

## On the Multigrid Method Based on Finite Difference Approximate Inverses

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**Abstract:** During the last decades, multigrid methods have been extensively used in order to solve large scale linear systems derived from the discretization of partial differential equations using the finite difference method. Approximate Inverses in conjunction with Richardson's iterative method could be used as smoothers in the multigrid method. Thus, a new class of smoothers based on approximate inverses could be derived. Effectiveness of explicit approximate inverses relies in the fact that they are close approximants to the inverse of the coefficient matrix and are fast to compute in parallel. Furthermore, the class of finite difference approximate inverses proposed in conjunction with the explicit preconditioned Richardson method present improved results against the classic smoothers such as Jacobi and Gauss – Seidel method. Moreover, a dynamic relaxation scheme is proposed based on the Dynamic Over / Under Relaxation (DOUR) algorithm. Furthermore, results for the multigrid preconditioned Bi-CGSTAB based on approximate inverse smoothing and a dynamic relaxation technique are presented for a class of model problems.

**Keywords:** multigrid method, explicit approximate inverse smoothing, finite differences, multigrid preconditioning, multigrid preconditioned Bi-CGSTAB, DOUR scheme.

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### 1 Introduction

Let us consider a class of problems defined by the following Partial Differential Equation (P.D.E.) subject to the following general boundary conditions:

$$\sum_{i,j=1}^{N=2} \frac{\partial}{\partial x_i} \left[ a_{i,j}(x) \frac{\partial u}{\partial x_j} \right] + \sum_{j=1}^{N=2} \left[ b_j(x) \frac{\partial u}{\partial x_j} \right] + c(x)u = f, x \in \Omega \tag{1a}$$

$$\alpha(x)u + \beta(x) \frac{\partial u}{\partial \eta} = \gamma(x), x \in \partial\Omega \tag{1b}$$

where  $\Omega$  is a closed bounded domain,  $\partial\Omega$  denotes the boundary of  $\Omega$ ,  $\partial\eta$  is the direction of the outward normal and  $a_{i,j}(x) > 0, b_j(x) > 0, c(x) \geq 0$  are sufficiently smooth functions on  $\Omega$ .

By applying the finite difference (FD) method to a PDE results in the following sparse linear system, i.e.

$$Au = s, \tag{2}$$

where the coefficient matrix A is a nonsingular large, sparse, unsymmetric, positive definite, diagonally dominant matrix of regular structure, viz.,

$$A \equiv \begin{matrix} \leftarrow & m & \rightarrow \\ \left[ \begin{array}{cccccccc} b_1 & c_1 & & & & & & w_1 \\ a_2 & \cdot & \cdot & & & & & \cdot \\ & \cdot & \cdot & \cdot & & & & \cdot \\ & & \cdot & b_{m-1} & c_{m-1} & & & \cdot \\ v_m & & a_m & b_m & c_m & & & \cdot \\ & \cdot & & a_{m+1} & \cdot & \cdot & & w_{n-m+1} \\ & & \cdot & & \cdot & \cdot & \cdot & \\ & & & \cdot & & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot & c_{n-1} \\ & & & & & v_n & a_n & b_n \end{array} \right] \end{matrix} \tag{3}$$

while u is the FD solution at the nodal points and s is a vector with components resulting from the combination of source terms and imposed boundary conditions. Explicit preconditioning methods have been extensively used for solving sparse linear systems on multiprocessor systems, and the preconditioned form of the sparse linear system (2) is

$$MAu = Ms, \tag{4}$$

where  $M$  is a suitable preconditioner. The preconditioner  $M$  has to be efficiently computed in parallel and  $MA$  should have a clustered spectrum. The effectiveness of explicit approximate inverse preconditioning relies on the use of suitable preconditioners that are close approximants to the inverse of the coefficient matrix and are fast to compute in parallel, [Gravvanis (2009); Gravvanis, Filelis-Papadopoulos, Giannoutakis and Lipitakis (2012); Giannoutakis and Gravvanis (2008); Gravvanis and Giannoutakis (2011)]. In this article we present a parameterized “smoother” based on the explicit approximate inverse preconditioner and the explicit preconditioned Richardson iterative method. Approximate inverses based on the minimization of the Frobenius norm of the error have been also proposed and used in conjunction with the multigrid method, [Bröker, Grote, Mayer and Reusken (2001); Frederickson (1996)].

During the last decades, multigrid methods, for solving large sparse linear systems, have gained substantial interest among the scientific community for both their efficiency and convergence behavior in many research fields, [Frederickson (1996); Furumura and Chen (2004); Haelterman, Viederndeels and Van Heule (2006); Zakhama, Abdalla, Smaoui and Gürdal (2009)]. Multigrid methods are based on the observation that the high frequency components of the error are damped effectively by a stationary iterative method (such as Jacobi or Gauss - Seidel), however the low - frequency components are not damped effectively. In order for low frequency components of the error to be handled, a series of coarser grids with higher discretization step are used as shown in Figure 1. In this series of coarser grids the low-frequency modes of the error are more oscillatory and can be damped efficiently by a stationary iterative method, [Hackbusch (1985a, 1985b); Trottenberg, Osterlee and Schuller (1970); Wesseling (1982)]. Multigrid methods are composed by four discrete elements: stationary iterative method, restriction operator, prolongation operator and cycle strategy. The stationary iterative methods are first order iterative methods such as Richardson, Jacobi and Gauss – Seidel method. Restriction and prolongation are transfer operators from finer to coarser grids and from coarser to finer grids respectively. The cycle strategy refers to the sequence in which the grids are visited until a solution with the prescribed tolerance is achieved.

In order to accelerate convergence and increase robustness, multigrid can be used as a preconditioner to the Krylov subspace iterative methods, [Trottenberg, Osterlee and Schuller (1970)]. Krylov subspace methods such as Bi-CGSTAB with multigrid preconditioning in conjunction with Approximate Inverse Smoothing are presented along with comparative results.

