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The Post-Buckling Behavior of A Tubular String in An Inclined Wellbore

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Abstract: A down-hole tubular string in an inclined wellbore, under variable axial and torsional loading, may simultaneously undergo a sinusoidal as well as helical buckling, at different sections. In this paper, the buckling equation for a tubular string, in an inclined wellbore, subjected to axial and torsional loading, is established by an equilibrium method. The analytical solutions for the buckling equations, for sinusoidal and helical configurations of buckled tubular string, are obtained by Galerkin and nonlinear scaling methods. Methods for computing the contact forces between the buckled tubular string and wellbore, are developed. The analytical solutions are in good accordance with the numerical results, for the nonlinear buckling equation. The critical loads for sinusoidal as well as helical buckling of a down-hole tubular string are determined, using the constraint condition under which the contact force is nonnegative. Thus, the post-buckling behavior of a tubular string, with different configurations, in an inclined wellbore, is determined by the presented analytical method.

1 Introduction

The buckling of down-hole tubular strings, such as a coiled tubing, a drill string, a casing string, or a tubing string, is a fundamental problem in petroleum engineering. Such a buckling may affect the normal down-hole operations, such as transmitting the loads or controlling the well trajectory, or may even result in the failure of the down-hole tubular strings, due to an excessive bending stress, excessive casing wear, and fatigue.

Since the first paper concerning the helical buckling of a drill-string in a vertical well, published by Lubinski (1962), the subject of down-hole tubular buckling has received a considerable attention in the field of petroleum engineering. Owing to the variable axial load under the action of its own weight, the buckling behavior of a tubular string in an inclined wellbore is more complicated than that in a horizontal

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well. Down-hole tubular-strings may have the different buckled configurations at different depths, including the initial configuration, a sinusoidal configuration, a transition from a sinusoidal to a helical configuration, and a helical configuration.

Down-hole tubular buckling in an inclined wellbore, under the action of its own weight, and of a compressive force applied at its upper end, has been analyzed by several authors since 1964. Down-hole tubular sinusoidal buckling was first studied by Paslay and Bogy (1964), and the critical load for the onset of a sinusoidal buckling was obtained by using an energy method [Miska et al., (1995, 1996), Wu et al. (1994)], which was also used to determine the critical load for the onset of helical buckling. The energy method was used to obtain the different critical loads, by assuming different buckling configurations. The equation for the buckling of a down-hole tubular, in inclined wellbore, under an axial load was developed by Mitchell(1995), who developed a simplified algebraic equation for an approximate analytical solution of the tubular which buckles helically. The numerical solution for the nonlinear buckling equation, derived for an arbitrary hole, was also used to obtain some results, by Mitchell(1997). The critical load for the helical buckling of a tubular string in an inclined wellbore was studied by Huang and Pattillo (2000).

The equation for the buckling of a tubular string, subjected to axial as well as torsional loads, in a horizontal wellbore, was developed by Gao(1998). In this equation, the axial load on the tubular string is invariable along the axial direction. The analytical solutions for a sinusoidal as well as a helical configuration of the tubular string, and the corresponding contact forces, were obtained by using the Galerkin method and a nonlinear scaling method. These analytical solutions are in good agreement with the numerical results obtained by solving the corresponding strongly nonlinear ordinary differential equations [Gao et al(1998) and (2002)] and Liu [Liu et al (1999)]. Based on the constraint condition under which the contact force is nonnegative [Liu et al (1999) and (2002)], the critical loads for sinusoidal as well as the helical configurations, were obtained separately. Therefore, the analytical solutions were obtained for down-hole tubular buckling behaviors, ranging from a sinusoidal to a helical configuration. Furthermore, analytical solutions were obtained for helical buckling of a down-hole tubular string, with hinged as well as fixed end conditions [Liu et al (1999)].

However, the loads on a tubular string in an inclined wellbore are variable along the axial direction. In a down-hole tubular string, the initial equilibrium configuration, a sinusoidal configuration, a transition from sinusoidal configuration to helical configuration, and the helical configuration, may all occur simultaneously. The equations governing the buckling of a down-hole tubular, subjected to axial as well as torsional loads, in an inclined wellbore, are developed in this paper. The analytical solutions for these equations governing the buckling of the tubular are obtained for sinusoidal as well as helical configurations, by the Galerkin method and a non-linear scaling method. The contact force between the buckled tubular string and the wellbore are obtained. The maximum sinusoidal buckling load, and the minimum helical buckling load, are determined by using the constraint conditions under which the contact force is nonnegative. Therefore, the various buckling behaviors of the down-hole tubular can be described for different buckled configurations, in an inclined wellbore. These research results enable a thorough understanding of the buckling characteristics of the down-hole tubular.

2 Buckling Equation for the Tubular

The following basic assumptions are invoked in the present research work:

(1) The elasticity theory for a slender beam can be used to define the relationship between the bending moment and the curvature;

(2) The down-hole tubular string is so long, that the end conditions are not considered in the analysis;

(3) The down-hole tubular string keeps a continuous contact with the wellbore;

(4) The wellbore is circular, straight, rigid and frictionless.

