 CMES, vol.88, no.2, pp.93-106, 2012

Application of Geometric Approach for Fuzzy Linear Systems to a Fuzzy Input-Output Analysis

Nizami Gasilov[1], Şahin Emrah Amrahov[2], Afet Golayoğlu Fatullayev[1]
Halil Ibrahim Karakaş[1] and Ömer Akın[3]

Abstract: Uncertainties in some parameters of problems of Leontief input-output analysis lead naturally to fuzzy linear systems. In this work, we consider input-output model, where the technology matrix is crisp and the vector of final outputs is fuzzy. The model is expressed by a fuzzy linear system with crisp matrix and with fuzzy right-hand side vector. We apply a geometric method for solving the system. The method finds the solution in the form of a fuzzy set of vectors. The solution set is shown to be a parallelepiped in coordinate space and is expressed by an explicit formula. The features of the proposed method are illustrated by examples.

Keywords: Fuzzy linear systems, fuzzy sets, input-output analysis, technology matrix, Leontief model.

1 Introduction

Fuzzy linear systems arise in many branches of science and technology such as economics, social sciences, telecommunications, image processing etc (Chen et al., 2006; Hu and Yang, 2000; Li et al., 2007; Shen et al., 2005; Trivedi and Singh, 2005; Wu and Chang, 2004; Zhang et al., 2003; Zhou et al., 2006). Leontief input-output model in economics is well-known when there is no uncertainty (Leontief, 1936). In the last 70 years, input-output analysis has proven itself as an effective and normative tool. Different generalizations have been given for this model to cover the cases when uncertainties occur (Buckley, 1989, 1992; Li and Liu, 2008; Sevastjanov and Dymova, 2009). Differences take place in modelling the uncertainties, in types of systems of equations obtained from the model or in approaches for solving the systems. For instance, an uncertainty can be modelled by using random values, fuzzy sets or interval analysis. Fuzzy models are also different from each

[1] Baskent University, Ankara, Turkey
[2] Ankara University, Computer Engineering Department, Ankara, Turkey
[3] TOBB ETU, Ankara, Turkey

other. Part of input data or all of them may be uncertain. Even fuzzy linear systems of the same type may exhibit differences according to methods of solution. Nowadays, fuzzy numbers are used for solutions of fuzzy linear systems, but this causes loss of data and also restricts the methods of solution. Recently, we have proposed a method, in which the searched solutions are fuzzy sets of vectors, not vectors of fuzzy numbers (Gasilov et al., 2011). This is the fundamental difference between our work and the previous ones (Abbasbandy et al., 2007; Allahviranloo, 2004, 2005; Amrahov and Askerzade, 2011; Asady et al., 2005; Ezzati, 2008; Friedman et al., 1998, 2000; Matinfar et al., 2008, 2009; Nasseri and Khorramizadeh, 2007; Peeva, 1992). The proposed method, which we call geometric approach, is applied in this work to fuzzy input-output analysis problems. In the model that we consider, the technology matrix is crisp, but the final demand matrix has fuzzy entries and with this model we aim at analyzing and forecasting an economical system.

This work is organized as follows. We give preliminaries about fuzzy numbers and fuzzy linear systems in section 2. In section 3, we explain the proposed solution method for fuzzy linear systems. In section 4, the method is applied to some problems of fuzzy input-output analysis. We summarize our results in section 5.

2 Preliminaries

We give the following definition according to (Cong-Xin and Ming, 1992).

Definition 1. A fuzzy number $\tilde{u}$ in parametric form is a pair $\tilde{u} = (u_L,\, u_R)$ of functions $u_L(r)$ and $u_R(r)$, $0 \le r \le 1$, which satisfy the following requirements:

1. $u_L(r)$ is a bounded monotonic increasing left continuous function over [0, 1]
2. $u_R(r)$ is a bounded monotonic decreasing left continuous function over [0, 1]
3. $u_L(r) \le u_R(r)$, $0 \le r \le 1$

For two arbitrary fuzzy numbers $\tilde{u}$ and $\tilde{v}$ we define the arithmetic operations as follows.

a) Addition: $\tilde{u} + \tilde{v} = (u_L(r) + v_L(r),\, u_R(r) + v_R(r))$

b) Multiplication by a real number k:

$$k\tilde{u} = \begin{cases} (ku_L(r),\, ku_R(r)), & k \ge 0 \\ (ku_R(r),\, ku_L(r)), & k < 0 \end{cases}$$

c) Subtraction: $\tilde{u} - \tilde{v} = \tilde{u} + (-1)\tilde{v}$

We will denote $\underline{u} = u_L(0)$ and $\bar{u} = u_R(0)$ to indicate the left and the right boundaries of $\tilde{u}$, respectively.