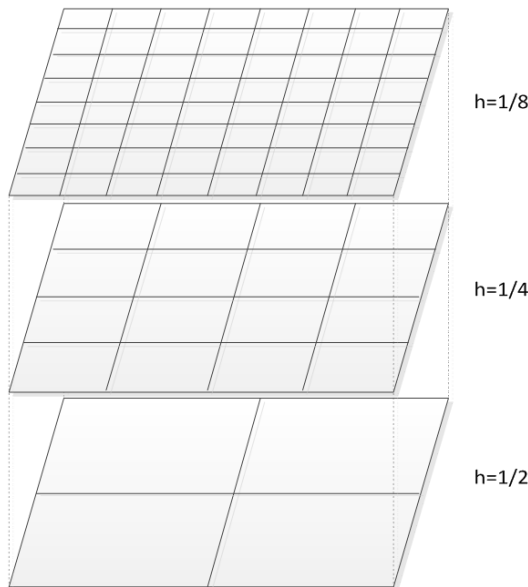


Figure 1: Square domain with different discretization step  $h$

## 2 Multigrid method based on Explicit Approximate Inverses

Let us consider the linear system derived from the discretization of a PDE on a unit square domain with mesh size  $h$ :

$$A_h u_h = s_h \quad (5)$$

The linear system (5) is then solved using the multigrid method based on Explicit Approximate Inverse smoothing in conjunction with the DOUR scheme, [Haelterman, Viederndeels and Van Heule (2006)]. In order to form the multigrid method for the solution of linear system (5), the basic components of the method should be introduced, [Bröker, Grote, Mayer and Reusken (2001); Hackbusch (1985a); Trottenberg, Osterlee and Schuller (2000); Wesseling (1982)].

These components are the prolongation and restriction operators as well as the respective coefficient matrices and cycle strategy, [Hackbusch (1985a); Trottenberg, Osterlee and Schuller (2000); Wesseling (1982)]. Another essential component is the stationary iterative method, namely smoother, which is used for  $v_1$  pre-smoothing steps and  $v_2$  post-smoothing steps on each level, [Hackbusch (1985a); Trottenberg, Osterlee and Schuller (2000); Wesseling (1982)]. The prolongation and restriction operators used are the bilinear interpolation and full – weighting

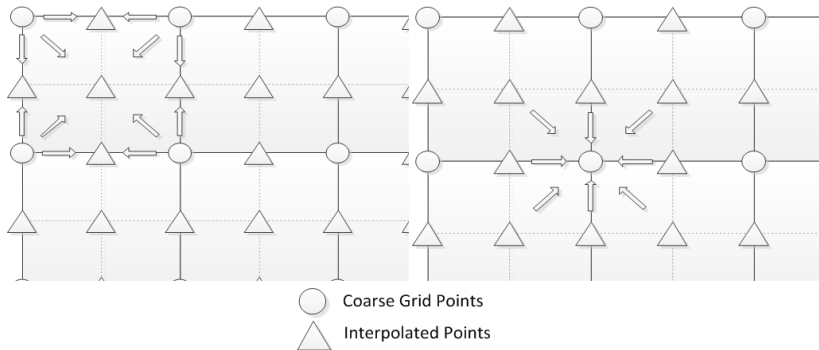


Figure 2: Representation of the prolongation and restriction procedure where the coarse grid consists of “circle” nodes and the finer grid with both “circle” and “triangle” nodes

respectively, [Hackbusch (1985a); Trottenberg, Osterlee and Schuller (2000); Wesseling (1982)]. The two transfer operators are schematically represented in Figure 2.

The solution of a model problem can be achieved by successive applications of a multigrid Cycle, such as the V, W or F Cycle, according to arbitrary termination criterion. The proposed multigrid scheme descends to the coarsest possible level, where the exact solution is obtained. The coarsest level has only one unknown at the center of the grid. The Cycle strategy used in the proposed multigrid schemes is the V - Cycle. The iterative algorithm for the V – Cycle multigrid method is given in [Trottenberg, Osterlee and Schuller (2000)]. Various cycle strategies and their corresponding algorithms are presented in [Hackbusch (1985a); Trottenberg, Osterlee and Schuller (2000); Wesseling (1982)].

An important component in multigrid methodology is a stationary iterative solver, namely smoother, that can be described by the following recurrence relation known as general iteration method:

$$x_\ell^{(k+1)} = x_\ell^{(k)} + M_\ell r_\ell, r_\ell = s_\ell - A_\ell x_\ell^{(k)}, \tag{6}$$

where  $k$  denotes the current iterative step,  $s_\ell, A_\ell$  are the right hand side and the coefficient matrix and  $x_\ell^{(k)}$  is the solution vector. Equation (6) describes a family of stationary iterative methods, according to the choice of the  $M_\ell$  matrix. The basic smoother for the multigrid iteration is the Jacobi method. The Jacobi iterative method can be derived from equation (6) by substituting  $M_\ell = D^{-1}$ . For the damped Jacobi method with relaxation parameter  $\omega$  the  $M_\ell$  matrix is of the form  $M_\ell =$

$\omega D^{-1}$ . Further discussions and proofs about classical smoothers can be found in [Hackbusch (1985a, 1985b); Wesseling (1982)].

Approximate inverses in conjunction with the general iterative method (6) can be used as smoothers for multigrid schemes, by choosing  $M_\ell = (M_\ell)_r^{\delta l}$ , where  $(M_\ell)_r^{\delta l}$  is a class of finite difference approximate inverse with “fill - in” parameter  $r$  and “retention” parameter  $\delta l$  at the  $\ell$ -th level of discretization, [Evans and Lipitakis (1979)]. The new class of smoothing methods proposed can be described as follows:

$$x_\ell^{(k+1)} = x_\ell^{(k)} + \omega (M_\ell)_r^{\delta l} (s_\ell - A_\ell x_\ell^{(k)}) \quad (7)$$

where  $\omega$  is the damping parameter with  $0 < \omega \leq 1$ .

