confirm the above findings.

#### 4 Numerical solutions

In this section, the mean value ( $I_1$ ) and the second moment ( $I_2$ ) of the spring elongation will be calculated through MCS, and numerical results compared with the solutions from direct integrations. The effect of three factors will be investigated: (1) the distribution type, both the normal and lognormal distribution models will be used; (2) the coefficient of variation, three values of 0.15, 0.20 and 0.25 will be considered; (3) the sample size in MCS, the range considered is from  $10^3$  to  $10^{11}$ .

Because the numerical results of  $I_1$  and  $I_2$  obtained with MCS are also random variables (see, for example, Ang and Tang (1975)), MCS for each sample size are repeated for ten times with different random series to get a more realistic assessment of simulation results.

##### 4.1 Random number generating

As ten different random series with sample sizes up to  $10^{11}$  will be used in the MCS, the random number generator used must be capable of producing a random number series with a period larger than  $10^{12}$ . The most commonly-used linear congruential generator proposed by Lehmer (1951) only has a maximum period of about  $2.1 \times 10^9$  on a 32-bits computer [Gentle (2003)]. L'Ecuyer (1988) has proposed a combined congruential generator with the maximum period in the order of  $10^{18}$  for a 32-bits computer. In this study, the ran2 generator presented in [Press, Flannery, Teukolsky, and Vetterling (1992)], which is based on the combined congruential generator, is applied to produce uniformly-distributed random numbers for the MCS computation.

**4.2 Singularity-removed integral**

By removing a small domain containing the singularity, we define the following integral

$$\tilde{I}_i(\varepsilon) = \int_{\substack{k \in \mathbb{R} \\ |k| > \varepsilon\mu}} \left(\frac{1}{k}\right)^i f_K(k) dk, \quad \varepsilon \in (0, +\infty) \tag{34}$$

which will be referred to as singularity-removed (SR) integral. Then the singular integral  $I_i$  in Eq. (3) can be approximated by  $\tilde{I}_i(\varepsilon)$  as [Neri (1971)]

$$I_i = \lim_{\varepsilon \rightarrow 0^+} \tilde{I}_i(\varepsilon) \tag{35}$$

When the axial stiffness  $K$  is a normal variable,  $\tilde{I}_i(\varepsilon)$  can be expressed in terms of non-dimensional variable as

$$\tilde{I}_i(\varepsilon) = \frac{1}{\sqrt{2\pi}\delta\mu^i} \int_{-\infty}^{-\varepsilon} \frac{1}{t^i} e^{-\frac{(t-1)^2}{2\delta^2}} dt + \frac{1}{\sqrt{2\pi}\delta\mu^i} \int_{\varepsilon}^{+\infty} \frac{1}{t^i} e^{-\frac{(t-1)^2}{2\delta^2}} dt \tag{36}$$

in which, both integrals can be computed by using Gauss-Laguerre quadrature.

When the axial stiffness  $K$  is a lognormal variable,  $\tilde{I}_i(\varepsilon)$  can be expressed as

$$\begin{aligned} \tilde{I}_i(\varepsilon) &= \int_{\varepsilon}^{+\infty} \frac{1}{(\mu t)^i} \frac{1}{\sqrt{2\pi}\sigma_{\ln T} t} e^{-\frac{(\ln t - \mu_{\ln T})^2}{2\sigma_{\ln T}^2}} dt = \frac{1}{\sqrt{2\pi}\sigma_{\ln T} \mu^i} \int_{\ln \varepsilon}^{+\infty} e^{-\frac{(s - \mu_{\ln T})^2}{2\sigma_{\ln T}^2} - is} ds \\ &= \frac{1}{2\mu^i} e^{\frac{i}{2}(\sigma_{\ln T}^2 - 2\mu_{\ln T})} \left[ 1 - \Phi \left( \frac{\sigma_{\ln T} i}{\sqrt{2}} - \frac{\mu_{\ln T}}{\sqrt{2}\sigma_{\ln T}} + \frac{\ln \varepsilon}{\sqrt{2}\sigma_{\ln T}} \right) \right] \\ &= \frac{1}{2\mu^i} (1 + \delta^2)^{\frac{i(i+1)}{2}} \left[ 1 - \Phi \left( \frac{1}{\sqrt{2\ln(1 + \delta^2)}} \ln \left[ \varepsilon (1 + \delta^2)^{i + \frac{1}{2}} \right] \right) \right] \end{aligned} \tag{37}$$

in which,  $s = \ln t$ , and  $\Phi(\cdot)$  is the error function defined as [Kreyszig (1993)]

$$\Phi(x) = \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv \tag{38}$$

**4.3 Numerical results**

The MCS is conducted for the mean value and the second moment of the spring axial elongation and numerical results are presented in Figs. 3 to 6. Because the

integrals involved have a singularity, the result by MCS is influenced significantly by the sample closest to the singular point among all the sample values generated for random variable. Therefore, the numerical results are presented in plots with horizontal axis for the minimum absolute normalised value of the samples,  $\varepsilon^*$ . Values of SR integral based on the expressions in Eqs. (36) and (37) are also shown in the plots with the parameter  $\varepsilon$  as  $\varepsilon^*$ .

#### 4.3.1 The normal distribution case

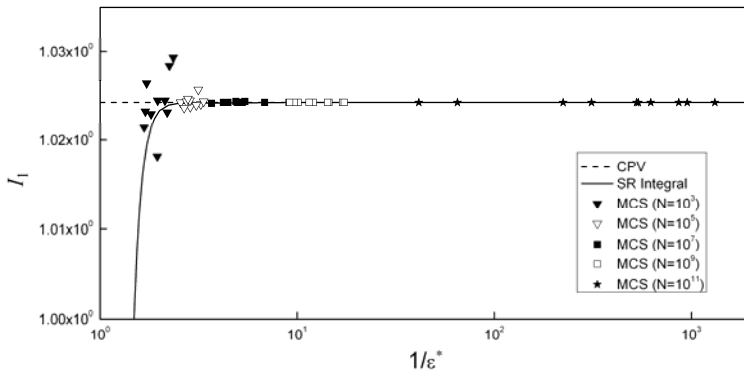
Figures 3 and 4 show that when a bigger variation of the spring stiffness is assumed or a larger sample size for MCS is used, the minimum absolute normalised value of the samples,  $\varepsilon^*$ , will be smaller. For a given sample size, the simulation results have bigger dispersion as CV increases from 0.15 to 0.25. For the case of CV=0.15, as the sample size increases, the numerical results of  $I_1$  seem to converge to the CPV, and those of  $I_2$  to a fixed value, at least for the range considered. For the other two cases with CV=0.20 and CV=0.25, however, the MCS results have different convergence behaviors. First, let us consider  $I_1$  (Fig. 3). When CV=0.20, the MCS results nearly converge to the CPV for  $N=10^7$ , but the results for  $N=10^9$  and  $N=10^{11}$  show bigger deviations. When CV=0.25, the MCS produces no converged solution at all. As for the second moment,  $I_2$ , MCS results increase continuously with  $N$ , and the SR integral gives a lower bound.