A down-hole tubular string, subjected to an axial load $\mathbf{F}$ and a toque $\mathbf{M}_n$, and constrained within an inclined wellbore, is considered. The coordinate system employed is shown in Fig.1. It is assumed that the axial line of the wellbore is in the vertical plane, and that the angle between the axial line of the wellbore and vertical line is denoted as α. The initial equilibrium state of the tubular string is in the bottom of wellbore, under its own weight.

As the loading on the tubular string is increased to some critical values, the tubular string will be buckled to depart from the initial equilibrium state. The buckled tubular string is assumed to keep continuous contact with the wellbore. The geometric contact between the buckled tubular string and wellbore, and the overall configuration, are shown in Fig.2,

where, N is the contact force between the buckled tubular string and the wellbore. θ is the angle between the buckled tubular string and the initial equilibrium position. r is the tubular string/wellbore radial clearance. According to the assumption of continuous contact, any point C in the axial line of the buckled tubular string is always on the cylindrical surface with a radius of r. The radius vector of any point C in the axial line is represented by $\mathbf{r}(s)$, where s is the arc length measured from one end of the tubular string.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1}$$

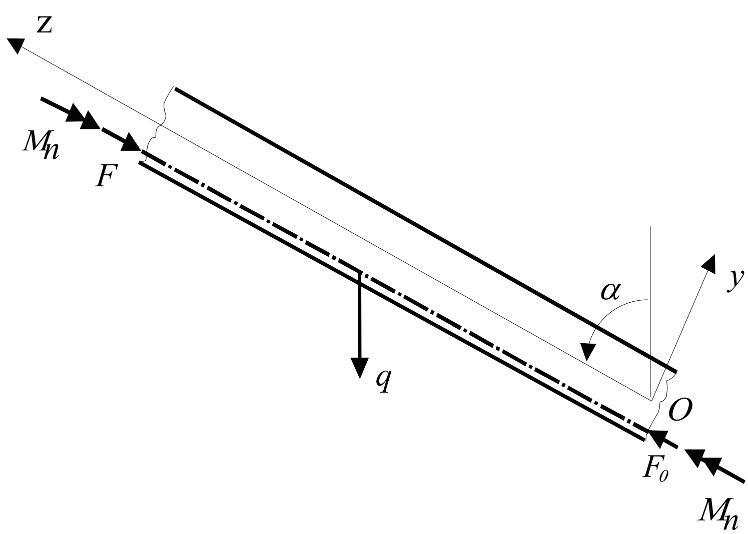


Figure 1: The Down-hole tubular string in an inclined wellbore (side view)

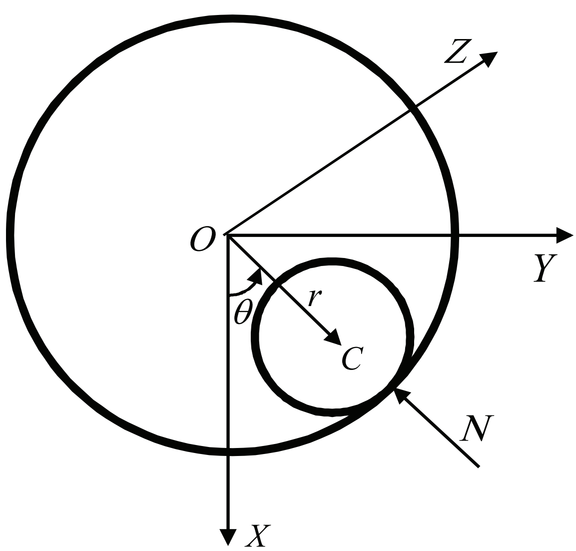


Figure 2: Down-hole tubular string in an inclined wellbore (cross section)

$$x = r_0 \cos\theta, \quad y = r_0 \sin\theta \tag{2}$$

$$\frac{d\mathbf{r}}{dz} = \frac{dx}{dz}\mathbf{i} + \frac{dy}{dz}\mathbf{j} + \mathbf{k} \tag{3}$$

The equilibrium equations of the buckled tubular string can be expressed as:

$$\frac{d\mathbf{F}}{dz} + \mathbf{f} = 0 \tag{4}$$

$$\frac{d\mathbf{M}}{dz} + \frac{d\mathbf{r}}{dz} \times \mathbf{F} = 0 \tag{5}$$

where $\mathbf{F}$ is the internal force vector in the tubular string, $\mathbf{M}$ is the moment of internal force, and $\mathbf{f}$ is the external force per unit length. $\mathbf{f}$ can be expressed as:

$$\mathbf{f} = (q\sin\alpha - N\cos\theta)\,\mathbf{i} - N\sin\theta\,\mathbf{j} - q\cos\alpha\,\mathbf{k} \tag{6}$$

The moments are related to the curvature and the twist of the tubular string, through the conventional relationships of the slender-beam theory, as generalized to three dimensions:

$$\mathbf{M} = EI(\frac{d\mathbf{r}}{dz} \times \frac{d^2\mathbf{r}}{dz^2}) + GJ\frac{d\gamma}{dz}\frac{d\mathbf{r}}{dz} \tag{7}$$

where E is the elastic modulus of tubular string; G is the shear modulus; I is the inertia of tubular string section; J is the polar moment of inertia; γ is the twist angle of tubular string.