Let us assume the sparse approximate factorization, such that

$$A \approx L_r U_r, r \in [1, m - 1] \quad (8)$$

where  $r$  is the so called “fill-in” parameter and  $L_r, U_r$  are upper and lower matrices, respectively, of the same profile as the coefficient matrix  $A$ , [Evans and Lipitakis (1979)]. The elements of the decomposition factors  $L_r, U_r$  can be computed by the ALUBOT algorithm, [Evans and Lipitakis (1979)].

Let  $M_r^{\delta l} = (\mu_{i,j})$ ,  $i \in [1, n]$ ,  $j \in [i - \delta l + 1, i + \delta l - 1]$  be the approximate inverse of the coefficient matrix  $A$ . The elements of a class of banded forms of the approximate inverse, by retaining  $\delta l$  elements, can be computed by solving recursively the following systems:

$$M_r^{\delta l} L_r = (U_r)^{-1} \quad \text{and} \quad U_r M_r^{\delta l} = (L_r)^{-1}, r = [1, m - 1], \delta l = [1, \rho m] \quad (9)$$

where  $\rho = 1, 2, \dots, m - 1$ .

Then, the elements of the optimized form, of the approximate inverse based on a shifted window from top to bottom of the generalized approximate inverse are computed by the Optimized Banded Generalized Approximate Inverse (OBGAIM) algorithm [Gravvanis (2000, 2002, 2009); Lipitakis and Evans (1987)]. The computational work of the OBGAIM algorithm is  $O(n \times \delta l \times r)$ . It has been shown that the value of the “retention” parameter  $\delta l$  can be chosen as multiples of the semi-bandwidth  $m$ , [Gravvanis (2000, 2002, 2009)].

In multigrid convergence theory two properties must be satisfied in order for the two-grid cycle to converge, [Hackbusch (1980, 1981, 1982, 1985a, 1985b); Wesseling (1979)]:

- The smoothing property, [Hackbusch (1980, 1981, 1982, 1985a, 1985b)]

$$\|A_\ell S_\ell^v\|_2 \leq \eta(v) \|A_\ell\|_2, 0 \leq v < \infty, \ell \geq 1 \tag{10}$$

where  $\eta(v)$  is any function with  $\lim_{v \rightarrow \infty} \eta(v) = 0$ .

- The approximation property, [Hackbusch (1980, 1981, 1982, 1985a, 1985b)]

$$\|A_\ell - PA_{\ell-1}R\|_2 \leq \frac{C_A}{\|A_\ell\|_2}, \ell \geq 1. \tag{11}$$

If the above two properties are satisfied convergence, independent of number of the levels  $\ell$ , for the W-Cycle is also implied. Hackbusch stated that for symmetric positive definite matrices, convergence for the V-Cycle independent of the levels  $\ell$  is also implied, [Hackbusch (1980, 1981, 1982, 1985b)]. The approximation property is independent of the smoother implied and depends only on the discretization, the prolongation and restriction operators. The approximation property has been proven, for various elliptic boundary value problems, [Hackbusch (1980, 1981, 1982, 1985b)]. Additionally, the smoothing property of the classical smoothers for symmetric positive definite problems, such as the Jacobi or Gauss-Seidel iterative method, is satisfied. Further details and proofs for the multigrid rate of convergence are given in [Hackbusch (1980, 1981, 1982); Wesseling (1979)]. Furthermore, sharp estimates of the multigrid rate of convergence with general smoothing and acceleration, based on V and W cycles, for elliptic boundary value problems have been presented by Bank and Douglas along with empirical results, [Bank and Douglas (1985)]. The aforementioned generalized estimates also apply for the case of explicit approximate inverse smoothing. Additionally, Notay has proved convergence bounds for perturbed two-grid and multigrid with general smoothers that also apply to approximate inverse smoothing [Notay (2007)]. Moreover, the smoothing property for S.P.D problems is satisfied for the classical smoothers such as the Jacobi and Gauss-Seidel iteration. For the Banded Approximate Finite Difference Inverse matrix the smoothing property is proven as follows:

Let

$$\begin{aligned} \|A_\ell S_\ell^v\|_2 &\leq \|A_\ell\|_2 \|S_\ell^v\|_2 = \|A_\ell\|_2 \left\| \left( I_\ell - \omega(M_\ell)_r^{\delta l} A_\ell \right)^v \right\|_2 \\ &\leq \left\| \left( I_\ell - \omega(M_\ell)_r^{\delta l} A_\ell \right) \right\|_2^v \|A_\ell\|_2 \end{aligned} \tag{12}$$

In order for the smoothing condition to be satisfied there has to be

$$\left\| \left( I_\ell - \omega(M_\ell)_r^{\delta l} A_\ell \right) \right\|_2 < 1$$

By substituting  $I_\ell = M_\ell A_\ell$  we have

$$\left\| \left( M_\ell A_\ell - \omega(M_\ell)_r^{\delta l} A_\ell \right) \right\|_2 = \left\| \left( M_\ell - \omega(M_\ell)_r^{\delta l} \right) A_\ell \right\|_2 \tag{13}$$

and by substituting  $M_\ell = (M_\ell)_r^{\delta l} + \tilde{M}_\ell$ , where  $\tilde{M}_\ell$  is the error matrix, [Gravvanis (1996)], we obtain

$$\begin{aligned} \left\| \left( M_\ell - \omega (M_\ell)_r^{\delta l} \right) A_\ell \right\|_2 &= \left\| \left( (M_\ell)_r^{\delta l} + \tilde{M}_\ell - \omega (M_\ell)_r^{\delta l} \right) A_\ell \right\|_2 \\ &= \left\| (I - \omega I) (M_\ell)_r^{\delta l} A_\ell + \tilde{M}_\ell A_\ell \right\|_2 \end{aligned} \tag{14}$$