#### 4.3.2 The lognormal distribution case

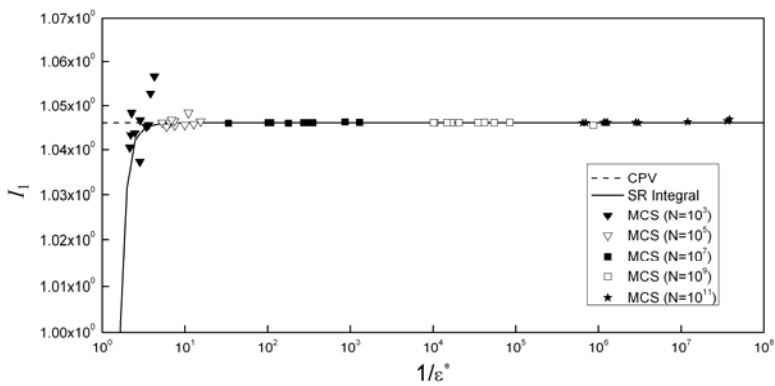
Figures 5 and 6 show that  $\varepsilon^*$  will be smaller for a bigger variation of the spring stiffness or a larger sample size, but the decreasing rate is much smaller than that in the normal distribution case. And, as the sample size increases, the numerical simulation results for both  $I_1$  and  $I_2$  converge steadily to the exact values. It is noteworthy that the CV of random stiffness has insignificant influence on the convergence rate of the simulation results for both  $I_1$  and  $I_2$ .

## 5 Asymptotical exactness of MCS

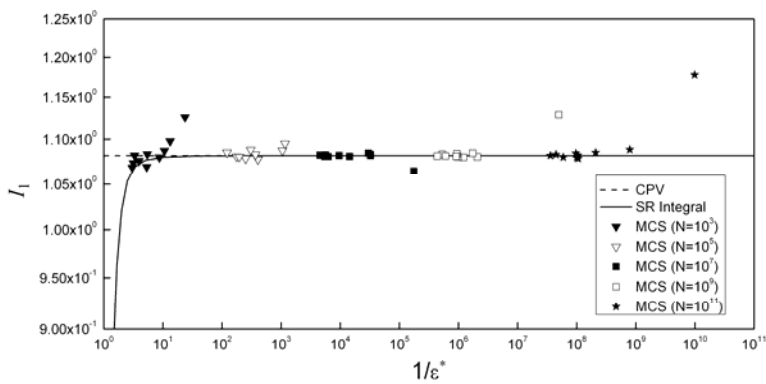
The numerical results presented in the last section show that, when the spring stiffness is a random variable with normal distribution, MCS fails to produce convergent results for  $I_1$  and  $I_2$ , because both of these two values actually do not exist at all. This finding may have significant implications for the modelling of random structures, considering that the MCS is a widely-used technique for deriving reference solutions in the stochastic structural analysis, and the normal distribution is often assumed for the random material properties (see, for example, Liu, Belytchko, and Mani (1986a; 1986b), Ghanem and Spanos (1991a; 1991b), Anders and Hori (1999), and Kim and Inoue (2004)).



(a) The case of CV = 0.15

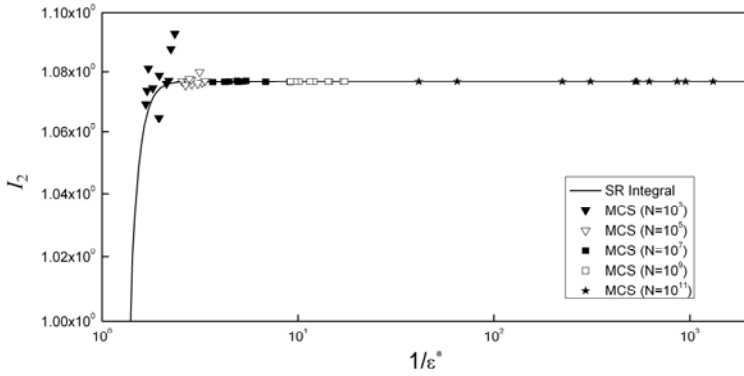


(b) The case of CV = 0.20

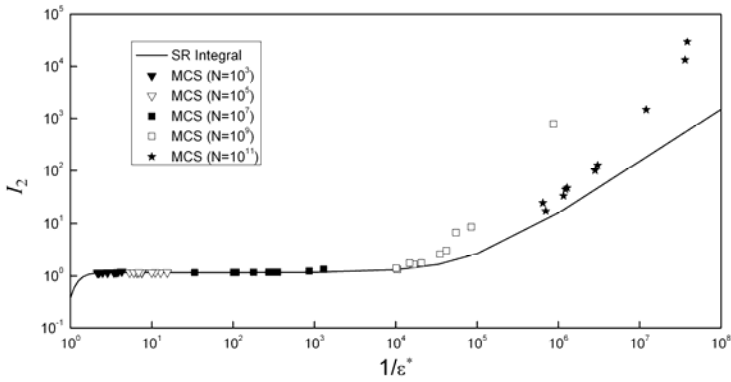


(c) The case of CV = 0.25

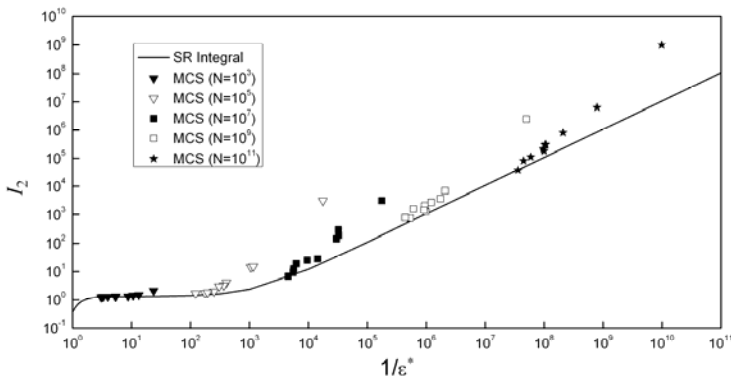
Figure 3: Results of  $I_1$  for the normal distribution



