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General distance transformation for the numerical evaluation of nearly singular integrals in BEM

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Abstract: The accurate and efficient evaluation of nearly singular integrals is one of the major concerned problems in the implementation of the boundary element method (BEM). Among the various commonly used nonlinear transformation methods, the distance transformation technique seems to be a promising method to deal with various orders of nearly singular integrals both in potential and elasticity problems. In this paper, some drawbacks of the conventional distance transformation, such as the sensitivity to the position of projection point, are investigated by numerical tests. A general distance transformation technique is developed to circumvent these drawbacks, which is aimed to remove or weaken the limitations of the projection point. Several numerical examples are presented for both straight and curved line elements to validate the accuracy and efficiency of presented method.

Keywords: Boundary element method, Nearly singular integrals, Numerical integration, Distance transformation technique.

1 Introduction

The nearly singular integrals arise when the source point is very close to but not on the integration element in the implementation of boundary element method (BEM). The conventional Gauss quadrature becomes inefficient or even inaccurate to evaluate such integrals. The accurate and efficient evaluation of nearly singular integrals plays an important role in many cases, especially involving problems of thin or shell-like structures [Cruse and Aithal (1993); Krishnasamy, Rizzo, and Liu (2005); Liu (1998)], the unknowns around crack tips [Aliabadi and Rooke (1991)], the contact problems [Karami (1993)] and the sensitivity problems [Zhang, Rizzo, and Rudolphi (1999)].

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Various numerical techniques have been proposed to remove the near singularities, such as the element subdivision technique [Eberwien, Duenser, and Moser (2005)], the rigid body displacement solutions [Chen, Lu, Huang, and Williams (1998)], global regularization method [Sladek, Sladek, and Tanaka (1993); Liu and Rudolphi (1999)], semi-analytical and analytical algorithms [Niu, Wendland, Wang, and Zhou (2005); Zhou, Niu, Cheng, and Guan (2008)], and various nonlinear transformation methods. The element subdivision technique is simple but not recommended because of its inefficient. The closer the computing point is to the integration element, the more subdivisions are needed, which consumes great computation effort and may increase the accumulative error. The rigid body displacement method constructs a nearly zero factor in the denominator of kernel function by the zero factor in density function using the regularization ideas, but the accuracy of the results are not satisfactory. The analytical and semi-analytical algorithms are effective but only limited to linear or planar elements. Curved elements must be divided into a large number of linear or planar elements, thus losing efficiency and accuracy. At present, the most widely used methods are various nonlinear transformations, such as the cubic polynomial transformation [Telles (2005)], the bi-cubic transformation [Cerrolaza and Alarcon (1989)], the sigmoidal and semi-sigmoidal transformation [Johnston (1999, 2000)], the coordinate optimization transformation [Sladek, Sladek, and Tanaka (2000)], the attenuation mapping method [Nagarajan and Mukherjee (1993)], the rational transformation [Huang and Cruse (1993)], the PART method [Hayami (2005)], the exponential transformation method [Zhang, Gu, and Chen (2009)] and the sinh transformation [Johnston and Elliott (2005)]. The basic ideas of the above transformations can be generalized into two categories: one is removing the nearly zero factor in the denominator of the kernels using zero factor, the other is converting the nearly zero factor in the denominator of the kernels to be part of the numerator. However, most nonlinear transformations are limited to certain order of singularities or specific boundary element. The distance transformation method [Ma and Kamiya (2001, 2002a,b, 2003)], which has been proposed by Ma, is a general strategy to deal with nearly singular integrals in BEM. This promising method is derived from Guiggiani's excellent work for dealing with singular boundary integrals [Guiggiani and Gigante (1990)]. It has been applied to two- and three-dimensional nearly singular integrals with various orders both in potential and elasticity problems, and attractive results have been presented.

However, as the definition of the projection point, finding the projection point is essential for each computation, which may lower the efficiency of the method. The numerical tests in Section 4.1 show that the local coordinate of the projection point must be calculated accurately, otherwise undesirable results will be obtained. Moreover, if the projection point is located on the tangential line through the projection point, the method failed and another transformation should be taken. In this paper, a general distance transformation is developed to circumvent these drawbacks.

The paper is organized as follows. The general form of nearly singular integrals is described in Section 2. The conventional distance transformation is briefly reviewed in Section 3. The drawbacks for conventional distance transformation are presented by numerical tests and a general distance transformation is developed in the Section 4. Some illustrative numerical examples are given to verify the efficiency and accuracy of presented method in Section 5. The paper ends with conclusions in Section 6.