When  $\delta l \rightarrow n$  and  $r \rightarrow m - 1$  then the error matrix  $\tilde{M}_\ell$  tends to zero. Since  $\omega \in (0, 1]$ , we have

$$\left\| (I - \omega I) (M_\ell)_r^{\delta l} A_\ell + \tilde{M}_\ell A_\ell \right\|_2 \approx \left\| (I - \omega I) (M_\ell)_r^{\delta l} A_\ell \right\|_2 \leq (1 - \omega) \left\| (M_\ell)_r^{\delta l} A_\ell \right\|_2 \tag{15}$$

For real symmetric positive definite matrices (Laplace equation with zero boundary values), it can be shown

$$\left\| \left( I_\ell - \omega (M_\ell)_r^{\delta l} A_\ell \right) \right\|_2 \leq (1 - \omega) \left\| (M_\ell)_r^{\delta l} A_\ell \right\|_2 = (1 - \omega) \rho \left( (M_\ell)_r^{\delta l} A_\ell \right) \tag{16}$$

In order for the smoothing property to be satisfied there has to be

$$(1 - \omega) \rho \left( (M_\ell)_r^{\delta l} A_\ell \right) < 1 \Leftrightarrow \omega > 1 - \frac{1}{\rho \left( (M_\ell)_r^{\delta l} A_\ell \right)} \tag{17}$$

Therefore,

$$\eta(v) = \left[ (1 - \omega) \rho \left( (M_\ell)_r^{\delta l} A_\ell \right) \right]^v, \text{ with } \lim_{v \rightarrow \infty} \eta(v) \rightarrow 0 \tag{18}$$

It should be stated that the theoretical estimates were found to be in qualitative agreement with the results presented in Figure 3 for model problem I. The choice of the relaxation parameter governs the smoothing properties of the inverse. In addition the value of the relaxation parameter should satisfy the smoothing condition, and hence the Dynamic Over / Under Relaxation (DOUR) algorithm is used.

The choice of the relaxation parameter governs the smoothing properties of the approximate inverse. In addition, the value of the relaxation parameter should satisfy the smoothing condition, and hence the Dynamic Over / Under Relaxation (DOUR) algorithm is used, [Haelterman, Viederndeels and Van Heule (2006)].

The proposed approximate inverse smoothing scheme (7) requires a relaxation parameter  $\omega$  in order to be more effective and efficient. The choice of the relaxation parameter is non-trivial for a wide variety of problems and various choices of the “retention” parameter.



From Figure 3, it can be observed that for greater values of the “retention” parameter  $\delta$  the optimal value of the relaxation parameter  $\omega$  tends to  $\omega=1$ . As the order  $n$  of the linear system (5) increases, finding the optimal value for the relaxation parameter by using different possible relaxation parameters is inefficient. A method to compute the relaxation parameter dynamically is a “predictor–corrector” like scheme. This scheme is based on the DOUR (Dynamic Over / Under Relaxation), [Haelterman, Viederndeels and Van Heule (2006)].

As the order  $n$  of the linear system (5) increases, finding the optimal value for the relaxation parameter by using different possible relaxation parameters is inefficient. A method to compute the relaxation parameter dynamically is a “predictor–corrector” like scheme. This scheme is based on the DOUR scheme, [Haelterman, Viederndeels and Van Heule (2006)].

Let us consider the equivalent expression for the relaxation scheme (7),

$$x_\ell^{(k+1)} = x_\ell^{(k)} + \omega \left( S \left( x_\ell^{(k)} \right) - x_\ell^{(k)} \right) \tag{19}$$

where  $S \left( x_\ell^{(k)} \right) = x_\ell^{(k)} + (M_\ell)_r^{\delta l} \left( s_\ell - A_\ell x_\ell^{(k)} \right)$ .

By applying the predictor–corrector scheme, [Haelterman, Viederndeels and Van Heule (2006)], we have

$$\tilde{x}_\ell^{(k)} = x_\ell^{(k)} + \omega \left( S \left( x_\ell^{(k)} \right) - x_\ell^{(k)} \right) \tag{20}$$

$$x_\ell^{(k+1)} = x_\ell^{(k)} + \kappa \left( \Delta x_\ell^{(k)} \right), \Delta x_\ell^{(k)} = \tilde{x}_\ell^{(k)} - x_\ell^{(k)} \tag{21}$$

where

$$\kappa = \frac{\left\langle \Delta x_\ell^{(k)}, s_\ell - A_\ell \tilde{x}_\ell^{(k)} \right\rangle}{\left\langle \Delta x_\ell^{(k)}, A_\ell \Delta x_\ell^{(k)} \right\rangle} \tag{22}$$

From (19), (20), (21) and (22) we obtain

$$x_\ell^{(k+1)} = x_\ell^{(k)} + \omega_e \left( S \left( x_\ell^{(k)} \right) - x_\ell^{(k)} \right), \omega_e = \omega (1 + \kappa) \tag{23}$$

where  $\omega_e$  is the effective relaxation parameter and equation (23) is the proposed iterative scheme. The equation (23) denotes a two stage non-stationary approximate inverse smoother. Further information and convergence analysis of the DOUR algorithm are given in [Haelterman, Viederndeels and Van Heule (2006)].