We can represent a crisp number a as $u_L(r) = u_R(r) = a$, $0 \le r \le 1$.

For a fuzzy number $\tilde{u} = (u_L(r),\ u_R(r))$ the membership function is

$$\mu_{\tilde{u}}(x) = \begin{cases} (u_R)^{-1}(x), & x > u_R(1) \\ 1, & u_L(1) \leq x \leq u_R(1) \\ (u_L)^{-1}(x), & x < u_L(1) \end{cases},$$

where u^{-1} denotes the inverse function of u.

A fuzzy number with $u_L(r) = a + (c-a)r$ and $u_R(r) = b + (c-b)r$ (where $a \leq c \leq b$) is called a triangular number, which we represent as a triple $\tilde{u} = (a, c, b)$.

The membership function of the triangular fuzzy number $\tilde{u} = (a, c, b)$ is

$$\mu_{\tilde{u}}(x) = \begin{cases} \frac{x-a}{c-a}, & a < x < c \\ 1, & x = c \\ \frac{b-x}{b-c}, & c < x < b \\ 0, & otherwise \end{cases}$$

A crisp number a may be regarded as the triangular fuzzy number (a, a, a).

Definition 2. Let $\tilde{u} = (a, c, b)$ be a triangular fuzzy number. The number $u_{cr} = c$ is called the crisp part of $\tilde{u}$. The uncertainty of $\tilde{u}$, denoted by $\tilde{u}_{un}$, is the triangular fuzzy number $\tilde{u}_{un} = (a-c,\ 0,\ b-c)$.

By Definition 2: $\tilde{u} = u_{cr} + \tilde{u}_{un}$ (crisp part + uncertainty).

Definition 3. Let $a_{ij},\ (1 \leq i, j \leq n)$ be crisp numbers and $\tilde{f}_i = (f_{i,L}(r),\ f_{i,R}(r))$, $(1 \leq i \leq n)$ be fuzzy numbers. The system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = \tilde{f}_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = \tilde{f}_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = \tilde{f}_n \end{cases} \tag{1}$$

is called a fuzzy linear system (FLS).

We note that, it is not known a priori $x_1,\ x_2,\ \ldots,\ x_n$ are fuzzy numbers or not. But in any case they restrict an n-dimensional fuzzy set we would like to obtain.

One can rewrite (1) as follows using matrix notation:

$$A\tilde{X} = \tilde{B} \tag{2}$$

where $A = [a_{ij}]$ is an $n \times n$ crisp matrix and $\tilde{B} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)^T$ is a vector of fuzzy numbers.

We will consider systems with nonsingular (invertible) matrices.

One can look for solutions of fuzzy linear systems in two ways: solutions as fuzzy number vectors, or solutions as fuzzy sets formed by vectors.

Definition 4 (*Fuzzy number vector solution*). A vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ is called a fuzzy number vector solution of the system (1), if the fuzzy numbers $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ satisfy the system, when they are substituted instead of $x_1, x_2, \ldots, x_n$, respectively.

Fuzzy number vectors are convenient tool for expressing the solution in a simple and effective form but it gives rise to some difficulties. First of all, fuzzy number vector solution exists for only some special matrices (Friedman et al., 1998, 2000). Secondly, even if such a solution exists, it may not expose the solution adequately. The following example explains this situation.

$$\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (-1,\ 4,\ 9) \\ (-2,\ 3,\ 8) \end{bmatrix} \tag{3}$$

This system has fuzzy number vector solution $\tilde{X} = (\tilde{x}, \tilde{y})$ with components $\tilde{x} = (4,\ 5,\ 6)$ and $\tilde{y} = (1,\ 2,\ 3)$. The vector $(x,\ y) = (3,\ 2)$ does not belong to the set expressed by the vector $(\tilde{x},\ \tilde{y})$. On the other hand, it can be seen that the same vector $(x,\ y) = (3,\ 2)$ with membership degree $\mu \equiv \alpha = 0.2$ is a solution of the system:

$$\left.\begin{array}{ll} 2\cdot 3 - 3\cdot 2 = 0 \in (-1,\ 4,\ 9); & \mu_1 = \mu_{(-1,\,4,\,9)}(0) = \frac{1}{5} \\ -1\cdot 3 + 4\cdot 2 = 5 \in (-2,\ 3,\ 8); & \mu_2 = \mu_{(-2,\,3,\,8)}(5) = \frac{3}{5} \end{array}\right. \Rightarrow \mu = \min\{\mu_1,\ \mu_2\} = 0.2$$

To overcome the difficulties caused by fuzzy number vector solutions, in this paper, we seek the solution as a fuzzy set of n-dimensional vectors.