Combining Eq.4 , Eq.5 and Eq.7, the differential equation for the buckled configuration, defined by the angle θ , can be expressed as follows:

$$\frac{\mathrm{d}^4\theta}{\mathrm{d}\,z^4} - 6(\frac{\mathrm{d}\,\theta}{\mathrm{d}\,z})^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\,z^2} + 3\frac{M_n}{EI}\frac{\mathrm{d}\,\theta}{\mathrm{d}\,z}\frac{\mathrm{d}^2\theta}{\mathrm{d}\,z^2} + \frac{\mathrm{d}}{\mathrm{d}\,z}(\frac{F}{EI}\frac{\mathrm{d}\theta}{\mathrm{d}\,z}) + \frac{q\sin\alpha}{EIr}\sin\theta = 0 \tag{8}$$

where q is the weight of the tubular string per unit length, F_0 is the axial load at the lower end of the tubular string. F is the axial load on the section of z, which can be expressed by:

$$F = F_0 - qz\cos\alpha \tag{9a}$$

The torque in the section at z is:

$$M_z = M_A + M_n \tag{9b}$$

where $M_A = EI\,r^2\left(\frac{\mathrm{d}\,\theta}{\mathrm{d}\,z}\right)^3$ is the additional torque with the buckling deformation (Liu and Gao,1999).

Furthermore, the contact force between the buckled tubular string and the wellbore can be described as follows:

$$N = EIr[4\frac{\mathrm{d}^3\theta}{\mathrm{d}\,z^3}\frac{\mathrm{d}\theta}{\mathrm{d}\,z} + 3(\frac{\mathrm{d}^2\theta}{\mathrm{d}\,z^2})^2 - (\frac{\mathrm{d}\theta}{\mathrm{d}\,z})^4] + M_n r[(\frac{\mathrm{d}\theta}{\mathrm{d}\,z})^3 - \frac{\mathrm{d}^3\theta}{\mathrm{d}\,z^3}] + Fr(\frac{\mathrm{d}\theta}{\mathrm{d}\,z})^2 + q\sin\alpha\cos\theta \quad (10)$$

The effect of the torque on the buckling behavior of the down-hole tubular has been discussed in the previous papers [Gao et al.(1998) and Liu et al (1999)]. For simplicity, the effect of the torque will be not considered below.

A parameter is introduced as:

$$\omega_0 = \sqrt{\frac{F_0}{2EI}} \quad (11a)$$

and the dimensionless parameters are introduced, as:

$$\xi = \omega_0 z, \quad n = \frac{N}{EIr\omega_0^4}, \quad \varepsilon = \frac{q}{F_0}\sqrt{\frac{2EI}{F_0}} \quad (11b)$$

where ξ is the dimensionless position, n is the dimensionless contact force, and ε is the dimensionless ratio of the variation of the axial load.

The buckling equation can be rewritten in a dimensionless form as:

$$\theta'''_{\xi} - 6\theta'^2_{\xi}\theta''_{\xi} + 2[(1-\varepsilon\xi\cos\alpha)\theta'_{\xi}]'_{\xi} + Q_0\sin\alpha\sin\theta = 0 \quad (12a)$$

where Q_0 is the dimensionless lateral load, and can be determined by:

$$Q_0 = \frac{q}{EIr\omega_0^4} \quad (12b)$$

The contact force, in its dimensionless form, can be expressed as:

$$n = 4\theta'''_{\xi}\theta'_{\xi} + 3\theta''^2_{\xi} - \theta'^4_{\xi} + 2(1-\varepsilon\xi\cos\alpha)\theta'^2_{\xi} + Q_0\sin\alpha\cos\theta \quad (13)$$

3 Solutions for tubular buckling equation, for a sinusoidal buckling configuration

As the change of the axial load of the tubular string, under its own weight, is very low, the dimensionless parameter ε in Eq.12a is a small quantity. The solutions of the buckling equation for a sinusoidal buckling configuration can be obtained by the Galerkin method.

For the sake of simplicity, the trial function for the buckling configuration can be assumed as:

$$\theta = A\sin(\eta) \tag{14}$$

where

$$\eta = \frac{2}{3\varepsilon\cos\alpha}[1-(1-\varepsilon\xi\cos\alpha)^{3/2}] \tag{15}$$