2 Statement of the problem

Considering the description of 2D potential problems in the domain Ω enclosed by boundary Γ, the two basic integral equations are written in terms of the flux q and the potential u on the boundary as follows:

$$c(\mathbf{y})u(\mathbf{y}) = \int_\Gamma q(\mathbf{x})u^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma u(\mathbf{x})q^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) \tag{1}$$

$$c(\mathbf{y})u_k(\mathbf{y}) = \int_\Gamma q(\mathbf{x})u_k^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma u(\mathbf{x})q_k^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) \tag{2}$$

where $\mathbf{y}$ and $\mathbf{x}$ are the source and the field points, respectively. c is a coefficient depending on the smoothness of the boundary at the source point $\mathbf{y}$. $u^*(\mathbf{x},\mathbf{y})$ represents the fundamental solution for 2D potential problems expressed as

$$u^*(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\log(\frac{1}{r}) \tag{3}$$

and $u_k^*(\mathbf{x},\mathbf{y})$, $q^*(\mathbf{x},\mathbf{y})$ and $q_k^*(\mathbf{x},\mathbf{y})$ are the derived fundamental solutions

$$u_k^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial x_k}, \quad q^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}}, \quad q_k^*(\mathbf{x},\mathbf{y}) = \frac{\partial q^*(\mathbf{x},\mathbf{y})}{\partial x_k} \tag{4}$$

where r denotes the Euclidean distance between the source and the field points and $\mathbf{n}$ is the unit outward normal on the boundary Γ.

To evaluate the boundary integrals numerically, the boundary Γ is discretized into a number of linear or quadratic elements and then the boundary integrations are performed on each element. When the source point is very close to but not on the integration element, nearly singular integrals arise with different orders.

In this paper, we deal with these boundary integrals with nearly singularity of the following forms:

$$\mathrm{I} = \int_{-1}^{1} O(1/r^{\chi}) f(\xi)\phi_i(\xi)G(\xi)\mathrm{d}\xi \tag{5}$$

where $O(1/r^{\chi})$ represents the nearly singular integral kernels, $\log(1/r)$ for nearly weak singular integrals, $1/r$ for nearly strong singular integrals and $1/r^2$ for nearly hyper-singular integrals. $f(\xi)$ is a bounded function for local coordinate ξ, $\xi \in [-1,1]$. $\phi_i(\xi)$ denotes the shape functions and $G(\xi)$ is the Jacobian of the transformation from $\mathrm{d}\Gamma$ to $\mathrm{d}\xi$. As the singular integrals over linear elements can be computed analytically, only quadratic elements are discussed in this paper.

3 Conventional distance transformation

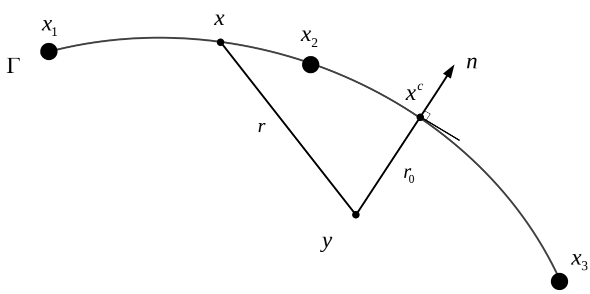

Figure 1: Definition of the projection point $\mathbf{x}^c$

In this section, the definition of the conventional distance function and the variable transformation technique are reviewed. As shown in Fig. 1, the minimum distance r_0 from the source point to the boundary element is defined perpendicular to the tangential line, through the projection point $\mathbf{x}^c$ and the source point $\mathbf{y}$. By employing the first-order Taylor expansion in the neighborhood of the projection point, we have

$$x_k - y_k = x_k - x_k^c + x_k^c - y_k = \frac{\partial x_k}{\partial \xi}|_{\xi=c}(\xi - c) + r_0 n_k(c) + O(|\xi - c|^2) \tag{6}$$

where c is the local coordinate of the projection point $\mathbf{x}^c$. The real distance can be expanded to the following form:

$$
\begin{aligned}
r^2(\xi) &= (x_k - y_k)(x_k - y_k) \\
&= r_0^2 + \frac{\partial x_k}{\partial \xi}\frac{\partial x_k}{\partial \xi}|_{\xi=c}(\xi - c)^2 + 2r_0\frac{\partial x_k}{\partial \xi}|_{\xi=c} n_k(c)(\xi - c) + O(|\xi - c|^3) \\
&= r_0^2 + G_c^2(\xi - c)^2 + O(|\xi - c|^3) \\
&= G_c^2 g^2(\xi) + O(|\xi - c|^3)
\end{aligned}
\tag{7}
$$

where G_c denotes the Jacobian at point c and $g(\xi)$ is the distance function defined as

$$g(\xi) = \sqrt{\alpha^2 + (\xi - c)^2} \tag{8}$$

This definition represents the distance in the local parametric plane and $\alpha = r_0/G_c$. When the projection point is inside of the boundary element, the integration span is split into two parts at point c, taking the following one-order transformation pairs for the integration variable:

$$\eta(\xi) = \log[g(\xi) + (\xi - c)] \tag{9}$$

$$\xi(\eta) = \frac{1}{2}[\exp(\eta) - \alpha^2\exp(-\eta)] + c \tag{10}$$

Substituting Eq. 9 and Eq. 10 into Eq. 5 yields

$$
\begin{aligned}
\mathrm{I} &= \int_{-1}^{1} O(1/r) f(\xi)\phi_i(\xi)G(\xi)\mathrm{d}\xi \\
&= \int_{\eta(-1)}^{\eta(c)} O(1/r^{\chi}) f[\xi(\eta)]\phi_i[\xi(\eta)]G[\xi(\eta)]\mathrm{d}\eta \\
&+ \int_{\eta(c)}^{\eta(1)} O(1/r^{\chi}) f[\xi(\eta)]\phi_i[\xi(\eta)]G[\xi(\eta)]\mathrm{d}\eta
\end{aligned}
\tag{11}
$$

It is easily can be seen that the distance function and the Jacobian of transformation play the role of damping out the nearly singularity of the kernels. For the possibility of unifying and simplifying the computer code, the one-order transformation is used for various orders of singularities, which can obtain an acceptable result even for the hyper-singular kernel [Ma and Kamiya (2002a)].

4 General distance transformation

4.1 Sensitivity to the position of projection point

As we know, finding the accurate position of the projection point is an essential step for the successful implementation of the distance transformation method when dealing with nearly singular integrals. The Newton's method is widely used to find approximate position of the projection point and an inevitable error will be produced. In this section, the influence of the position of the projection point on the accuracy of the distance transformation method is investigated. Here we assume the source point is fixed and the local coordinate ξ_a^c of the approximate projection point is determined by an offset parameter k with the following equation:

$$\xi_a^c = \xi^c + k\xi^c \tag{12}$$

where ξ^c is the accurate local coordinate of the projection point and k indicates the offset caused by the error during finding the projection point. Obviously, the approximate projection point is coincident with the accurate one when $k = 0$.

Considering the first example in Ref. [Ma and Kamiya (2002a)], the relative distance describing the closeness of the near singular point to the boundary is taken as 10^{-4} and ten points Gauss quadrature is used for all the computations. The integrals with kernel u^* and q^* corresponding to different offset values of k have been computed using the conventional distance transformation and the reference values are obtained by subdivision method with enough subelements. Numerical results are shown in Fig. 2 and Fig. 3, and it can be seen that the results obtained with conventional distance transformation is very sensitive to the position of the projection point and poor results are obtained even with a very little deviation of the position of the projection point. Besides, the results get much worse for high order singular integrals.

The drawbacks of the distance transformation method are very obvious: the computation of the position of the projection point should be very rigorous and the process of finding the projection point is time-consuming but essential for each source point, which may lower the computational efficiency. Is the projection point really essential? The work presented later is tried to overcome the shortcomings of the conventional distance transformation method.