Definition 5 (*Fuzzy set solution*). Let $\underline{f_i}$ and $\overline{f_i}$ be the left and the right boundaries of $\tilde{f}_i$, $1 \le i \le n$. The set

$$\tilde{X} = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) |\ \underline{f_i} \le a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le \overline{f_i}, \quad 1 \le i \le n\}$$

is called a fuzzy set solution of the system (1). The membership degree of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is defined as

$$\mu_{\tilde{X}}(\mathbf{x}) = \min\{\mu_1(\mathbf{x}),\ \mu_2(\mathbf{x}),\ \ldots,\ \mu_m(\mathbf{x})\}, \tag{4}$$

where $\mu_i(\mathbf{x}) = \mu_{\tilde{f}_i}(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)$.

We note that $\mu_i(\mathbf{x})$ is the membership degree of the value obtained in the left-hand side of i-th equation with respect to the fuzzy number $\tilde{f}_i$.

We also note that the membership degree of a vector $\mathbf{x}$ in the solution set $\tilde{X}$ is determined by the equation satisfied with least membership degree.

3 The geometric approach for solving FLS

Consider FLS (2). Fig.1 exhibits the linear transformation corresponding to the system $A\tilde{X} = \tilde{B}$.

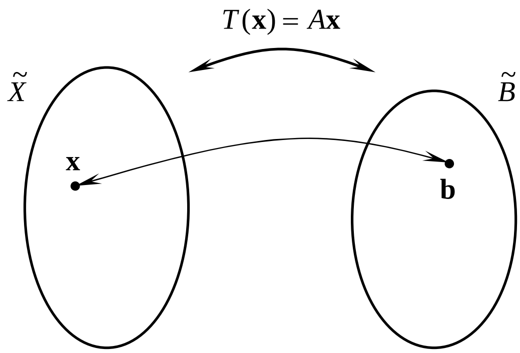

Figure 1: Linear transformation $T(\mathbf{x}) = A\mathbf{x}$ from $\tilde{X}$ to $\tilde{B}$

If the matrix A is invertible, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ is a bijection (one-to-one correspondence). Therefore, then we can interchange the places of the sets $\tilde{X}$ and $\tilde{B}$ (Fig. 2).

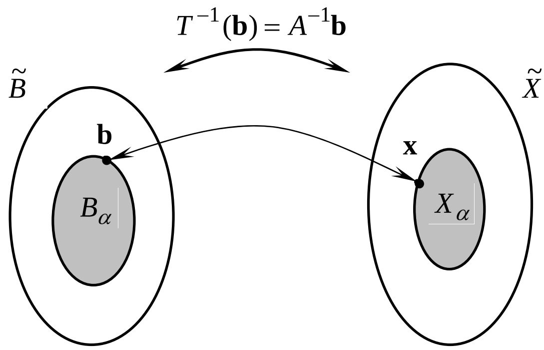

Figure 2: Linear transformation $T^{-1}(\mathbf{b}) = A^{-1}\mathbf{b}$ from $\tilde{B}$ to $\tilde{X}$

If a vector $\mathbf{b}$ has a membership degree $\alpha = \mu_{\tilde{B}}(\mathbf{b})$, then corresponding vector $\mathbf{x}$ is a solution of $A\tilde{X} = \tilde{B}$, with membership degree α.

According to the explanations above, to find the solution set, the following two questions should be answered:

1) How is the structure (α-cuts) of the set $\tilde{B}$?

2) How does the transformation $T^{-1}(\mathbf{b}) = A^{-1}\mathbf{b}$ change the structure of $\tilde{B}$?

Taking into consideration the answers to these questions, we recently have proposed a new method to find a solution in the form of a fuzzy set of vectors (Gasilov et al., 2011). In this method we use two facts from linear algebra (Anton and Rorres, 2005): The first fact: A vector of fuzzy numbers forms a fuzzy region in the form of a rectangular prism in n-dimensional space. The α-cuts of the region are rectangular prisms nested within one another. In the case, where components of the vector are triangular fuzzy numbers, these prisms are similar. The second fact: A linear transformation maps a pair of parallel straight lines to a pair of parallel straight lines (thus a pair of parallel faces to a pair of parallel faces). Consequently, a linear transformation maps a parallelepiped to a parallelepiped.

These facts bring us to the following conclusion: Since the right-hand side vector of the system (2) determines nested rectangular prisms (or more generally speaking, parallelepipeds), then the solution forms a nested parallelepipeds, too.

In particular, for $n = 2$, rectangular prism and parallelepiped turn into rectangle and parallelogram, respectively. According to the discussion above, solution set makes up nested parallelograms.

Formulas for the fuzzy solution set $\tilde{X}$ of the system (2) are obtained as follows.