(a) The case of CV = 0.15

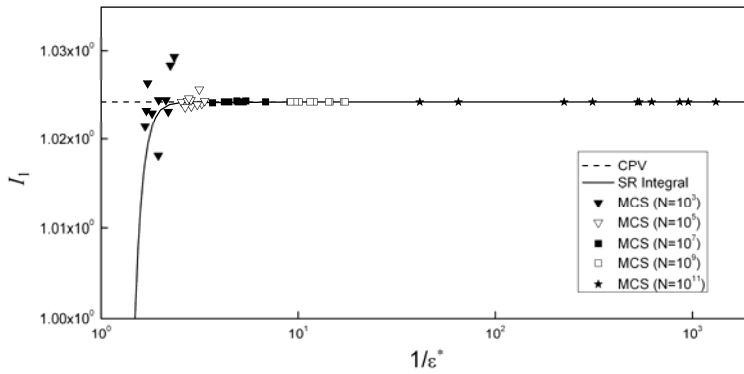


(b) The case of CV = 0.20

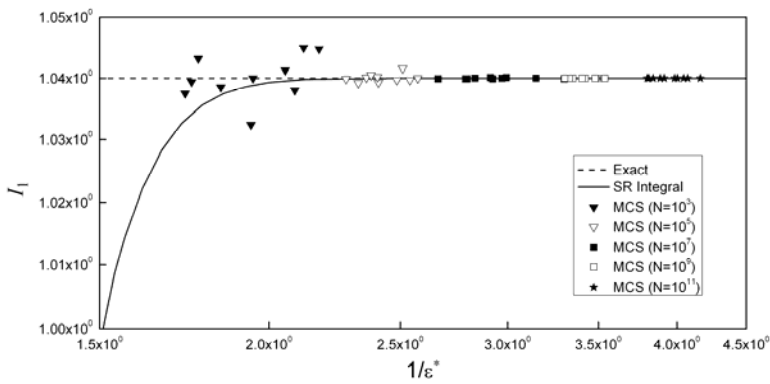


(c) The case of CV = 0.25

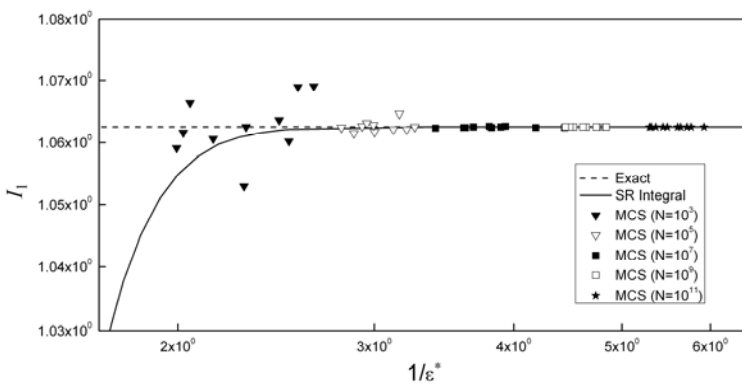
Figure 4: Results of  $I_2$  for the normal distribution



(a) The case of CV = 0.15



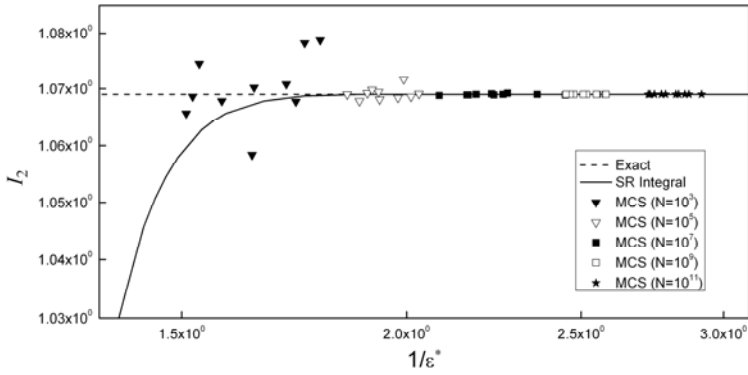
(b) The case of CV = 0.20



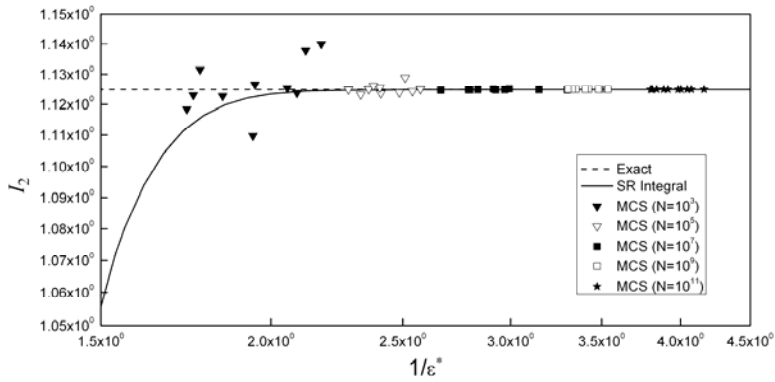
(c) The case of CV = 0.25

Figure 5: Results of  $I_1$  for the lognormal distribution

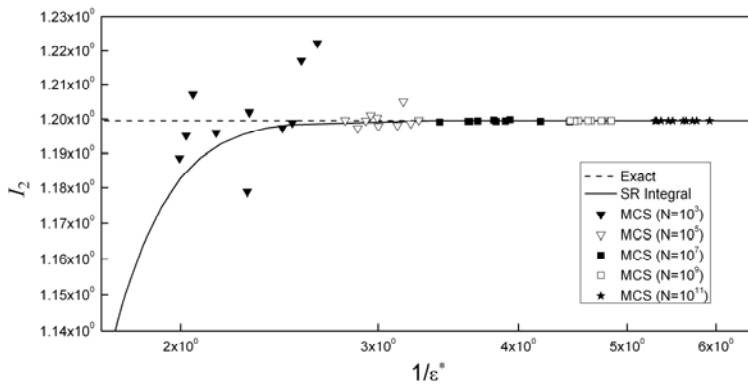




(a) The case of  $CV = 0.15$



(b) The case of  $CV = 0.20$



(c) The case of  $CV = 0.25$

Figure 6: Results of  $I_2$  for the lognormal distribution

The asymptotical exactness of MCS results is one of the main reasons for its wide use. In some reported studies, only 5,000 to 10,000 samples were used in the MCS to derive reference solutions. It has been shown analytically in Section 3.1 that, when normal distribution is used the moments of the structural response do not exist for any CV the random material property may take. The numerical results in last section have further shown that when the integrals do not exist, the MCS may produce plausible results of finite magnitude no matter how many samples are used, and while these results may seem reasonable, they are actually meaningless.

One may wonder how this can happen and whether the general statement about the asymptotical exactness of MCS is valid for all stochastic analysis. To answer these questions, it may be helpful to look into the basis of MCS.