4.2 Definition of general distance function

In this section, a general projection point $\mathbf{x}^{c_0}$ is defined to construct a new distance function as shown in Fig. 4. The general projection point $\mathbf{x}^{c_0}$ can be located inside the integration element or on one node of the element. τ and $\mathbf{n}$ are the unit tangential and outward normal vector, respectively. A new vector $\mathbf{d}$ from the source point

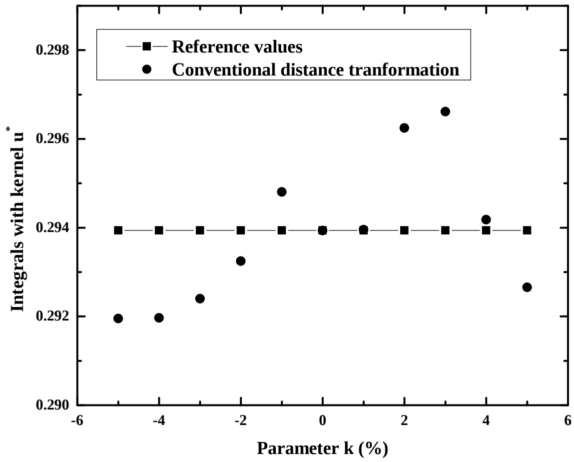

Figure 2: Various integrals with kernel u^*

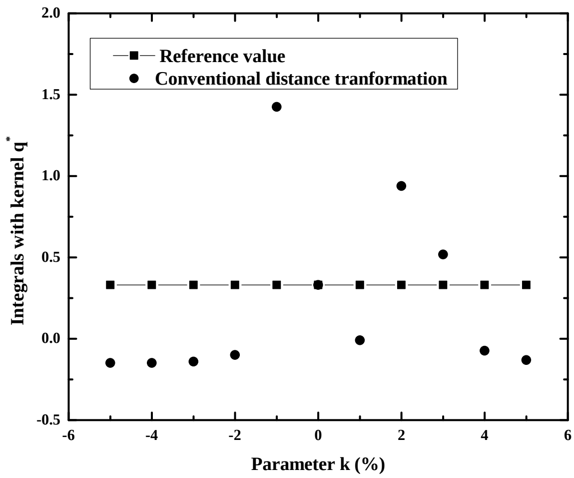

Figure 3: Various integrals with kernel q^*

y to the general projection point $\mathbf{x}^{c_0}$ is defined additionally, which is not required to be perpendicular to the tangential line through $\mathbf{x}^{c_0}$. By applying the first-order Taylor expansion in the neighborhood of point $\mathbf{x}^{c_0}$, we have

$$x_k - y_k = x_k - x_k^{c_0} + x_k^{c_0} - y_k = \frac{\partial x_k}{\partial \xi}|_{\xi=c_0}(\xi - c_0) + d_k + O(|\xi - c_0|^2) \tag{13}$$

where c_0 is the local coordinate of the general projection point $\mathbf{x}^{c_0}$, and d_k is one of the components of **d**. The real distance can also be expanded to the following

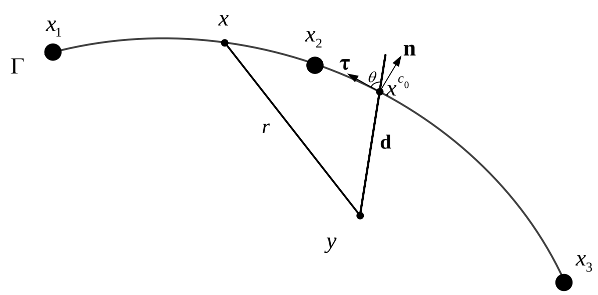

Figure 4: General definition of the projection point $\mathbf{x}^{c_0}$

form:

$$\begin{aligned} r^2(\xi) &= (x_k - y_k)(x_k - y_k) \\ &= d^2 + \frac{\partial x_k}{\partial \xi}\frac{\partial x_k}{\partial \xi}|_{\xi=c_0}(\xi - c_0)^2 + 2d_k\frac{\partial x_k}{\partial \xi}|_{\xi=c_0}(\xi - c_0) + O(|\xi - c_0|^3) \end{aligned} \tag{14}$$

Noted that

$$2d_k\frac{\partial x_k}{\partial \xi}|_{\xi=c_0}(\xi - c_0) = 2G_{c_0}(\mathbf{d}\cdot\tau) = 2G_{c_0}d\cos\theta \tag{15}$$

where G_{c_0} denotes the Jacobian at point c_0 and θ is the angle between $\mathbf{d}$ and τ , which is only related to the position of $\mathbf{x}^{c_0}$ and $\mathbf{y}$. The real distance can be rewritten as