Omitting the terms of higher-order in ε, Eq.12a can be rewritten as:

$$\theta''''_\eta - 6\theta'^2_\eta\theta''_\eta + 2\theta''_\eta + Q_1\sin\theta = 0 \tag{16a}$$

where

$$\theta'_\eta = \frac{\mathrm{d}\theta}{\mathrm{d}\,\eta};\quad \theta''_\eta = \frac{\mathrm{d}^2\theta}{\mathrm{d}\,\eta^2};\quad \theta'''_\eta = \frac{\mathrm{d}^3\theta}{\mathrm{d}\,\eta^3};\quad \theta''''_\eta = \frac{\mathrm{d}^4\theta}{\mathrm{d}\,\eta^4};$$

$$Q_1 = \frac{Q_0\sin\alpha}{(1-\varepsilon\xi\cos\alpha)^2} \tag{16b}$$

Applying the Galerkin method to Eq.16a, we can obtain the weighted-residual expression as:

$$\int_0^\pi(\theta''''_\eta - 6\theta'^2_\eta\theta''_\eta + 2\theta''_\eta + Q_1\sin\theta)\sin\eta d\eta = 0$$

As the half-wave-length of the sinusoidal buckling configuration is very small, relative to a long tubular string, $\varepsilon\xi$ in the half wavelength of the sinusoidal configuration can be regarded as being invariant. We can obtain:

$$A = \sqrt{\frac{8(1-Q_1)}{12-Q_1}} \tag{17}$$

The sinusoidal buckling configuration can be expressed by:

$$\theta = \sqrt{\frac{8[(1-\varepsilon\xi\cos\alpha)^2 - Q_0\sin\alpha]}{12(1-\varepsilon\xi\cos\alpha)^2 - Q_0\sin\alpha}}\sin(\eta) \tag{18}$$

Based on numerical analysis, the lateral buckling amplitudes were been obtained by Mitchell (1997), using the least-squares method, as:

$$\theta_{\max} = 1.12271[(1-\varepsilon\xi\cos\alpha) - \sqrt{Q_0\sin\alpha}]^{0.460}(1-\varepsilon\xi\cos\alpha)^{0.040} \tag{19}$$

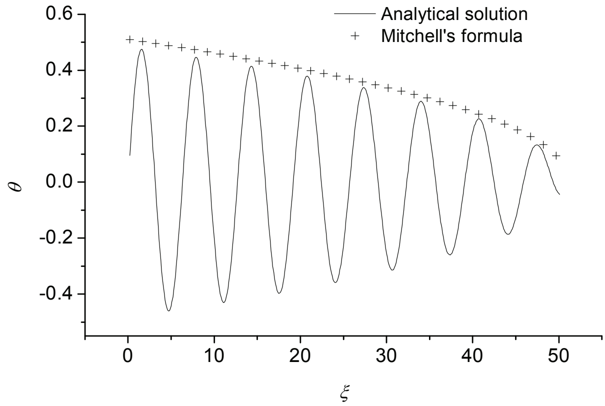


Figure 3: Comparison between the present analytical results, and Mitchell's formula

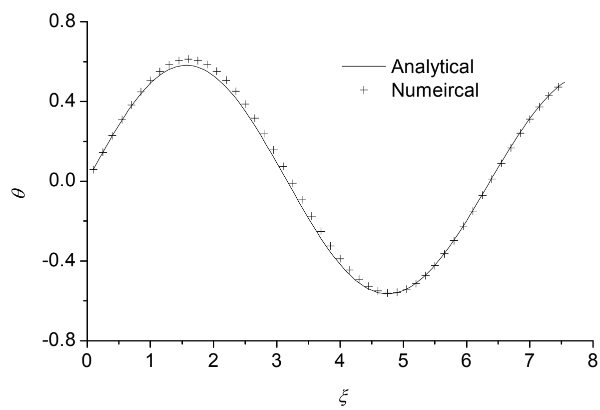


Figure 4: Comparison between the present analytical and numerical solutions for a sinusoidal buckling configuration

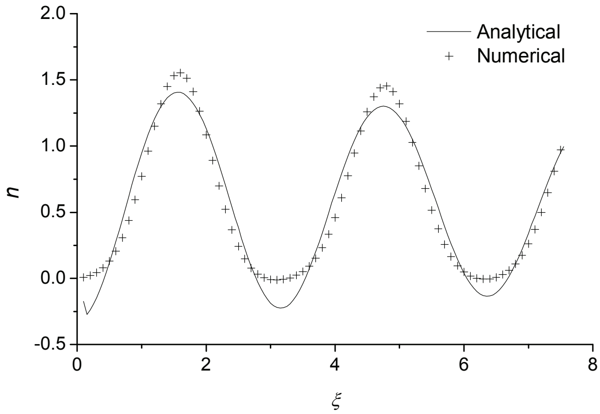


Figure 5: Comparison between the present analytical and numerical solutions for the dimensionless contact force caused by a sinusoidal buckling

The comparison between Eq.18 and Mitchell's formula Eq.19 is shown as Fig.3.

Based on the analytical solution Eq.18, the numerical solution of Eq.12a for sinusoidal buckling configuration can be obtained. The comparison between the present analytical and numerical solutions for a sinusoidal buckling configuration is shown in Fig.4, and the comparison between their corresponding contact forces is shown in Fig.5, where, $Q_0 = 1.0$, $\alpha = 30^{o}$, $\varepsilon = 0.01$.