Let  $Y$  be a random variable of interest, and samples of  $Y$  obtained from  $N$  simulated realization are denoted by  $\{y_i\}, i = 1, \dots, N$ . The first moment  $\mu_Y = E[Y]$  and the second central moment  $\sigma_Y^2 = E[(Y - \mu_Y)^2]$  of random variable  $Y$  can be estimated from the following sample mean value and sample second central moment

$$\hat{\mu}_Y = \frac{1}{N} \sum_{i=1}^N y_i \tag{39}$$

$$\hat{\sigma}_Y^2 = \frac{1}{N-1} \sum_{i=1}^N (y_i - \hat{\mu}_Y)^2 \tag{40}$$

Both  $\hat{\mu}_Y$  and  $\hat{\sigma}_Y^2$  can be treated as random variable, and it can be proven (see Ang and Tang (1975)) that  $\hat{\mu}_Y$  and  $\hat{\sigma}_Y^2$  are unbiased estimators for  $\mu_Y$  and  $\sigma_Y^2$  respectively, that is

$$E[\hat{\mu}_Y] = \mu_Y \tag{41}$$

$$E[\hat{\sigma}_Y^2] = \sigma_Y^2 \tag{42}$$

Furthermore,  $\hat{\mu}_Y$  and  $\hat{\sigma}_Y^2$  have variances given by (see Ang and Tang (1975))

$$\text{Var}[\hat{\mu}_Y] = \frac{\sigma_Y^2}{N} \tag{43}$$

$$\text{Var}[\hat{\sigma}_Y^2] = \frac{\sigma_Y^4}{N} \left( \frac{\mu_{4,Y}}{\sigma_Y^4} - \frac{N-3}{N-1} \right) \tag{44}$$

where  $\mu_{4,Y}$  is the fourth central moment of the random variable  $Y$ ,  $\mu_{4,Y} = E[(Y - \mu_Y)^4]$ .

From Eqs. (41) and (43), the CV of  $\hat{\mu}_Y$  is

$$\frac{\sqrt{\text{Var}[\hat{\mu}_Y]}}{E[\hat{\mu}_Y]} = \frac{1}{\sqrt{N}} \frac{\sigma_Y}{\mu_Y} \tag{45}$$

And similarly, from Eqs. (42) and (44), the CV of  $\hat{\sigma}_Y^2$  is

$$\frac{\sqrt{\text{Var} [\hat{\sigma}_Y^2]}}{\text{E} [\hat{\sigma}_Y^2]} = \frac{1}{\sqrt{N}} \sqrt{\frac{\mu_{4,Y}}{\sigma_Y^4} - \frac{N-3}{N-1}} \tag{46}$$

For a given distribution of random variable  $Y$ ,  $\mu_Y$ ,  $\sigma_Y$  and  $\mu_{4,Y}$  are all fixed values. Therefore, Eqs. (45) and (46) clearly show that the ‘‘uncertainties’’ in the estimators of  $\mu_Y$  and  $\sigma_Y^2$  (i.e.  $\hat{\mu}_Y$  and  $\hat{\sigma}_Y^2$ ) will decrease as the sample size  $N$  increases. When  $N$  is a large number, the CVs of  $\hat{\mu}_Y$  and  $\hat{\sigma}_Y^2$  are inversely proportional to  $\sqrt{N}$ , indicating that simulation results will approach the exact values as  $N$  approaches infinity.

It should be pointed out that, however, the above expressions and statements are for the simulation of the random variable  $Y$ , but not for an arbitrary function of  $Y$ ; in other words, the asymptotical exactness is a feature of MCS when applied for the simulation of the variable itself.

For a linear function of the random variable  $Y$ , it can be easily found that similar relationships as given in Eqs. (41) to (46) will hold for the sample mean value and sample second central moment of the function. Then the general statement made on the simulation accuracy of the random variable can also be made for the simulation accuracy of the function. Therefore, MCS is applicable for estimating the first moment and second central moment of any linear function of  $Y$ , and the results will be asymptotically exact.

For a nonlinear function of random variable  $Y$ , however, the above statement does not necessarily stand. Taking the sample problem of this paper as an example, the spring elongation is proportional to the reciprocal of random axial stiffness  $K$ . Based on the ten repeated MCS, we can roughly check whether the relationships in Eqs. (45) and (46) still hold for the spring elongation  $g(K)$ . For each set of the simulation results of  $g(K)$ , the first moment  $\hat{\mu}_{g(K)}$  and the second central moment  $\hat{\sigma}_{g(K)}^2$  can be calculated using the following expressions

$$\hat{\mu}_{g(K)} = \frac{1}{N} \sum_{i=1}^N g(k_i) \tag{47}$$

$$\hat{\sigma}_{g(K)}^2 = \frac{1}{N-1} \sum_{i=1}^N (g(k_i) - \hat{\mu}_{g(K)})^2 \tag{48}$$

Based on  $J$  sets of simulation results, the mean values and variances of the estimators can be calculated with  $\hat{\mu}_{g(K)}^{(j)}$  and  $\hat{\sigma}_{g(K)}^2{}^{(j)}$  ( $j = 1, 2, \dots, J$ ) as

$$\bar{\text{E}} [\hat{\mu}_{g(K)}] = \frac{1}{J} \sum_{j=1}^J (\hat{\mu}_{g(K)}^{(j)}) \tag{49}$$

$$\overline{\text{Var}} [\hat{\mu}_{g(K)}] = \frac{1}{J-1} \sum_{j=1}^J \left( (\hat{\mu}_{g(K)})^{(j)} - \bar{E} [\hat{\mu}_{g(K)}] \right)^2 \tag{50}$$

$$\bar{E} [\hat{\sigma}_{g(K)}^2] = \frac{1}{J} \sum_{j=1}^J \left( \hat{\sigma}_{g(K)}^2 \right)^{(j)} \tag{51}$$

$$\overline{\text{Var}} [\hat{\sigma}_{g(K)}^2] = \frac{1}{J-1} \sum_{j=1}^J \left( \left( \hat{\sigma}_{g(K)}^2 \right)^{(j)} - \bar{E} [\hat{\sigma}_{g(K)}^2] \right)^2 \tag{52}$$

Using the above expressions and the ten sets of simulation results, mean values and variances of the estimators can be obtained. Curves of the corresponding CVs of the estimators versus  $\sqrt{N}$  are then drawn in Fig. 7. The curves should be straight lines with negative slopes if the CV is inversely proportional to  $\sqrt{N}$ , which can be easily found in the case of lognormal distribution. In the case of normal distribution, however, significant deviations from the straight lines are observed. It is noticed that the deviation becomes bigger for larger sample sizes and CV values. On the other hand, the deviation for  $\hat{\sigma}^2$  is always bigger than that for  $\hat{\mu}$ .