$$\begin{aligned} r^2(\xi) &= d^2 + G_{c_0}^2(\xi - c_0)^2 + 2G_{c_0}d\cos\theta(\xi - c_0) + O(|\xi - c_0|^3) \\ &= G_{c_0}^2[\alpha^2 + (\xi - c_0)^2 + 2\alpha\cos\theta(\xi - c_0)] + O(|\xi - c_0|^3) \\ &= G_{c_0}^2 g^2(\xi) + O(|\xi - c_0|^3) \end{aligned} \tag{16}$$

where $g(\xi)$ is the general distance function defined as follows:

$$g(\xi) = \sqrt{\alpha^2 + (\xi - c_0)^2 + 2\alpha\cos\theta(\xi - c_0)} \tag{17}$$

with α being d/G_{c_0}. $g(\xi)$ represents the distance in the local parametric plane as shown in Fig. 5, which can be proved using the cosine law.

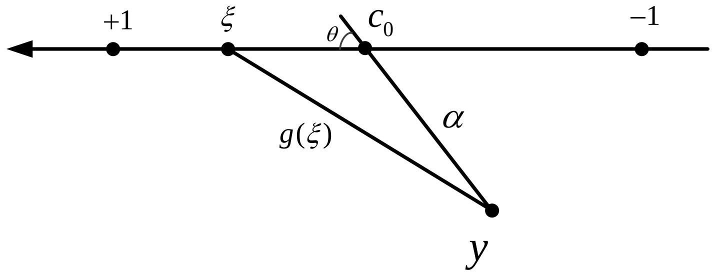

Figure 5: The distance function $g(\xi)$ in the local parametric plane

Now we introduce a similar pair of transformations for the integration variable, which is expressed as

$$\eta(\xi) = \log[g(\xi) + (\xi - c_0) + \alpha\cos\theta] \tag{18}$$

$$\xi(\eta) = \frac{1}{2}\exp(-\eta)[(\exp(\eta) - \alpha\cos\theta)^2 - \alpha^2] + c_0 \tag{19}$$

After splitting the integration element into two parts at point $(c_0 - \alpha\cos\theta)$, which is unnecessary if the general projection point is located at the vertex of the integration element, we can obtain the distance-transformed form of the near singular boundary integrals as Eq. 20. Now the nearly singular integrations with various orders can be computed accurately even if $\mathbf{x}^{c_0}$ is a little far away from the conventional projection point.

$$\begin{aligned}
\mathrm{I} &= \int_{-1}^{1} O(1/r) f(\xi)\phi_i(\xi)G(\xi)\mathrm{d}\xi \\
&= \int_{-1}^{(c_0-\alpha\cos\theta)} O(1/r) f(\xi)\phi_i(\xi)G(\xi)\mathrm{d}\xi + \int_{(c_0-\alpha\cos\theta)}^{1} O(1/r) f(\xi)\phi_i(\xi)G(\xi)\mathrm{d}\xi \\
&= \int_{\log[g(-1)+(-1-c_0)+\alpha\cos\theta]}^{\log[g(c_0-\alpha\cos\theta)]} O(1/r^{\chi}) f[\xi(\eta)]\phi_i[\xi(\eta)]G[\xi(\eta)]\mathrm{d}\eta \\
&+ \int_{\log[g(c_0-\alpha\cos\theta)]}^{\log[g(1)+(1-c_0)+\alpha\cos\theta]} O(1/r^{\chi}) f[\xi(\eta)]\phi_i[\xi(\eta)]G[\xi(\eta)]\mathrm{d}\eta
\end{aligned} \tag{20}$$

5 Numerical examples

In this section, a number of numerical examples including straight and curved lines are presented to validate the accuracy and efficiency of our method. The relative distance is given in terms of r_0/l to describe the influence of the nearly singular integrals over each element, where r_0 is the minimum distance as shown in Fig. 1 and l stands for the length of the element. For the purpose of error estimation, the relative error is defined as follows:

$$error = \frac{\mathrm{I}_{num} - \mathrm{I}_{ref}}{\mathrm{I}_{ref}} \tag{21}$$

where the subscripts num and ref refer to the numerical and reference solutions, respectively. The reference solutions are obtained by subdivision method with enough subelements. Ten Gauss points are used in all cases for the convenience of comparisons.