In order to accelerate the convergence, multigrid methods are used in conjunction with Krylov subspace iterative solvers. For complex applications the use of

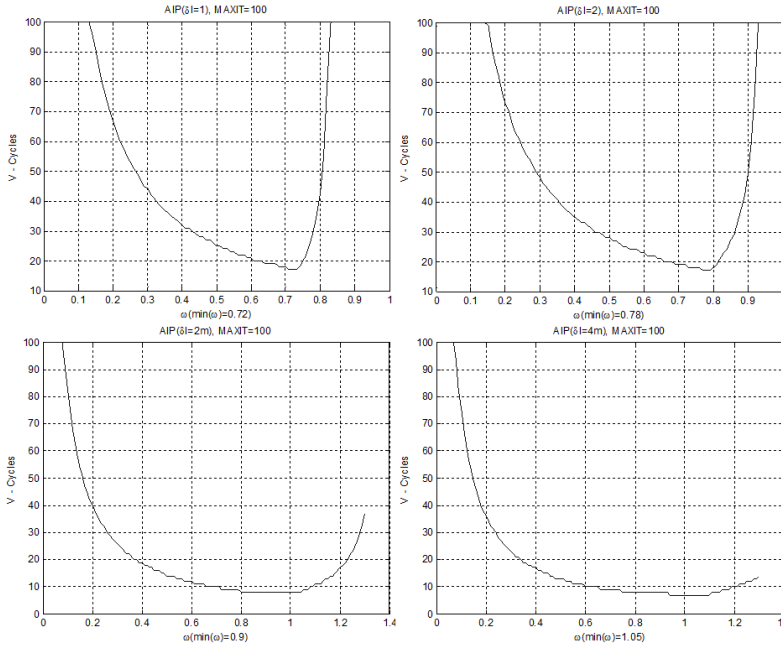


Figure 3: Convergence behavior for the approximate inverse smoother for different values of the “retention” parameter  $\delta l$  and the relaxation parameter  $\omega$

multigrid as a preconditioner to a Krylov subspace method results in an efficient iterative solution method, [Trottenberg, Osterlee and Schuller (2000)]. The proposed V-Cycle multigrid scheme is used in conjunction with Bi-Conjugate Gradient Stabilized method resulting in  $MGV(v_1, v_2, v_3)$  Bi-CGSTAB, where  $v_1$  denotes the pre-smoothing steps,  $v_2$  denotes the post-smoothing steps and  $v_3$  denotes the number of cycles performed during the preconditioning. The algorithm of the  $MGV(v_1, v_2, v_3)$  Bi-CGSTAB can be expressed as follows:

Let  $u_0$  be an arbitrary initial approximation to the solution vector  $u$  and  $r_0$  the residual vector for this initial approximation. Then,

$$\text{compute } r_0 = s - Au_0 \tag{24}$$

$$\text{set } r'_0 = r_0, \rho_0 = \alpha = \omega_0 = 1 \text{ and } v_0 = p_0 = 0 \tag{25}$$

Then, for  $i = 1, \dots$ , (until convergence) compute the vectors  $u_i, r_i, z_i, y_i, p_i, x_i, t_i$  and the scalar quantities  $\alpha, \beta, \omega_i, \rho_i$  as follows:

$$\text{compute } \rho_i = (r'_0, r_{i-1}), \text{ and } \beta = (\rho_i / \rho_{i-1}) / (\alpha / \omega_{i-1}) \tag{26}$$

*compute*  $p_i = r_{i-1} + \beta (p_{i-1} - \omega_{i-1}v_{i-1}),$  (27)

*perform*  $v_3$  Cycles of  $V(v_1, v_2)$  method  
with  $(M_\ell)_r^{\delta l}$  for the linear system  $Ay_i = p_i,$  (28)

*compute*  $v_i = Ay_i$  (29)

*compute*  $\alpha = \rho_i / (r'_0, v_i)$  ,and  $x_i = r_{i-1} - \alpha v_i,$  (30)

*perform*  $v_3$  Cycles of  $V(v_1, v_2)$  method  
with  $(M_\ell)_r^{\delta l}$  for the linear system  $Az_i = x_i,$  (31)

*compute*  $t_i = Az_i,$  (32)

*set*  $\omega_i = (t_i, x_i) / (t_i, t_i)$  (33)

*compute*  $u_i = u_{i-1} + \alpha y_i + \omega_i z_i,$  and  $r_i = x_i - \omega_i t_i.$  (34)

### 3 Numerical Results

In this section numerical results will be presented for the proposed multigrid schemes. Furthermore, comparative results of the proposed schemes against the classic smoothers will be presented. The results were obtained using the MATLAB environment. The convergence factor depends on the required number of iterations for convergence, [Bröker, Grote, Mayer and Reusken (2001); Hackbusch (1985a); Trottenberg, Osterlee and Schuller (2000)]. The convergence factor with respect to the 2-norm is defined as:

$$q = \sqrt[m]{\|r_m\|_2 / \|r_0\|_2} \tag{35}$$

where  $r_m$  is the residual vector at the m-th iteration. The termination criterion for all model problems was  $\|r_m\|_2 / \|r_0\|_2 < 10^{-10}$  and the numbering of the grid was lexicographical. The pre-smoothing and post-smoothing iterations for the model problems were chosen by default at  $v_1 = 2$  and  $v_2 = 1$  respectively. Furthermore, the number of cycles for preconditioning in MG $V(v_1, v_2, v_3)$  Bi-CGSTAB method was chosen by default at  $v_3 = 1$ .

**Model Problem I:** Let us consider the following P.D.E.:

$$-\Delta u = 2 [(1 - 6x^2)y^2 (1 - y^2) + (1 - 6y^2)x^2 (1 - x^2)], (x, y) \in \Omega \tag{36a}$$

$$u(x, y) = 0, (x, y) \in \partial\Omega \tag{36b}$$

where  $\Delta$  is the Laplace operator and  $\Omega$  is the unit square and  $\partial\Omega$  denotes the boundary of  $\Omega$ .

The five point stencil on all required grids of the multigrid method was used. In Table 1, the convergence factors and the convergence behavior of the V(2,1) multigrid method for various smoothers and mesh sizes are presented. In Table 2, the convergence factors and convergence behavior of the V(2,1) multigrid method with Approximate Inverse Preconditioner (AIP) smoothing for mesh size  $h = 1/128$  and different values of the “retention” parameter  $\delta l$ , are given. In Figure 4, behavior of the error measures  $\|r_i\|_2$  of the V(2,1) multigrid method for various smoothers with mesh size  $h=1/128$ , is depicted. In Table 3, the convergence factors and convergence behavior of the MGV(2,1,1) Bi-CGSTAB for various mesh sizes and smoothers are presented. In Table 4, the convergence factors and convergence behavior of the MGV(2,1,1) Bi-CGSTAB method for various choices of the “retention” parameter  $\delta l$  and mesh size are given. In Figure 5, behavior of the error measures  $\|r_i\|_2$  of the MGV(2,1,1)-BiCGSTAB for various smoothers and mesh size  $h=1/128$  is presented.