The results presented demonstrate that the MCS can yield convergent and reasonable estimations and may also give meaningless values, depending on the random variable model used. When the analysis problem to be solved involves integrals which are non-integrable, the MCS may not produce any reference solutions, because they simply do not exist at all. This study has shown that this may occur for the simple spring model when the axial stiffness is treated as a normal variable.

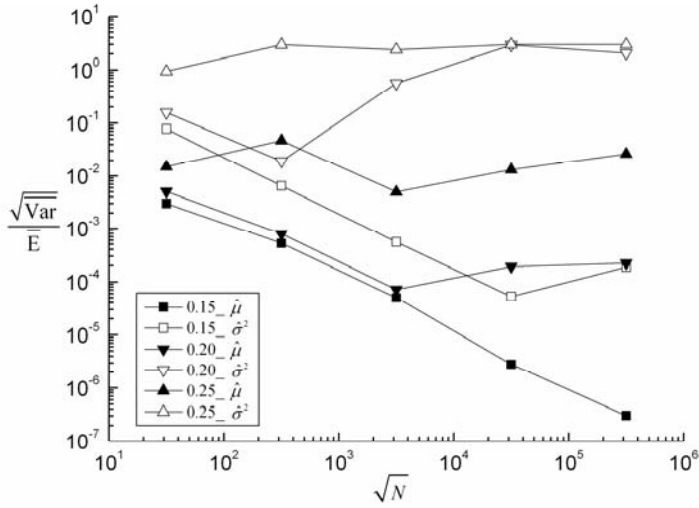
## 6 Systems with multiple random material properties

It has been found that, for a single spring with normally distributed axial stiffness, MCS may fail in stochastic structural analysis because the moments of the axial elongation do not exist. In this section, structural systems with normally distributed and correlated axial stiffness will be considered and this finding is to be extended to the stochastic analysis of general structures.

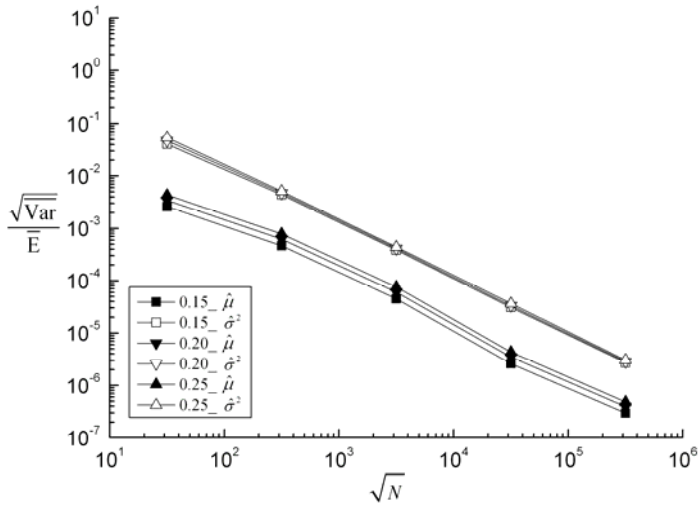
For the sake of simplicity, a parallel system and a series system will be investigated first, and the existence of the moments of system's response will be examined analytically and then checked numerically by MCS.

### 6.1 Parallel system

Figure 8 shows a parallel spring system with  $n$  springs of axial stiffness,  $K_j$  ( $j = 1, 2, \dots, n$ ). Assume  $K_j$  are  $n$ -dimensional normal variables, i.e.  $\{K_1, K_2, \dots, K_n\} \sim N(\{\mu_1, \mu_2, \dots, \mu_n\}, \Sigma)$ , where  $\Sigma$  is the covariance matrix.



(a) Case of normal distribution



(b) Case of lognormal distribution

Figure 7: Coefficients of variations of estimators obtained by MCS

The  $i$ -th moment of axial elongation of the system can be computed as

$$I_i = E \left[ \left( \frac{1}{K_1 + K_2 + \dots + K_n} \right)^i \right] = E \left[ \left( \frac{1}{K_{sum}} \right)^i \right] \tag{53}$$

where  $K_{sum} = K_1 + K_2 + \dots + K_n$  is known to be a new normal variable with a mean value of  $\mu_1 + \mu_2 + \dots + \mu_n$  and a variance of  $\{1, 1, \dots, 1\} \Sigma \{1, 1, \dots, 1\}^T$  (see Rao (1973)).

Consequently, all the conclusions drawn for the single spring with normally distributed axial stiffness hold for the case of parallel systems with multiple correlated springs, and therefore the MCS will fail for stochastic structural analysis in this case.

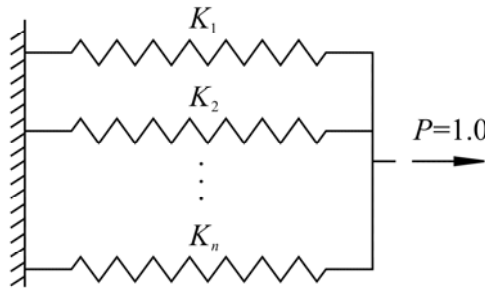


Figure 8: A parallel spring system subjected to a unit tension

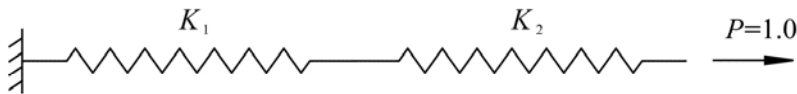


Figure 9: A series spring system subjected to a unit tension

### 6.2 Series system

Figure 9 shows a series spring system with two normally distributed axial stiffness,  $K_1$  and  $K_2$ , between which the correlation coefficient is  $r$ .