5.1 Numerical examples for straight line

The first example considers the nearly singular integrals on a straight line with the node coordinates of (0.0, 0.0), (0.5, 0.5) and (1.0, 1.0). The local coordinate of the conventional projection point c is located at 0.5 and the general projection point is put inside the element interval and $c_0 = 0.0$. The relative errors of nearly singular integrals using general and conventional distance transformation are listed in Table Tab. 1. It can be seen that the method using general distance transformation can get results of the same precision as the conventional distance transformation for various orders of singularity. To investigate the influence of the position of the general projection point, c_0 varies from -1.0 to 1.0 by the increment of 0.5 and r_0/l is taken as 10^{-4}. Table Tab. 2 presents the relative error of nearly singular integrals when the general projection point moves along the element. Results with high accuracy can be obtained, even when the general projection point is located on the vertex of the element as $c_0 = \pm 1.0$. These attractive results have demonstrated the efficiency of our method.

Remark 1. For straight lines, the conventional projection point is unessential and the accuracy of results is independent of the projection point. Results with high accuracy can be obtained using general projection point, and the general projection point can be specified arbitrarily, even on the vertex of the element.

5.2 Numerical examples for curved line

The numerical examples in Ref. [Ma and Kamiya (2002a)] is taken as the second example. The example is computed over a curved boundary element with the node

Table 1: Relative error of nearly singular integrals using different distance transformation methods

r_0/l	Methods	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
	Reference	-0.00734757	-0.01275862	-0.01352487	-0.01360412	-0.01361208	-0.01361287
$\phi_1 u^*$	General	8.2539E-09	-5.8075E-08	-1.0094E-05	-2.1425E-04	-1.6457E-03	-6.8727E-03
	Conventional	8.2539E-09	-5.8076E-08	-1.0094E-05	-2.1425E-04	-1.6447E-03	-6.9133E-03
	Reference	-0.01902442	0.00046255	0.00307982	0.00334907	0.00337607	0.00337492
$\phi_1 u_1^*$	General	-1.0156E-07	1.6990E-05	-6.3737E-05	1.7618E-04	1.6887E-03	4.7819E-03
	Conventional	-1.0156E-07	1.6990E-05	-6.3738E-05	1.7638E-04	1.6631E-03	4.4544E-03
	Reference	-0.05979469	-0.08206333	-0.08470973	-0.08497927	-0.08500628	-0.08500766
$\phi_1 u_2^*$	General	-2.7573E-08	-1.0347E-07	-1.5062E-07	9.7391E-06	3.0752E-05	-4.5424E-06
	Conventional	-2.7573E-08	-1.0347E-07	-1.5059E-07	9.7309E-06	3.1754E-05	-2.2208E-05
	Reference	-0.02882893	-0.05835461	-0.06207659	-0.06245757	-0.06249576	-0.06249592
$\phi_1 q^*$	General	6.9486E-09	-7.6565E-09	-2.3814E-06	1.6050E-05	9.4085E-05	1.6572E-04
	Conventional	6.9486E-09	-7.6565E-09	-2.3814E-06	1.6050E-05	9.4069E-05	1.6124E-04

Table 2: Relative error of nearly singular integrals with various general projection points

c_0	-1.0	-0.5	0.0	0.5	1.0
$\phi_2 u^*$	-2.9028E-05	-2.9025E-05	-2.9026E-05	-2.9026E-05	-2.9026E-05
$\phi_2 u_1^*$	7.4074E-06	7.4078E-06	7.4066E-06	7.4064E-06	7.4066E-06
$\phi_2 u_2^*$	8.3820E-05	8.3852E-05	8.3840E-05	8.3839E-05	8.3840E-05
$\phi_2 q^*$	1.6041E-05	1.6045E-05	1.6042E-05	1.6042E-05	1.6042E-05

coordinates of (2.0, 0.0), (1.0, 1.0), and (0.0, 0.5). The local coordinate of the conventional projection point c is set outside of the element interval and $c = 1.01$. The relative error of nearly singular integrals using general and conventional distance transformation is presented in Table Tab. 3. The local coordinate of the general projection point c_0 is located at 0.0. For results obtained with the conventional distance transformation, the precision will decline as the range of r_0/l. The results using the general distance transformation can keep high precision in a wide range of r_0/l, which is better than the results obtained by conventional distance transformation.

The influence of the location of the general projection point is also studied for curved boundary element. As the general projection point moves along the element, computations for nearly singular integrals with r_0/l being 10^{-4} are performed and the relative error is given in Table Tab. 4. As the general projection point is located at the middle of the element, best results can be obtained compared with other locations. All the results are acceptable relative to those obtained with the conventional distance transformation.