As shown in Fig.4, the analytical solution for sinusoidal configuration, obtained from Eq.14 has a good correlation with the numerical solution.

Assuming an appropriate amplitude of the sinusoidal buckling configuration, the initial force of sinusoidal buckling can be obtained as below:

$$Q_1 = 1 \tag{20}$$

Using the critical condition $Q_1 = 1$, the location ξ_{crs} at which the sinusoidal buckling starts, can be determined by Eq.16b.

As shown in Fig.5, the difference between the analytical and numerical results for the contact force increases, as Q_1 decreases. To improve the accuracy of analytical solution, an improved trial function for the sinusoidal buckling configuration is assumed as below:

$$\theta = A\sin(\eta) + B\sin(3\eta) \tag{21}$$

We substitute Eq.21 into the buckling equation, Eq.16a and obtain the weighted residual equations:

$$\begin{cases} \int_0^{\pi}(\theta'''_{\eta} - 6\theta'^2_{\eta}\theta''_{\eta} + 2\theta''_{\eta} + Q_1 \sin\theta)\sin\eta \mathrm{d}\eta = 0 \\ \int_0^{\pi}(\theta'''_{\eta} - 6\theta'^2_{\eta}\theta''_{\eta} + 2\theta''_{\eta} + Q_1 \sin\theta)\sin(3\eta)\mathrm{d}\eta = 0 \end{cases} \tag{22}$$

For a given Q_1, the undetermined coefficients A, B can be determined from Eq.22. Furthermore, the sinusoidal buckling configuration can be obtained from Eq.21.

The comparison of the analytical solution with the numerical solution is shown in Fig.6. The results demonstrate that the analytical results for both θ and n are all in good coincidence with the numerical results, where α=90°C and ε=0 [Liu et al (1999) and (2002)].

It is known that the negative contact force has no physical significance. Based on this constraint condition, the critical condition for the onset of the sinusoidal configuration can be determined by the nonlinear algebra equations as below:

$$\begin{cases} \frac{3}{2}A^3 + 27AB^2 + \frac{9}{2}A^2B - A + Q_1(A - \frac{1}{8}A^3\frac{1}{4}AB^2\frac{1}{8}A^2B) = 0 \\ 63B + \frac{3}{2}A^3 + 27A^2B + \frac{243}{2}B^3 + Q_1(B\frac{1}{4}A^2B\frac{1}{24}A^3) = 0 \\ -2A^2 - 108AB - 306B^2 - A^4 - 12A^3B - 54A^2B^2 - 108AB^3 - 81B^4 + Q_1 = 0 \end{cases} \tag{23}$$

Eq.23 is solved to obtain:

$$Q_{s\min} = 0.5266, \quad A = 0.5808, \quad B = -0.004108 \tag{24}$$

where $Q_{s\min}$ is the maximum load for the sinusoidal buckling configuration. Thus, the range of load for a stable sinusoidal configuration can be obtained as follows:

$$0.5266 < Q_1 < 1 \tag{25}$$

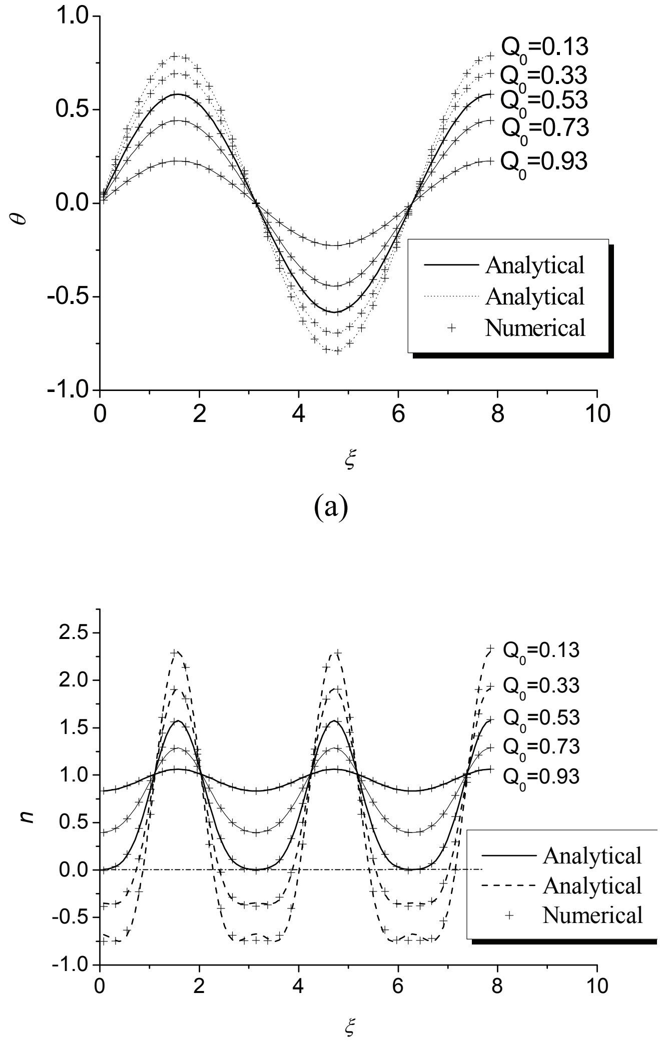
Furthermore, the portion of the tubular string which buckles sinusoidally can be determined. Thus, the entire sinusoidal buckling behavior of the tubular string in an inclined wellbore has been determined.