Table 1: Convergence factors and convergence behavior of V(2,1) multigrid method, for various mesh sizes of the finer grid and smoothers, for model problem I.

<b>h</b>	<b>1/128</b>	<b>1/256</b>	<b>1/512</b>	<b>1/1024</b>	<b>1/2048</b>
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>				
<b>Damped Jacobi</b> $\omega=4/5$	0.2840(18)	0.2901(18)	0.2954(18)	0.2993(19)	0.3087(19)
<b>Gauss - Seidel</b>	0.1463(11)	0.1464(12)	0.1471(12)	0.1489(12)	0.1570(12)
<b>AIP (<math>\delta l=1</math>)</b>	0.2097(14)	0.2168(15)	0.2209(15)	0.2089(14)	0.2073(14)
<b>AIP (<math>\delta l=2</math>)</b>	0.2145(15)	0.1961(14)	0.2070(14)	0.2233(15)	0.2544(16)

**Model Problem II:** Let us also consider a convection–diffusion problem given by:

$$-\varepsilon (\Delta u) + \alpha \frac{\partial u}{\partial x} = A \sin(\ell \pi y) \left( (-\varepsilon \ell^2 \pi^2) x^2 + (\varepsilon \ell^2 \pi^2 - 2\alpha) x + (\alpha + 2\varepsilon) \right),$$

$$(x, y) \in \Omega \quad (37a)$$

$$u(x, y) = 0, (x, y) \in \partial\Omega \quad (37b)$$

where  $\Delta$  is the Laplace operator,  $\Omega$  is the unit square and  $\partial\Omega$  denotes the boundary of  $\Omega$ . The five point stencil was used to discretize the second order partial derivatives and the first order partial derivative was discretized using the downwind stable

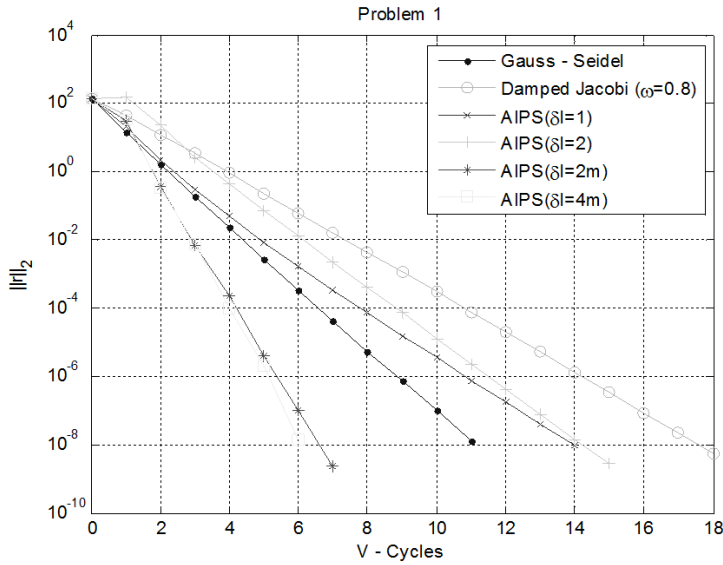


Figure 4: Behavior of the error measures  $\|r_i\|_2$  of the V(2,1) multigrid method for various smoothers and mesh size  $h=1/128$ , for model problem I.

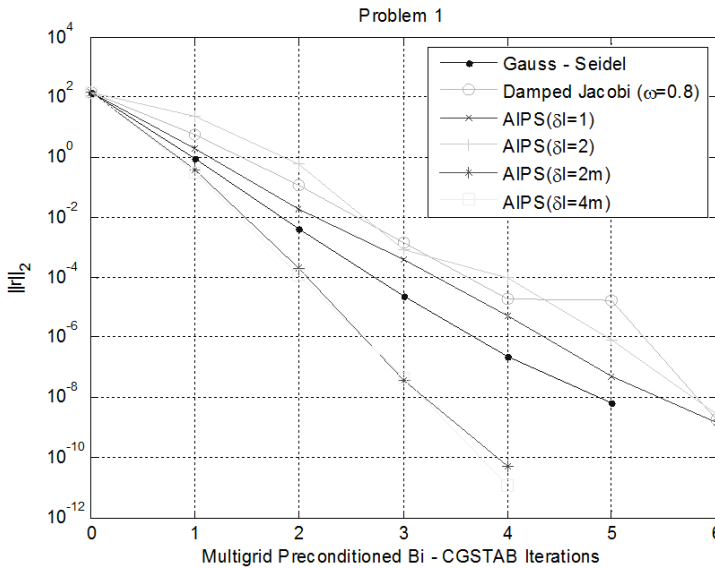


Figure 5: Behavior of the error measures  $\|r_i\|_2$  of the MGV(2,1,1)-BiCGSTAB for various smoothers with mesh size  $h=1/128$ , for model problem I.

Table 2: Convergence factors and convergence behavior of V(2,1) multigrid method, for various values of the “retention” parameter  $\delta l$  with mesh size  $h=1/128$ , for model problem I.

$\delta l$	1	2	2m	4m
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>			
<b>AIP</b>	0.2097(14)	0.2145(15)	0.0451(7)	0.0372(6)

Table 3: Convergence factors and behavior of MGV(2,1,1)-BiCGSTAB, for various mesh sizes of the finer grid and smoothers, for model problem I.