The  $i$ -th moment of axial elongation is given by

$$I_i = \int_{R_{K_2}} \int_{R_{K_1}} \left( \frac{1}{k_1} + \frac{1}{k_2} \right)^i f_{K_1, K_2}(k_1, k_2) dk_1 dk_2$$

$$= \sum_{j=0}^i C_i^j \int_{R_{K_2}} \int_{R_{K_1}} \left( \frac{1}{k_1} \right)^j \left( \frac{1}{k_2} \right)^{i-j} f_{K_1, K_2}(k_1, k_2) dk_1 dk_2 \tag{54}$$

where  $C_i^j$  is the binomial coefficient [Kreyszig (1993)], and  $f_{K_1, K_2}(k_1, k_2)$  is the joint normal PDF given by [Rao (1973)]

$$f_{K_1, K_2}(k_1, k_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)} \left[ \frac{(k_1-\mu_1)^2}{\sigma_1^2} - 2r\frac{(k_1-\mu_1)(k_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(k_2-\mu_2)^2}{\sigma_2^2} \right]}$$

(|r| < 1) (55)

The PDF  $f_{K_1, K_2}(k_1, k_2)$  can be rewritten as

$$f_{K_1, K_2}(k_1, k_2) = \sigma_2 \left( \frac{1}{\sqrt{2\pi\bar{\sigma}_1}} e^{-\frac{(\bar{k}_1-\bar{\mu}_1)^2}{2\bar{\sigma}_1^2}} \right) \left( \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{(k_2-\mu_2)^2}{2\sigma_2^2}} \right) = \sigma_2 f_{\bar{K}_1}(\bar{k}_1) f_{K_2}(k_2)$$

(56)

where  $\bar{K}_1$  is an artificial random variable

$$\bar{K}_1 = (\sigma_2 K_1 - r\sigma_1 K_2) \sim N(\bar{\mu}_1, \bar{\sigma}_1^2) \quad \mathbf{R}_{\bar{K}_1} = (-\infty, +\infty)$$

(57)

with

$$\bar{\mu}_1 = \sigma_2\mu_1 - r\sigma_1\mu_2$$

(58)

$$\bar{\sigma}_1^2 = (1-r^2)\sigma_1^2\sigma_2^2$$

(59)

Substituting Eqs. (56) and (57) into Eq. (54) yields

$$\begin{aligned} I_i &= \sum_{j=0}^i C_i^j \int_{\mathbf{R}_{K_2}} \left( \frac{1}{k_2} \right)^{i-j} \left[ \int_{\mathbf{R}_{\bar{K}_1}} \left( \frac{\sigma_2}{\bar{k}_1 + r\sigma_1 k_2} \right)^j f_{\bar{K}_1}(\bar{k}_1) d\bar{k}_1 \right] f_{K_2}(k_2) dk_2 \\ &= \sum_{j=0}^i C_i^j \sigma_2^j \int_{\mathbf{R}_{K_2}} \left( \frac{1}{k_2} \right)^{i-j} H_j(k_2) f_{K_2}(k_2) dk_2 \\ &= \sum_{j=0}^i C_i^j \sigma_2^j Q_{i-j, j} \end{aligned}$$

(60)

where

$$H_j(k_2) = \int_{\mathbf{R}_{\bar{K}_1}} \left( \frac{1}{\bar{k}_1 + r\sigma_1 k_2} \right)^j f_{\bar{K}_1}(\bar{k}_1) d\bar{k}_1$$

(61)

$$Q_{i-j, j} = \int_{\mathbf{R}_{K_2}} \left( \frac{1}{k_2} \right)^{i-j} H_j(k_2) f_{K_2}(k_2) dk_2$$

(62)

Similar as the checking of the existence of  $I_i$  for the single spring case in Section 3.1, it can also be proven that  $H_j(k_2)$  ( $j \geq 1$ ) does not exist but  $H_1(k_2)$  has a CPV. In the following, the existence of moments  $I_i$  as shown in Eq. (60) will be investigated.

6.2.1 The existence of  $I_1$

From Eq.s (60),  $I_1$  can be computed as

$$I_1 = Q_{1,0} + \sigma_2 Q_{0,1} \tag{63}$$

with

$$Q_{1,0} = \int_{\mathbb{R}_{K_2}} \frac{1}{k_2} H_0(k_2) f_{K_2}(k_2) dk_2 = \int_{\mathbb{R}_{K_2}} \frac{1}{k_2} f_{K_2}(k_2) dk_2 \tag{64}$$

$$Q_{0,1} = \int_{\mathbb{R}_{K_2}} H_1(k_2) f_{K_2}(k_2) dk_2 = \int_{\mathbb{R}_{K_2}} H_1(k_2) dF_{K_2}(k_2) \tag{65}$$

where  $F_{K_2}(k_2)$  is the CDF of random variable  $k_2$ . Similar to  $I_1$  for the single spring,  $Q_{1,0}$  does not exist but has a CPV. Since  $H_1(k_2)$  does not exist,  $Q_{0,1}$  does not exist either. However, because  $F_{K_2}(k_2)$  is a monotonic function and  $H_1(k_2)$  has a finite CPV over  $\mathbb{R}_{K_2}$ , it can be drawn from the properties of Riemann-Stieltjes integral (see, e.g. Stroock (1994)) that  $Q_{0,1}$  also has a CPV.

Therefore,  $I_1$  does not exist but has a CPV.

6.2.2 The existence of  $I_2$

From Eq. (60),  $I_2$  can be expressed as

$$I_2 = Q_{2,0} + 2\sigma_2 Q_{1,1} + \sigma_2^2 Q_{0,2} \tag{66}$$

where

$$Q_{2,0} = \int_{\mathbb{R}_{K_2}} \left(\frac{1}{k_2}\right)^2 H_0(k_2) f_{K_2}(k_2) dk_2 = \int_{\mathbb{R}_{K_2}} \left(\frac{1}{k_2}\right)^2 f_{K_2}(k_2) dk_2 \tag{67}$$

$$Q_{1,1} = \int_{\mathbb{R}_{K_2}} \frac{1}{k_2} H_1(k_2) f_{K_2}(k_2) dk_2 \tag{68}$$

$$Q_{0,2} = \int_{\mathbb{R}_{K_2}} H_2(k_2) f_{K_2}(k_2) dk_2 \tag{69}$$

Similar to  $I_2$  for the single spring,  $Q_{2,0}$  does not exist. Since  $H_1(k_2)$  and  $H_2(k_2)$  do not exist,  $Q_{1,1}$  and  $Q_{0,2}$  do not exist either. However, it can be proven that  $Q_{1,1}$  has a CPV (see Appendix).

Therefore,  $I_2$  does not exist. By following the similar procedures, the non-existence of higher-order moments  $I_i (i \geq 3)$  can also be shown.

Thus, for the series spring system considered, moments of axial elongation do not exist. Further, for series systems of more than two springs with correlated random



axial stiffness values, the same conclusions can be drawn by following the similar procedure.

For illustrative purpose, MCS are conducted on the series spring system with normally distributed and correlated axial stiffness  $K_1$  and  $K_2$  (see Fig. 9), and numerical results for  $I_1$  and  $I_2$  are calculated. Effects of three factors are investigated: (1) the correlation coefficient between  $K_1$  and  $K_2$ ,  $r = 0, 0.5, \text{ and } 1.0$ ; (2) the CVs of  $K_1$  and  $K_2$ ,  $\text{CV} = 0.15, 0.20 \text{ and } 0.25$ ; (3) the simulation sample size, the number of samples varies from  $10^3$  to  $10^{11}$ .