4 Solution for a helical buckling configuration

As the loads on the tubular string increase to a critical value, the tubular string will buckle helically. The helical configuration can also be obtained by solving the Eq.12a. The nonlinear scaling method is adopted, for the small parameter ε in the Eq.12a.

New variables are introduced as:

$$\lambda = \varepsilon\xi \tag{26}$$



(a)

(b)

Figure 6: Comparison of the analytical and numerical solutions for (a)sinusoidal buckling configuration, and (b) the normal contact force

$$\eta = \frac{g(\lambda)}{\varepsilon} \tag{27}$$

and for a helical buckling configuration is assumed as:

$$\theta = \theta_0(\lambda,\eta) + \varepsilon\theta_1(\lambda,\eta) + \varepsilon^2\theta_2(\lambda,\eta) + O\theta_3(\lambda,\eta) \tag{28}$$

Substituting Eq.28 into Eq.12a, the coefficient equation of ε^0 can be obtained as:

$$\varepsilon^0:\ g'^4(\lambda)\frac{\partial^4\theta_0}{\partial\eta^4} - 6g'^4(\lambda)(\frac{\partial\theta_0}{\partial\eta})^2\frac{\partial^2\theta_0}{\partial\eta^2} + 2(1-\lambda\cos\alpha)g'^2(\lambda)\frac{\partial^2\theta_0}{\partial\eta^2} = 0 \tag{29}$$

The solution of Eq.29 can be expressed as [Gao et al (1998)]:

$$\frac{\partial\theta_0}{\partial\eta} = \pm\frac{\sqrt{1-\lambda\cos\alpha}}{g'(\lambda)} \tag{30a}$$

Assume that $g(\lambda)$ satisfies the relation as:

$$g'(\lambda) = \sqrt{1-\lambda\cos\alpha} \tag{30b}$$

Then:

$$g(\lambda) = -\frac{2C_1}{3\cos\alpha}(1-\lambda\cos\alpha)^{3/2} + C_2 \tag{30c}$$

Assuming C_1=1, and $\eta(\xi$=0)=0, we can obtain:

$$C_2 = \frac{2}{3\cos\alpha} \tag{30d}$$

$$g(\lambda) = \frac{2}{3\cos\alpha}[1-(1-\lambda\cos\alpha)^{3/2}] \tag{31}$$

$$\eta = \frac{2}{3\varepsilon\cos\alpha}[1-(1-\lambda\cos\alpha)^{3/2}] \tag{32}$$

Therefore, the zeroth-order perturbed solution for Eq.12a can be obtained as:

$$\theta_0(\lambda,\eta) = \pm\eta \tag{33}$$

Substituting Eq.33 into the coefficient equation of order ε^1, we can obtain:

$$\frac{\partial^4\theta_1}{\partial\eta^4} - 4\frac{\partial^2\theta_1}{\partial\eta^2} + \frac{h\sin\alpha\sin\eta}{(1-\lambda\cos\alpha)^2} = 0 \tag{34}$$

The solution of the Eq.34 can be obtained as:

$$\theta_1 = -\frac{1}{5}\frac{h\sin\alpha\sin\eta}{(1-\lambda\cos\alpha)^2} \tag{35}$$

Substituting Eq.35 into the coefficient equation of ε^2, we can obtain:

$$\frac{\partial^4\theta_2}{\partial\eta^4}-4\frac{\partial^2\theta_2}{\partial\eta^2}+\frac{7}{50}\frac{h^2\sin^2\alpha\sin(2\eta)}{(1-\lambda\cos\alpha)^4}+\frac{3h\sin(2\alpha)\cos\eta}{2(1-\lambda\cos\alpha)^{7/2}}=0 \tag{36}$$

The solution of the Eq.36 can be obtained as:

$$\theta_2=-\frac{7}{1600}\frac{h^2\sin^2\alpha\sin2\eta}{(1-\lambda\cos\alpha)^4}-\frac{3}{10}\frac{h\sin2\alpha\cos\eta}{(1-\lambda\cos\alpha)^{7/2}} \tag{37}$$

Together, the asymptotic solution of buckling equation for helical configuration can be determined as:

$$\theta=\pm[\eta-\frac{1}{5}\frac{Q_0\sin\alpha\sin\eta}{(1-\lambda\cos\alpha)^2}-\frac{7}{1600}\frac{Q_0^2\sin^2\alpha\sin2\eta}{(1-\lambda\cos\alpha)^4}-\frac{3}{10}\frac{\varepsilon Q_0\sin2\alpha\cos\eta}{(1-\lambda\cos\alpha)^{7/2}}] \tag{38a}$$

Considering $\varepsilon << Q_0$, Eq.38a can be simplified as:

$$\theta=\pm(\eta-\frac{1}{5}Q_1\sin\eta-\frac{7}{1600}Q_1^2\sin2\eta) \tag{38b}$$

For verifying the accuracy of the analytical solution by non-linear scaling method, the numerical results of buckling equation for helical configuration have been obtained. The comparison between the analytical and numerical solutions is shown as in Fig.7 and Fig.8. The broken lines in the figures indicate that the corresponding contact forces are negative, and the solutions in the region have no physical significances. The analytical and numerical solutions show a good agreement as seen in the figures.