Let us represent the right-hand side vector as $\tilde{B} = \mathbf{b}_{cr} + \tilde{\mathbf{b}}$, where $\mathbf{b}_{cr}$ is a vector with membership degree 1 and denotes the vertex (crisp part) of the fuzzy region $\tilde{B}$, and $\tilde{\mathbf{b}}$ denotes the uncertain part the vertex of which is at the origin.

Note that the solution of the system (2) is in the form $\tilde{X} = \mathbf{x}_{cr} + \tilde{\mathbf{x}}$ (crisp solution + uncertainty), where $\mathbf{x}_{cr}$ is the solution of $A\mathbf{x}_{cr} = \mathbf{b}_{cr}$ and $\tilde{\mathbf{x}}$ is the solution of $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. The crisp solution is defined as $\mathbf{x}_{cr} = A^{-1}\mathbf{b}_{cr}$.

In the case when right-hand side of (2) consists of triangular fuzzy numbers $\tilde{f}_i = (l_i, m_i, r_i)$, we have $(b_{cr})_i = m_i$ and $\tilde{b}_i = (l_i - m_i,\ 0,\ r_i - m_i) = (\underline{b_i},\ 0,\ \overline{b_i})$. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be standard basis vectors. Then the rectangular prism, determined by $\tilde{B}$, can be represented as follows:

$$\tilde{B} = \{\mathbf{b} = \mathbf{b}_{cr} + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \ldots + c_n\mathbf{e}_n \mid\ \underline{b_i} \le c_i \le \overline{b_i}\}. \tag{5}$$

An α-cut of $\tilde{B}$ can be expressed in following way:

$$B_\alpha = \{\mathbf{b} = \mathbf{b}_{cr} + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \ldots + c_n\mathbf{e}_n \mid\ (1-\alpha)\underline{b_i} \le c_i \le (1-\alpha)\overline{b_i}\}. \tag{6}$$

Let $\mathbf{g}_i = A^{-1}\mathbf{e}_i$. It can be seen from (6) that an α-cut of the solution and the solution itself are represented as follows:

$$X_\alpha = \{\mathbf{x} = \mathbf{x}_{cr} + c_1\mathbf{g}_1 + c_2\mathbf{g}_2 + \ldots + c_n\mathbf{g}_n \mid (1-\alpha)\underline{b_i} \le c_i \le (1-\alpha)\overline{b_i}\}; \tag{7}$$

$\tilde{X} = X_0$ with $\mu_{\tilde{X}}(\mathbf{x}) = \min\limits_{1 \le i \le n} \alpha_i$, where

$$\alpha_i = \begin{cases} 1 - c_i/\overline{b_i}, & c_i \ge 0 \\ 1 - c_i/\underline{b_i}, & c_i < 0 \end{cases} \tag{8}$$

In the general case, the right-hand side of (2) consists of parametric fuzzy numbers $\tilde{f}_i = (f_{i,L}(r),\ f_{i,R}(r))$. We note that in this case the vector $\mathbf{b}_{cr}$ may be not unique. We can choose the components of $\mathbf{b}_{cr}$ arbitrarily to the extent that $f_{i,L}(1) \le (b_{cr})_i \le f_{i,R}(1)$. For instance, we can put $(b_{cr})_i = (f_{i,L}(1) + f_{i,R}(1))/2$. For the considered case the solution can be represented as follows:

$$\begin{aligned} X_\alpha = \{&\mathbf{x} = \mathbf{x}_{cr} + c_1\mathbf{g}_1 + c_2\mathbf{g}_2 + \ldots + c_n\mathbf{g}_n | \\ &f_{i,L}(\alpha) - (b_{cr})_i \le c_i \le f_{i,R}(\alpha) - (b_{cr})_i\}; \end{aligned} \tag{9}$$

$\tilde{X} = X_0$ with

$$\mu_{\tilde{X}}(\mathbf{x}) = \min_{1 \le i \le n} \alpha_i, \tag{10}$$

where

$$\alpha_i = \begin{cases} (f_{i,R})^{-1}(k_i), & k_i > f_{i,R}(1) \\ 1, & f_{i,L}(1) \le k_i \le f_{i,R}(1) \\ (f_{i,L})^{-1}(k_i), & k_i < f_{i,L}(1) \end{cases} \text{; and } k_i = (b_{cr})_i + c_i \tag{11}$$

Note that $\mathbf{g}_i = A^{-1}\mathbf{e}_i$ is the same as the i-th column of matrix A^{-1}, so no additional calculations are required to construct the parallelepipeds X_α and $\tilde{X} = X_0$. Hence, the complexity of the proposed method is determined by calculation of inverse matrix.