Results of MCS are presented in Figs. 10 and 11. Note that in each MCS the smallest absolute value of random stiffness,  $K_1$  and  $K_2$ , is taken as  $\varepsilon^*$ .

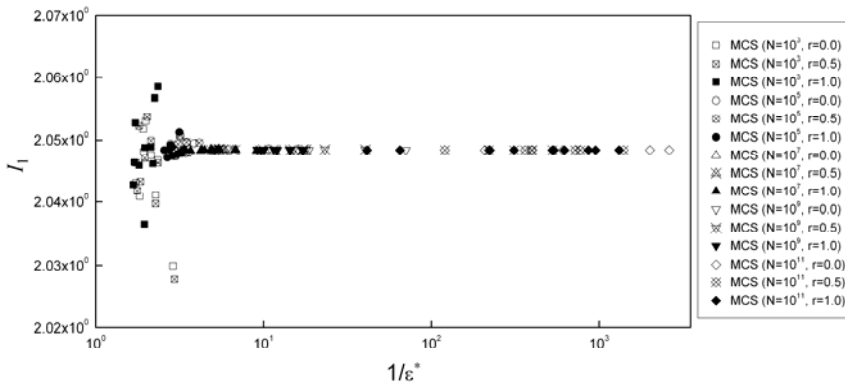
From Figs. 10 and 11, it can be seen that the correlation coefficient may influence the simulation results, see Fig. 11(a) for example, but it has limited effect on the convergence of simulation results. The CV and the simulation sample size have significant effects on the simulation, which are the same as for the model with a single spring (Section 4.3).

It is important to note that, as any structure may be considered as a combination of parallel and series sub-structures, all the findings are generally valid.

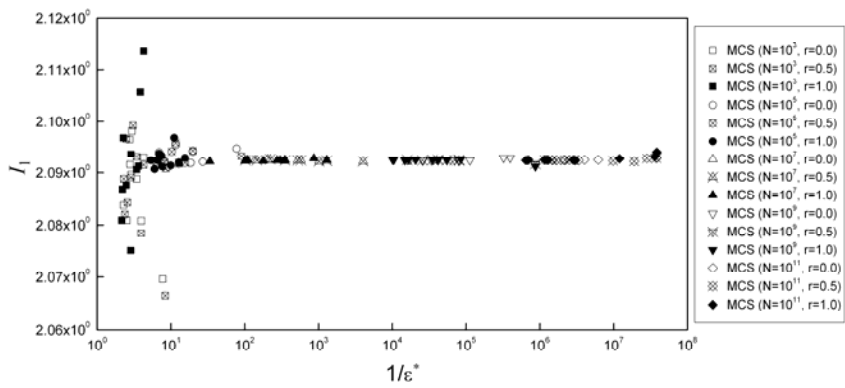
Some researchers used special techniques when applying the normal distribution to represent random material properties. For example, Shinozuka and his co-researchers [Shinozuka (1987); Bucher and Shinozuka (1988); Kardara, Bucher, and Shinozuka (1989)] used the normal distribution model for the random flexibility, and Anders and Hori (1999) adopted a truncating procedure in the MCS to cut off tail distribution from the standard normal distribution, which may remove possible nearly-zero samples. Anders and Hori (1999) mentioned that the tail distribution simulation may cause “destabilizing effect”, but they did not elaborate on this effect and presented no discussion on the consequences of the truncation on the MCS results. Based on the findings of this paper, the singularity removal could be the common rationale for both of the above two treatments.

## 7 Conclusions

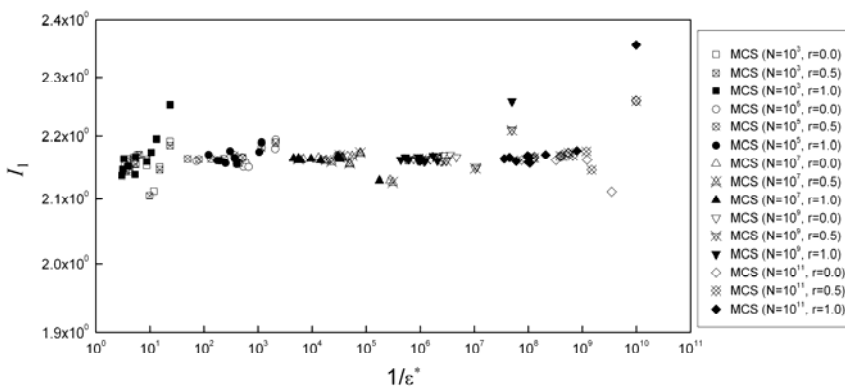
To investigate the accuracy and applicability of Monte Carlo simulations in stochastic structural analysis, simple problems of a single spring and spring systems under a deterministic unit tensile load have been investigated in details. For the single spring model, it has been found that when the spring axial stiffness is a normally distributed random variable, integrals for the moments of its elongation do not exist, and it would be impossible for the MCS to produce meaningful results. In contrast, if the lognormal distribution is assumed, all integrals for moments of the spring elongation are finite and the MCS can produce asymptotically exact solutions. The