Substituting Eq.38a into Eq.13, the critical condition for helical buckling can be obtained by the condition $n_{\min}\geq 0$:

$$Q_{crh}=0.5290 \tag{39}$$

As $Q_1<0.5290$, the tubular string will be buckled helically.Therefore, the entire helical buckling behaviors of the tubular string in an inclined wellbore have been determined.

5 Results and discussion

(1) The deformation of buckled tubular string

The configuration of a tubular string which buckles sinusoidally can be expressed as:

$$\theta=\sqrt{\frac{8\,(1\text{-}Q_1)}{12\text{-}Q_1}}\sin\eta \tag{40}$$

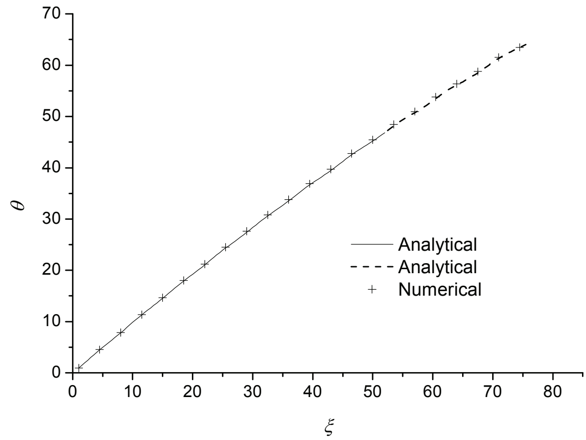


Figure 7: Comparison between the analytical and numerical solutions of the buckling equations, for helical buckling configuration

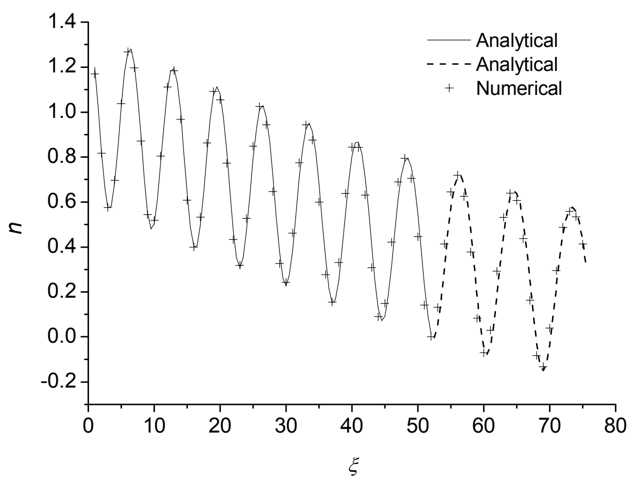


Figure 8: Comparison between the analytical and numerical solutions for the contact force, for tubular string which buckles helically

and the configuration of a tubular string which buckles helically can be expressed as:

$$\theta = \pm(\eta - \frac{1}{5}Q_1 \sin\eta - \frac{7}{1600}Q_1^2 \sin 2\eta) \quad (41)$$

For a given tubular string in an inclined wellbore, Q_0 can be determined from α, ε and F_0. Q_1 can also be represented as the dimensionless lateral load at any location ξ of buckled tubular string. Therefore, the relation between the configuration of buckled tubular string, and the force at any location, can be determined by Eq.40 or Eq.41.

When α=90°, the inclined wellbore degenerates into a horizontal wellbore, and Eq.15 changes into $\eta = \xi$ and Eq.16b changes into $Q_1 = Q_0$. The results are the same as those obtained by Liu et al (1999) and Gao et al (2002).

When $\alpha = 0^o$,the inclined wellbore degenerates into a vertical wellbore, and Eq.15 changes into $\eta = \frac{2}{3\varepsilon}[1-(1-\varepsilon\xi)^{3/2}]$ and Eq.38a changes into $\theta = \pm\eta$. The first order derivative of θ is:

$$\theta'_\xi = \sqrt{1-\varepsilon\xi} \quad (42)$$

The result is same as that of Mitchell(1988).

Based on Eq.40 or Eq.41, the stresses in a buckled tubular string, and the change of the axial length of a buckled tubular string can be determined [Gao et al (2002)].

(2) Critical conditions for different configurations of the buckled tubular string

The range of the force for a sinusoidal configuration to result, is $0.5266 < Q_1 < 1$ from Eq.25, and the range of the force for a helical configuration to result is $Q_1 < 0.5290$ from Eq.39. Thus, it can be seen that the maximum load for a sinusoidal configuration to exist, is very close to the minimum force for a helical configuration to result. The results show that there is no transition from a sinusoidal to a helical configuration, for the buckled tubular string in an inclined wellbore.