<b>h</b>	<b>1/128</b>	<b>1/256</b>	<b>1/512</b>	<b>1/1024</b>	<b>1/2048</b>
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>				
<b>Damped Jacobi</b> $\omega=4/5$	0.0277(6)	0.0345(6)	0.0330(6)	0.0350(7)	0.0352(6)
<b>Gauss - Seidel</b>	0.0188(5)	0.0172(5)	0.0146(5)	0.0133(5)	0.0135(5)
<b>AIP (<math>\delta l=1</math>)</b>	0.0269(6)	0.0547(8)	0.0455(7)	0.0651(8)	0.0407(7)
<b>AIP (<math>\delta l=2</math>)</b>	0.0259(6)	0.0281(6)	0.0469(8)	0.0502(7)	0.0378(7)

Table 4: Convergence factors and convergence behavior of MGV(2,1,1)-BiCGSTAB, for various values of  $\delta l$  with mesh size  $h=1/128$ , for model problem I.

$\delta l$	1	2	2m	4m
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>			
<b>AIP</b>	0.0269(6)	0.0295(6)	0.0033(4)	0.0024(4)

discretization scheme. The values of the parameters for the convection–diffusion P.D.E were set arbitrarily to  $\varepsilon = 0.1, \alpha = 2, \ell = 3$ .

In Table 5, the convergence factors and convergence behavior of the V(2,1) multigrid method for various mesh sizes and smoothers are presented. In Table 6, the convergence factors and convergence behavior of the V(2,1) multigrid method with AIP Smoothing for different values of the “retention” parameter  $\delta l$  are given. In Figure 6, behavior of the error measures  $\|r_i\|_2$  of the V(2,1) multigrid method, for

various smoothers with mesh size  $h=1/128$ , is depicted. In Table 7, the convergence factors and convergence behavior of the MG $V(2,1,1)$  Bi-CGSTAB method for various mesh sizes and smoothers are presented. In Table 8, the convergence factors and convergence behavior of the MG $V(2,1,1)$  Bi-CGSTAB in conjunction with AIP smoothing for various values of the “retention” parameter  $\delta l$  with mesh size  $h= 1/128$  are given. In Figure 7, behavior of the error measures  $\|r_i\|_2$  of the MG $V(2,1,1)$ -BiCGSTAB for various smoothers with mesh size  $h=1/128$ , is presented.

In Table 9, the convergence factors and convergence behavior of the  $V(2,1)$ -Cycle with the AIP smoother for various values of the perturbation parameter  $\varepsilon$  and mesh size  $h=1/128$ , are presented.

Table 5: Convergence factors and convergence behavior of  $V(2,1)$  multigrid method, for various mesh sizes of the finer grid and smoothers, for model problem II.

<b>h</b>	<b>1/128</b>	<b>1/256</b>	<b>1/512</b>	<b>1/1024</b>	<b>1/2048</b>
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>				
<b>Damped Jacobi</b> $\omega=4/5$	0.3974(24)	0.4086(25)	0.4152(26)	0.4179(26)	0.4202(26)
<b>Gauss - Seidel</b>	0.2297(15)	0.2434(16)	0.2513(16)	0.2555(16)	0.2579(16)
<b>AIP (<math>\delta l=1</math>)</b>	0.2430(16)	0.2408(16)	0.2363(16)	0.2410(16)	0.2444(16)
<b>AIP (<math>\delta l=2</math>)</b>	0.1793(13)	0.2112(14)	0.2282(15)	0.2359(16)	0.2404(16)

Table 6: Convergence factors and convergence behavior of  $V(2,1)$  multigrid method, for various choices of the “retention” parameter  $\delta l$  with mesh size  $h= 1/128$ , for model problem II.

$\delta l$	<b>1</b>	<b>2</b>	<b>2m</b>	<b>4m</b>
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>			
<b>AIP</b>	0.2430(16)	0.1793(13)	0.0907(9)	0.0814(9)

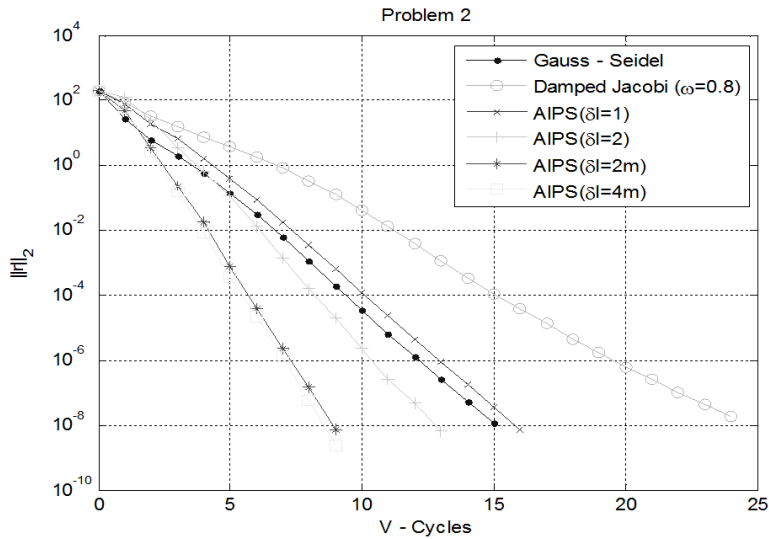


Figure 6: Behavior of the error measures  $\|r_i\|_2$  of the V(2,1) multigrid method for various smoothers with mesh size  $h=1/128$ , for model problem II.