(a) The case of CV = 0.15

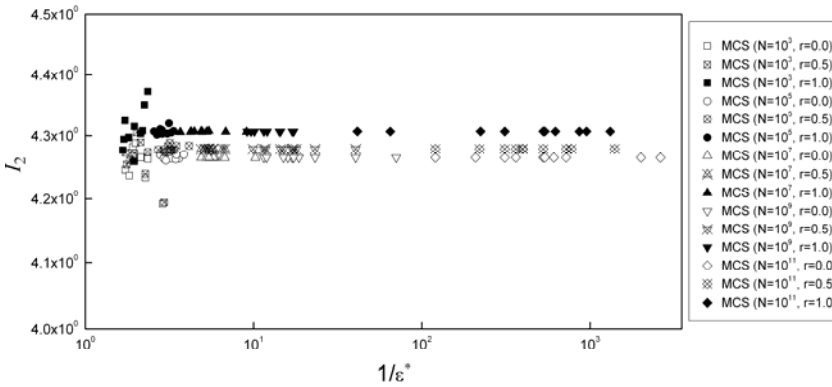


(b) The case of CV = 0.20

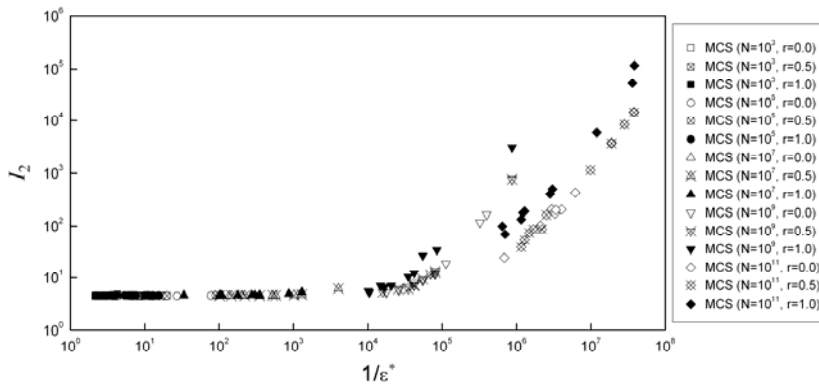


(c) The case of CV = 0.25

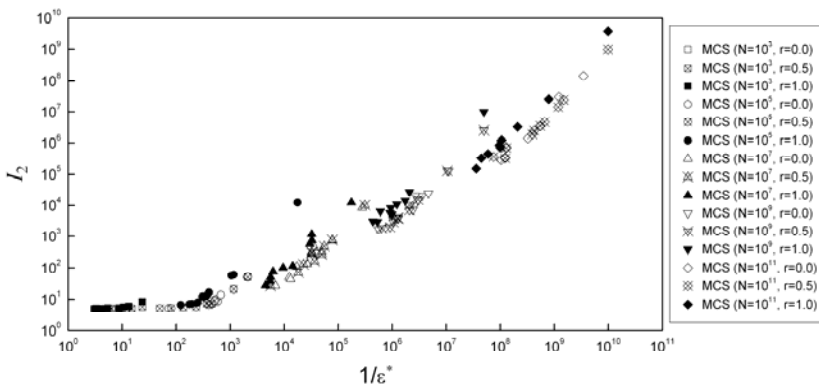
Figure 10: Results of  $I_1$  for series spring system



(a) The case of  $CV = 0.15$



(b) The case of  $CV = 0.20$



(c) The case of  $CV = 0.25$

Figure 11: Results of  $I_2$  for series spring system

above findings have subsequently been extended to the parallel and series spring systems.

As any structure may be considered as a combination of parallel and series sub-structures, the following general conclusions can be drawn:

1. Before the MCS is used in a stochastic structural analysis, it is important to check the existence of the integrals involved, especially when the integrals have singular integrands. The lack of integrability may result in “failure” of the MCS technique as shown in this study.
2. The normal distribution model is not a good choice for representing random material parameters such as spring stiffness and material Young’s modulus. In addition to the well-known problem that it allows negative values, the moments of a structural response may not exist if a normal distribution is used. As the lognormal distribution model does not have these drawbacks, it is a preferred model and more research effort should be made on its applications.

While only simple structural models have been considered in the current study, the findings are meaningful for studies on stochastic analysis of other engineering systems as well.

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## Appendix

From Eq. (68), by letting  $K_2 = \mu_2 T$ , the CPV of  $Q_{1,1}$  can be computed as

$$\begin{aligned}
 \text{p.v. } Q_{1,1} &= \text{p.v. } \int_{\mathbb{R}} \frac{1}{\mu_2 t} H_1(\mu_2 t) f_T(t) dt \\
 &= \int_{\mathbb{R}} \frac{1}{\mu_2 t} [\text{p.v. } H_1(\mu_2 t)] f_T(t) dt \\
 &= \frac{1}{\mu_2} \int_{-\infty}^{-c} \frac{1}{t} [\text{p.v. } H_1(\mu_2 t)] dF_T(t) + S_2 \int_{-c}^c \frac{1}{t} [\text{p.v. } H_1(\mu_2 t)] e^{-\frac{(t^2-2t)}{2\delta_2^2}} dt \\
 &\quad + \frac{1}{\mu_2} \int_c^{+\infty} \frac{1}{t} [\text{p.v. } H_1(\mu_2 t)] dF_T(t)
 \end{aligned} \tag{70}$$

where

$$f_T(t) = \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{(t-1)^2}{2\delta_2^2}} \tag{71}$$

$$S_2 = \frac{1}{\sqrt{2\pi\sigma_2\mu_2}} e^{-\frac{1}{2\delta_2^2}} \tag{72}$$

$$\delta_2 = \frac{\sigma_2}{\mu_2} \tag{73}$$

and  $F_T(t)$  is the CDF of non-dimensional random variable  $t$ , and  $c$  is a small positive value introduced so that over the domain  $[-c, c]$  the exponential term in  $f_T(t)$  can be expanded with convergent power series as follows

$$e^{-\frac{(t^2-2t)}{2\delta_2^2}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{t^2-2t}{2\delta_2^2}\right)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(t^2-2t)^m}{2^m \delta_2^{2m}} \tag{74}$$

On the right-hand side of Eq. (70), [p.v.  $H_1(\mu_2 t)$ ] /  $t$  is finite and  $F_T(t)$  is monotonic over the integration domains,  $(-\infty, -c]$  and  $[c, +\infty)$ , so both the first and the third integrals are finite (see, e.g. Stroock (1994)).

Substituting Eq. (74) into the second integral on the right-hand side of Eq. (70) yields

$$\begin{aligned} S_2 \int_{-c}^c \frac{1}{t} [\text{p.v. } H_1(\mu_2 t)] e^{-\frac{(t^2-2t)}{2\delta_2^2}} dt \\ = S_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{\delta_2^{2m}} \left[ \int_{-c}^c [\text{p.v. } H_1(\mu_2 t)] t^{m-1} \left(\frac{t}{2} - 1\right)^m dt \right] = S_2 \sum_{m=0}^{\infty} Q_m \end{aligned} \tag{75}$$

which will be shown to be a convergent series for a small value of  $c$ . When  $c$  is small, the ratio between two consecutive terms in the series can be determined as

$$\frac{|Q_{m+1}|}{|Q_m|} = \frac{\left| \frac{(-1)^{m+1}}{(m+1)!} \frac{1}{\delta_2^{2m+2}} \left[ \int_{-c}^c [\text{p.v. } H_1(\mu_2 t)] t^m \left(\frac{t}{2} - 1\right)^{m+1} dt \right] \right|}{\left| \frac{(-1)^m}{m!} \frac{1}{\delta_2^{2m}} \left[ \int_{-c}^c [\text{p.v. } H_1(\mu_2 t)] t^{m-1} \left(\frac{t}{2} - 1\right)^m dt \right] \right|} = \frac{1}{(m+1) \delta_2^2} O(c) \tag{76}$$

Therefore, there must exist a positive integer,  $L$ , which can guarantee

$$\frac{|Q_{m+1}|}{|Q_m|} < 1 \quad \text{when } m > L \tag{77}$$

Hence, the series on the right-hand side of Eq. (75) converges to a finite value and the second integral on the right-hand side of Eq. (70) is finite.

As all the three integrals on the right-hand side of Eq. (70) are finite, [p.v.  $Q_{1,1}$ ] must be finite.