Remark: Fuzzy set solution of the system (3) can be obtained by using the formulas (7)-(8). Since $A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, the fuzzy set solution of the system is as follows:

$$\tilde{X} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \;\middle|\; \begin{matrix} c_1 \in [-1,\ 1] \\ c_2 \in [-1,\ 1] \end{matrix} \right\}$$

One can easily check that the vector $(x,\ y) = (3,\ 2)$, mentioned above (after Definition 4), belongs to the set $\tilde{X}$ (the vector corresponds to the values $c_1 = -0.8,\ c_2 = 0.4$ of the coefficients). This example can be considered as a justification of the fact that the proposed method is more appropriate in general.

4 Fuzzy set solution of fuzzy input-output analysis problem

Leontief input–output analysis is a scientific method and effective tool to model and study the economical balance, as well as to forecast the development of future economy.

Suppose that there are n sectors $S_1,S_2,\ldots,S_n$ in an economical system. Let the entry $(M)_{ij}$ of matrix M be the input required from S_i to produce a dollar's worth of output for S_j, D be the vector of final outputs or demands (usually sales and inventory) and X be the total output products vector. Then the Leontief input-output matrix equation is $X = MX + D$ or $(I-M)X = D$. Here I is the identity matrix and M is the technological coefficients matrix.

Leontief's model in the above form does not take into consideration the natural uncertainties that occur in real-world economy. Input data (entries of M and D) are in general uncertain and they can be modeled by intervals, or fuzzy numbers. The fuzzy extension of Leontief's model was first proposed by Buckley (1989, 1992). Cao (1993) considered fuzzy input-output analysis using a defined cone index. Beynon et al. (2005) developed the fuzzy input-output system. They explained the notion of fuzzy output multipliers, and demonstrated a method of ranking them (as part of key sector identification in a regional economy). Beynon and Munday (2006) investigated analogous fuzzy multipliers and their moments within fuzzy closed Leontief input-output systems, utilizing two forms for the levels of fuzziness inherent in the systems. In the present work, to model the uncertainty, the entries of the final output vector D are taken as fuzzy numbers. Thus we consider the following form of Leontief's input-output model:

$\tilde{X} = M\tilde{X} + \tilde{D}$ or $(I-M)\tilde{X} = \tilde{D}$,

where $\tilde{D}$ is the vector of fuzzy final outputs and $\tilde{X}$ is the fuzzy set of total output vectors.

Below, we apply the geometric approach, proposed for solving linear systems, to the fuzzy input-output problems obtained by fuzzifying the final output products vector of some examples given by Barnett et al. (2002).

Example 1 (Two-industry problem): Solve the two-industry problem $\tilde{X} = M\tilde{X} + \tilde{D}$ with the given technology matrix M and fuzzy final outputs $\tilde{D}$:

$$M = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}; \quad \tilde{D} = \begin{bmatrix} (11,\ 12,\ 14) \\ (7,\ 8,\ 9) \end{bmatrix}$$

Find α-cut of the solution set for $\alpha = 0.6$.

Solution:

We have to determine $\tilde{X}$.

We denote $A = I - M$ and rewrite the equation in the form of $A\tilde{X} = \tilde{D}$. Then $A = \begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.6 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1.50 & 0.50 \\ 0.25 & 1.75 \end{bmatrix}$.

We represent $\tilde{D}$ as the sum of crisp part and uncertainty: $\tilde{D} = \mathbf{d}_{cr} + \tilde{\mathbf{d}} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} + \begin{bmatrix} (-1,\ 0,\ 2) \\ (-1,\ 0,\ 1) \end{bmatrix}$.

The crisp solution is $\mathbf{x}_{cr} = A^{-1}\mathbf{d}_{cr} = \begin{bmatrix} 1.50 & 0.50 \\ 0.25 & 1.75 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix}$.

According to the formulas (7)-(8) the solution set itself and its $\alpha = 0.6$-cut $X_{0.6}$ can be represented as follows:

$$\tilde{X} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix} + c_1 \begin{bmatrix} 1.50 \\ 0.25 \end{bmatrix} + c_2 \begin{bmatrix} 0.50 \\ 1.75 \end{bmatrix} \;\middle|\; \begin{matrix} c_1 \in [-1,\ 2] \\ c_2 \in [-1,\ 1] \end{matrix} \right\} \tag{12}$$

with membership function

$$\mu_{\tilde{X}}(\mathbf{x}) = \min\{\alpha_1,\ \alpha_2\},$$

where $\alpha_1 = \begin{cases} 1 - c_1/2, & c_1 \geq 0 \\ 1 - c_1/(-1), & c_1 < 0 \end{cases}$ and $\alpha_2 = \begin{cases} 1 - c_2/1, & c_2 \geq 0 \\ 1 - c_2/(-1), & c_2 < 0 \end{cases}$

$$X_{0.6} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix} + c_1 \begin{bmatrix} 1.50 \\ 0.25 \end{bmatrix} + c_2 \begin{bmatrix} 0.50 \\ 1.75 \end{bmatrix} \;\middle|\; \begin{matrix} c_1 \in [-0.4,\ 0.8] \\ c_2 \in [-0.4,\ 0.4] \end{matrix} \right\}$$

Now let us see how the obtained solution set can be interpreted geometrically.