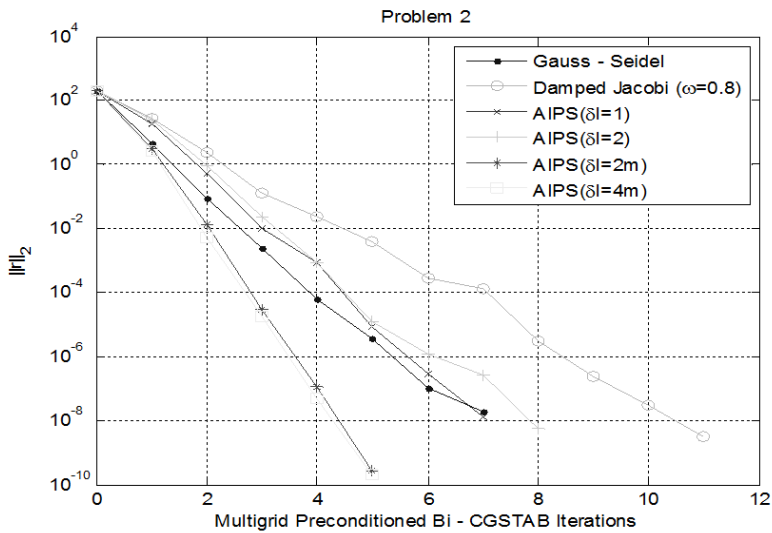


Figure 7: Behavior of the error measures  $\|r_i\|_2$  of the MG(2,1,1)-BiCGSTAB for various smoothers and mesh size  $h=1/128$ , for model problem II.



Table 7: Convergence factors and convergence behavior of MG(2,1,1)-BiCGSTAB, for various mesh sizes of the finer grid and smoothers, for model problem II.

<b>h</b>	<b>1/128</b>	<b>1/256</b>	<b>1/512</b>	<b>1/1024</b>	<b>1/2048</b>
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>				
<b>Damped Jacobi</b> $\omega=4/5$	0.1259(11)	0.1342(11)	0.1388(11)	0.1412(11)	0.1431(11)
<b>Gauss - Seidel</b>	0.0559(7)	0.0468(8)	0.0515(9)	0.0525(9)	0.0528(9)
<b>AIP (<math>\delta l=1</math>)</b>	0.0534(7)	0.0647(8)	0.0725(8)	0.1135(10)	0.0810(9)
<b>AIP (<math>\delta l=2</math>)</b>	0.0672(8)	0.0486(7)	0.0904(8)	0.0548(7)	0.0756(7)

Table 8: Convergence factors and convergence behavior of MG(2,1,1)-BiCGSTAB, for various values of the “retention” parameter  $\delta l$  with mesh size  $h= 1/128$ , for model problem II.

$\delta l$	<b>1</b>	<b>2</b>	<b>2m</b>	<b>4m</b>
<b>Smoother</b>	<b>Convergence factor (Iterations)</b>			
<b>AIP</b>	0.0534(7)	0.0672(8)	0.0106(5)	0.0101(5)

Table 9: Convergence factors and convergence behavior of the V(2,1)-Cycle with the AIP smoother for various values of the perturbation parameter and with  $h=1/128$ .

$\epsilon \delta l$	$10^{-1}$		$10^{-3}$		$10^{-6}$	
	<b>q</b>	<b>iter</b>	<b>q</b>	<b>iter</b>	<b>q</b>	<b>iter</b>
<b>1</b>	0.2430	16	0.6252	49	0.6519	54
<b>2</b>	0.1793	13	0.6255	49	0.6523	54
<b>2m</b>	0.0907	9	0.3314	22	0.4317	27
<b>4m</b>	0.0814	9	0.2147	14	0.3341	24

**Model Problem III:** Let us consider the Bratu nonlinear PDE in two dimensions:

$$-\Delta u(x, y) + \gamma u(x, y) e^{u(x, y)} = 2((x - x^2) + (y - y^2)) + \gamma(x - x^2)(y - y^2) e^{(x - x^2)(y - y^2)}, (x, y) \in \Omega \tag{38a}$$

$$u(x, y) = 0, (x, y) \in \partial\Omega \tag{38b}$$

where  $\Omega$  is the unit square,  $\partial\Omega$  denote the boundary of  $\Omega$  and the value of  $\gamma$  was arbitrarily set to  $\gamma=10$ . The analytical solution is

$$u(x, y) = (x - x^2)(y - y^2), (x, y) \in \Omega \tag{39}$$

The Bratu non-linear problem can be solved by using a linearization method such as Newton’s method in conjunction with the multigrid method, based on Finite Difference approximate inverses, [Gravvanis (2000); Ortega and Rheinboldt (1970)]. The outer stopping criterion for Newton’s method was set to  $\frac{\|r_m\|_2}{\|r_0\|_2} < 10^{-8}$ .

In Table 10, the convergence behavior of the V(2,1) multigrid method for various mesh sizes of the finer grid, for model problem III, is presented. In Table 11, the convergence behavior of the MGV(2,1)-BiCGSTAB for various mesh sizes of the finer grid, for model problem III, is given.

Table 10: Convergence behavior of the V(2,1) multigrid method for various mesh sizes of the finer grid for model problem III.

$\delta l h$	1	2	2m	4m
	Outer(Inner) Iterations			
<b>1/32</b>	4(58)	4(57)	4(53)	4(28)
<b>1/64</b>	4(61)	4(57)	4(54)	4(30)
<b>1/128</b>	4(63)	4(58)	4(54)	4(32)

#### 4 Conclusion

The proposed schemes have been shown to have better convergence behavior and convergence factors compared to damped-Jacobi and Gauss-Seidel smoothing schemes, which are considered as classical smoothing schemes. Moreover, it is known that the approximate inverse smoothing scheme possesses high level of parallelism and does not require complex ordering and coloring schemes such as the Gauss–Seidel iterative scheme. Furthermore, the DOUR algorithm increases the robustness and effectiveness of the approximate inverse smoothing schemes. Finally, we state that

the proposed multigrid scheme can be efficiently used for solving highly non-linear initial/boundary value problems.

Table 11: Convergence behavior of the MG $V(2,1,1)$ -BiCGSTAB for various mesh sizes of the finer grid for model problem III.

$\delta l h$	1	2	2m	4m
	Outer(Inner) Iterations			
1/32	4(25)	4(25)	4(24)	4(15)
1/64	4(25)	4(25)	4(25)	4(16)
1/128	4(28)	4(25)	4(23)	4(16)

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