Based on the stability of numerical solution,Mitchell(1997) had given an approximate range for the critical force. Because the solution of the non-linear buckling solution is not unique, both the sinusoidal and helical configurations are all the solutions of buckling equation Eq.12a. The method by Mitchell may have difficulties in assuring the accuracy of the critical forces.

Using different assumptions for the buckled configuration, Wu and Juvkam-Wold(1994) and Miska et al (1995, 1996) had obtained the critical loads for the sinusoidal buckling and the helical buckling, by an energy method. The different critical loads had been obtained by using different hypothetical configurations. The actual buckled configuration could not be determined by the energy method. The difference between different results is obvious. Several typical results are list in Table.1, where A is the coefficient of F_{cr}=$A\sqrt{EIq\sin\alpha/r}$, which is a general formula of the critical load for buckling of the tubular string in an inclined wellbore.

(3) Contact force between helically buckled tubular and wellbore

The contact force exerted by the helically buckled tubular string, on wellbore is considered to be the main cause of the locking-up of the tubular string. The contact force can be obtained by substituting Eq.38a into Eq.13. A simplified form of the contact force can be determined by omitting the high-orders of small quantities.

$$n=(1-\lambda\cos\alpha)^2-\frac{9}{5}Q_0\sin\alpha\cos\eta-\left(\frac{1}{50}-\frac{7}{25}\cos^2\eta\right)\sin^2\alpha \tag{43}$$

When α=0, the inclined wellbore degenerates into a vertical wellbore, and the dimensionless contact force changes into:

$$n=(1-\lambda\cos\alpha)^2 \tag{44}$$

Furthermore, the actual contact force can be expressed as:

$$N=\frac{r\left(F_0-qz\right)^2}{4EI}=\frac{rF^2}{4EI} \tag{45}$$

The form of this formula for the contact force, is the same as that for a weightless tubular string buckled helically [Lubinski *et al.*, (1962)], but the axial load here is variable instead of a constant.

6 Sample Calculation

The example problem [Mitchell (1997)] consists of a 2-7/8-in tubing(6.5 lbm/ft) buckled inside a 7 in casing(32 lbm/ft). For this case, q=0.640 lbf/in, I=1.61 in.^{4} and r=1.61 in. The calculated results are shown in table 2.

7 Conclusions

1. The equation for the buckling of a down-hole tubular in inclined wellbore, under the action of axial as well as torsional loads, has been developed. The tubular buckling behavior has been illustrated by solving the strongly nonlinear ordinary differential equation.

Table 1: Comparison of different coefficients in critical load formula

Authors	Method	Sinusoidal buckling	Transition	Helical buckling
Dawson and Paslay (1984)	Energy method	2∼		
Wu and Juvkam-Wold (1994)	Energy method	2	3.657∼	
Miska *et al* (1995, 1996)	Energy method	2∼3.75	3.75∼4	5.657∼
Mitchell (1997)	Stability of numerical solution of buckling equation	2∼2.83	2.83∼5.66	5.66∼
This paper	Constraint condition: contact forces are nonnegative	2∼2.75	no	2.75∼

Table 2: Buckling criteria for sample problem

Hole angle (degrees)	Minimum critical load of sinusoidal buckling(lbf) by Eq.20	Minimum critical load of sinusoidal buckling(lbf) by Eq.25	Minimum critical load of helical buckling(lbf) by Eq.39	Minimum critical load of helical buckling(lbf) (Mitchell, 1997)
0	0	0	0	0
10	3,652	5,032	5,021	10,329
30	6,197	8,539	8,520	17,528
50	7,670	10,569	10,545	21,694
70	8,495	11,706	11,680	24,027
90	8,763	12,076	12,049	24,786

2. The analytical solution of the tubular buckling equation is obtained for a sinusoidal configuration by using the Galerkin method.The minimum load for sinusoidal configuration to exist, is determined by assuming that the amplitude of sinusoidal configuration is equal to zero. The maximum load for the sinusoidal configuration to exist, is determined by using the constraint condition under which the contact force between the buckled tubular and the wellbore is nonnegative. The down-hole tubular string from the initial sinusoidal buckling to its end can be located in an inclined wellbore by the critical loads. Furthermore, the contact force between the tubular string buckled sinusoidally, and the inclined wellbore has been obtained. Thus, the entire sinusoidal buckling behavior of a down-hole tubular string in an inclined wellbore can be described quantitatively.

3. The perturbed solution of the buckling equation, for a helical buckling configuration, has been obtained by using a nonlinear scaling perturbation method, and the corresponding critical buckling load is determined by using the constraint condition under which the contact force between the buckled tubular string and inclined wellbore is nonnegative. The results show that there is almost no transition from a sinusoidal configuration to a helical configuration for the buckled tubular string, in an inclined wellbore.

4. The analytical solutions are in good accordence with the corresponding numerical results, for the down-hole tubular buckling equation, in an inclined wellbore.

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