Denote $\tilde{D} = \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \begin{bmatrix} (11,\ 12,\ 14) \\ (7,\ 8,\ 9) \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

The vector $\tilde{D} = (\tilde{a}, \tilde{b})$ on the right-hand side of the system $A\tilde{X} = \tilde{D}$ forms a rectangular region in the coordinate plane. The boundary of this region is shown by the rectangle $ABCD$ in Fig. 3. Coordinates of the points are: $A(11, 9)$, $B(14, 9)$, $C(14, 7)$, $D(11, 7)$ and $P(12, 8)$.

For $\alpha = 0.6$ we have: $\begin{matrix} \underline{a}(0.6) = 11.6; & \bar{a}(0.6) = 12.8 \\ \underline{b}(0.6) = 7.6; & \bar{b}(0.6) = 8.4 \end{matrix}$. The rectangular boundary of the corresponding α-cut of $\tilde{D}$ is depicted by dotted line in the figure.

Under the multiplication by the inverse matrix A^{-1}, one can find the images of the points A, B, C, D and the vertex P: $A'(21.00, 18.50)$, $B'(25.50, 19.25)$, $C'(24.50, 15.75)$, $D'(20.00, 15.00)$ and the crisp solution is $P'(22.00,\ 17.00)$.

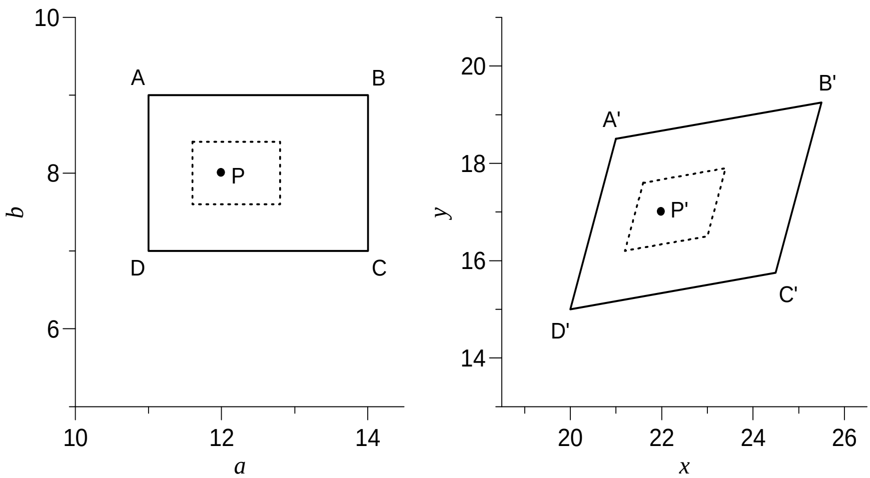

Figure 3: The points P and P' represent the crisp values of the right-hand side and the corresponding crisp solution of the system $A\tilde{X} = \tilde{D}$, respectively. *The left part of the figure*: The rectangular boundary $ABCD$ of the fuzzy region determined by the right-hand side of the system and the boundary of its α-cut for $\alpha = 0.6$ (dotted line). *The right part of the figure*: The parallelogram boundary $A'B'C'D'$ of the fuzzy region determined by the solution and the boundary of its α-cut.

Since the right-hand side of the system is in the form of a vector of triangular fuzzy numbers, one can determine an α-cut of the fuzzy solution from A', B', C', D' and P' by taking P' as a center and using geometric similarity with coefficient $k = 1-\alpha$. For $\alpha = 0.6$, we work out the vertices of the parallelogram that bounds the α-cut.

$$O'\hat{A} = O'P' + (1-\alpha)P'A' = (22.0,\ 17.0) + 0.4\cdot(-1.0,\ 1.5) \quad \Rightarrow \quad \hat{A}(21.6,\ 17.6).$$

Similarly: $\hat{B} = (23.4,\ 17.9)$, $\hat{C} = (23.0,\ 16.5)$ and $\hat{D} = (21.2,\ 16.2)$.

According to the computations above, for $\mathbf{d}_{cr} = (12,\ 8)$ we have the solution $\mathbf{x}_{cr} = (22,\ 17)$. This result means that an output of 22 units (of money) for sector S_1 and 17 units for sector S_2 will meet the given final demands. Since the vector $\tilde{D}$ is a neighborhood of $\mathbf{d}_{cr}$, it would be natural to expect that the solution $\tilde{X}$ be a neighborhood of $\mathbf{x}_{cr}$. The obtained solution set $\tilde{X}$ (12) shows how small or how large this neighborhood is. At the same time, for a given $\tilde{D}$, the solution set $\tilde{X}$ indicates which vectors $\mathbf{x} = (x,y)$ (and with what membership degree) fulfill the requirements and which of them do not fulfill the requirements at all. One can see from Figure 3 that $x \in [20,\ 25.5]$ and $y \in [17,\ 18.5]$. However, one should be careful, because the solution set is in the form of a parallelogram (not a rectangle): some values from the obtained intervals will not fulfill the requirements. For example, the final output vector $(x, y) = (20.5,\ 18.0)$ does not belong to the solution set (although it is in intervals). For this reason, the choice of this vector as a solution may cause the economy to lose its balance.

Example 2 (Three-industry problem): Consider a three-industry problem with the technology matrix M and fuzzy final outputs $\tilde{D}$:

$$M = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.4 & 0.2 & 0.1 \\ 0 & 0.4 & 0.3 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} (18,\ 20,\ 22) \\ (9,\ 10,\ 11) \\ (27,\ 30,\ 33) \end{bmatrix}.$$

Solution:

We have the input-output equation $\tilde{X} = M\tilde{X} + \tilde{D}$ or $(I - M)\tilde{X} = \tilde{D}$ with

$$I - M = A = \begin{bmatrix} 0.8 & 0 & -0.1 \\ -0.4 & 0.8 & -0.1 \\ 0 & -0.4 & 0.7 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{bmatrix}.$$

We represent $\tilde{D} = \mathbf{d}_{cr} + \tilde{\mathbf{d}} = \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix} + \begin{bmatrix} (-2,\ 0,\ 2) \\ (-1,\ 0,\ 1) \\ (-3,\ 0,\ 3) \end{bmatrix}$

Then according to the formulas (7)-(8) the solution set can be represented as follows:

$$\tilde{X} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 33 \\ 37 \\ 64 \end{bmatrix} + c_1 \begin{bmatrix} 1.3 \\ 0.7 \\ 0.4 \end{bmatrix} + c_2 \begin{bmatrix} 0.1 \\ 1.4 \\ 0.8 \end{bmatrix} + c_3 \begin{bmatrix} 0.2 \\ 0.3 \\ 1.6 \end{bmatrix} \;\middle|\; \begin{matrix} c_1 \in [-2,\ 2] \\ c_2 \in [-1,\ 1] \\ c_3 \in [-3,\ 3] \end{matrix} \right\}$$

The membership function is determined by the formula (8). Geometrically, the solution forms a fuzzy parallelepiped.

Comments similar to those after Example 1 can be given here, too. For $\mathbf{d}_{cr} = (20,\ 10,\ 30)$, we have $\mathbf{x}_{cr} = (33,\ 37,\ 64)$. For vectors in the solution set $\tilde{X}$ one can see that $29.7 \le x \le 36.3$, $33.3 \le y \le 40.7$ and $57.6 \le z \le 70.4$. However, one should be careful, not all vectors satisfying these conditions belong to the solution set. For example, $\mathbf{x} = (36,\ 34,\ 58) \notin \tilde{X}$.

5 Conclusion

We have proposed a new method for solving FLS based on geometric representation of linear transformations. The method uses the fact that a vector of fuzzy triangular numbers forms a rectangular prism in n-dimensional space and that the image of a parallelepiped is also a parallelepiped under a linear transformation. We have applied this method for solving two-industry and three-industry problems of Leontief's input-output analysis. The method can also be applied to high-dimensional industry problems.

References

Abbasbandy, S.; Allahviranloo, T.; Ezzati, R. (2007): A method for solving fuzzy linear general systems. *The Journal of Fuzzy Mathematics*, 15 (4), 881-889.

Allahviranloo, T. (2004): Numerical methods for fuzzy system of linear equations. *Applied Mathematics and Computation*, 155, 493-502.

Allahviranloo, T. (2005): The Adomian decomposition method for fuzzy system of linear equations. *Applied Mathematics and Computation*, 163, 555-563.

Amrahov, Ş. E.; Askerzade, I. N. (2011): Strong solutions of the fuzzy linear systems. *CMES: Computer Modeling in Engineering & Sciences*, 76 (4), 207-216.

Anton, H.; Rorres, C. (2005): *Elementary Linear Algebra, Applications Version*. (9th ed.). New York: John Wiley & Sons.

Asady, B.; Abbasbandy, S.; Alavi, M. (2005): Fuzzy general linear systems. *Applied Mathematics and Computation*, 169 (1), 34-40.

Barnett, R. A,; Ziegler, M. R.; Byleen, K. E. (2002): *Applied Mathematics for Business, Economics, Life Sciences and Social Sciences*. (8th ed.). New Jersey: Pearson Education.

Beynon, M. J.; Munday, M.; Roberts, A. (2005): Ranking sectors using fuzzy output multipliers. *Econom. Systems Res.*, 17(3), 237-254.

Beynon, M. J; Munday, M. (2006): The elucidation of multipliers and their moments in fuzzy closed Leontief input-output systems. *Fuzzy Sets and Systems*, 157, 2482-2494.

Buckley, J. J. (1989): Fuzzy input-output analysis. *European Journal of Operational Research*, 39, 54-60.

Buckley, J. J. (1992): Solving fuzzy equations in economics and finance. *Fuzzy Sets and Systems*, 48, 289-296.

Cao, B.-Y. (1993): Input-output mathematical model with T-fuzzy data. *Fuzzy Sets and Systems*, 59, 15-23.

Chen, Sh..; Li, Zh.; Xu, Q. (2006): Grey target theory based equipment condition monitoring and wear mode recognition. *Wear*, 260, 438-449.

Cong-Xin, W.; Ming. M. (1992): Embedding problem on fuzzy number space: Part III. *Fuzzy Sets and Systems*, 46, 281-286.

Ezzati R. (2008): A method for solving dual fuzzy general linear systems. *Applied* *and Computational Mathematics*, 7 (2), 235-241.

Friedman, M.; Ming, M.; Kandel, A. (1998): Fuzzy linear systems. *Fuzzy Sets and Systems*, 96, 201-209.

Friedman, M.; Ming, M.; Kandel, A. (2000): Duality in fuzzy linear systems. *Fuzzy Sets and Systems*, 109, 55-58.

Gasilov, N.; Amrahov, Ş. E.; Fatullayev, A. G.; Karakaş, H. I.; Akın, Ö. (2011): A geometric approach to solve fuzzy linear systems. *CMES: Computer Modeling in Engineering & Sciences*, 75 (3), 189-203.

Hu, W.; Yang, C. (2000): Grey model of direct solar radiation intensity on the horizontal plane for cooling loads calculation. *Building and Environment*, 35, 587-593.

Leontief, W. (1936): Quantitative input-output relations in the economic system of the United States. *Review of Economic Statistics*, 18, 105-125.

Li, G.-D.; Yamaguchi, D.; Nagai, M. (2007): A GM(1,1)-Markov chain combined model with an application to predict the number of Chinese international airlines. *Technological Forecasting and Social Change*, 74 (8), 1465-1481.

Li, Q.-X.; Liu, S.-F. (2008): The foundation of the grey matrix and the grey input-output analysis. *Applied Mathematical Modelling*, 32, 267-291.

Matinfar, M.; Nasseri, S. H.; Sohrabi, M. (2008): Solving fuzzy linear system of equations by using Householder decomposition method. *Applied Mathematical Sciences*, 2 (52), 2569-2575.

Matinfar, M.; Nasseri, S. H.; Alemi, M. (2009): A new method for solving of rectangular fuzzy linear system of equations based on Greville's algorithm. *Applied Mathematical Sciences*, 3 (2), 75-84.

Nasseri, S. H.; Khorramizadeh, M. (2007): A new method for solving fuzzy linear systems. *International Journal of Applied Mathematics*, 20 (4), 507-516.

Peeva, K. (1992): Fuzzy linear systems. *Fuzzy Sets and Systems*, 49, 339-355.

Sevastjanov, P.; Dymova, L. (2009): A new method for solving interval and fuzzy equations: Linear case. *Information Sciences*, 179, 925-937.

Shen, V. R. L.; Chen, T. S.; Shai, K. Q. (2005): A novel cryptosystem based on grey system theory and genetic algorithm. *Applied Mathematics and Computation*, 170, 1290-1302.

Trivedi, H. V.; Singh, J. K. (2005): Application of grey system theory in the development of a runoff prediction model. *Biosystems Engineering*, 92 (4), 521-526.

Wu, C. C.; Chang, N. B. (2004): Corporate optimal production planning with varying environmental costs: a grey compromise programming approach. *European Journal of Operational Research*, 155, 68-95.

Zhang, H.; Li, Zh.; Chen, Zh. (2003): Application of grey modeling method to fitting and forecasting wear trend of marine diesel engines. *Tribology International*, 36 (10), 753-756.

Zhou, P.; Ang, B. W.; Poh, K. L. (2006): A trigonometric grey prediction approach to forecasting electricity demand. *Energy*, 31, 2839-2847.