As the local coordinate of the conventional projection point c changes from 1.1 to 1.000001, the source point $\mathbf{y}$ becomes increasingly closer to the element, which may lead to poor results during computation of nearly singular integrals. Here we assume $c_0 = 0$ and $r_0/l = 10^{-4}$ to verify the effectiveness of the conventional and general distance transformation. The results with kernel u^* as c varies from 1.1 to 1.000001 are listed Table Tab. 5. It can be easily seen that our method is less sensitive to different values of c and better results can be obtained than the conventional distance transformation.

Remark 2. When the conventional projection point is outside of the curved boundary element, the presented method can get results with high accuracy in a wide range of r_0/l, even in some adverse cases. As the general projection point moves along the element, acceptable results can be obtained compared with the conventional method.

5.3 Sensitivity to the position of projection point

Now we consider the case of curved boundary element with $|c| < 1$, namely, the conventional projection point is located inside the element interval. The curved boundary element in Ref. [Ma and Kamiya (2002a)] is considered and the conventional projection point c is also set at -0.5 with $r_0/l = 10^{-4}$.Giving a set of values of k using in Eq. 12, the relative error for results with kernels u^* and q^* are shown in Fig. 6 and Fig. 7, respectively. The sensitivity to the position of the conventional projection point has been studied in detail, of which the results have been presented in Section 4.1. It is obviously seen that acceptable results can be obtained when there is a little offset. The presented method is less sensitive to the

Table 3: Relative error of nearly singular integrals using different distance transformation methods

r_0/l	Methods	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
	Reference	-0.08013504	-0.07159177	-0.07055153	-0.07044348	-0.07043263	-0.07043155
$\phi_1 u^*$	General	-1.3931E-07	-4.8909E-08	-3.7454E-08	-3.4041E-08	-3.3901E-08	-3.3889E-08
	Conventional	-2.7597E-06	6.2638E-05	8.6432E-07	-3.7262E-07	-1.7228E-05	2.1675E-03
	Reference	0.03240507	0.03656118	0.03868258	0.03896392	0.03899269	0.03899557
$\phi_1 u_1^*$	General	-2.9928E-06	-2.9649E-07	-5.0784E-06	-3.8277E-06	-3.6064E-06	-3.5835E-06
	Conventional	2.9397E-06	-3.4924E-04	-6.5327E-04	-6.5951E-04	-6.7930E-04	1.8681E-03
	Reference	-0.01156755	-0.02226277	-0.02430147	-0.02448001	-0.02449723	-0.02449894
$\phi_1 u_2^*$	General	-4.1133E-05	8.0888E-07	2.1044E-06	6.5314E-06	6.8976E-06	6.9330E-06
	Conventional	-1.5371E-05	-3.5901E-04	-9.5886E-05	-8.7302E-05	-9.4695E-05	8.8743E-04
	Reference	-0.04679418	-0.04497265	-0.04301120	-0.04278013	-0.04275696	-0.04275464
$\phi_1 q^*$	General	8.2062E-06	-2.3720E-07	2.6763E-06	2.4552E-07	-4.2287E-08	-7.1289E-08
	Conventional	5.2246E-06	-2.3952E-04	-4.5426E-04	-4.6333E-04	-4.7736E-04	1.2603E-03

Table 4: Relative error of nearly singular integrals with various general projection points

c_0	-1.0	-0.5	0.0	0.5	1.0	Conventional
$\phi_2 u^*$	-3.1992E-03	-5.8251E-04	-3.9583E-07	1.8035E-05	3.1392E-03	3.1624E-03
$\phi_2 u_1^*$	2.1904E-03	7.7142E-04	2.4198E-06	1.2023E-04	1.7605E-04	1.7676E-04
$\phi_2 u_2^*$	4.3266E-03	1.4958E-03	4.2367E-06	2.2400E-04	1.9356E-04	1.9587E-04
$\phi_2 q^*$	-4.4855E-05	-3.7130E-05	-4.1349E-07	-1.6025E-05	2.4758E-04	2.4803E-04

Table 5: Relative error of weakly singular integrals with different values of c

c	Methods	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\phi_1 u^*$	General	-3.7832E-08	-3.4041E-08	7.3708E-07	1.1750E-06	1.2146E-06	1.2184E-06
	Conventional	-3.0558E-06	-3.7262E-07	-2.1542E-03	-1.0368E-02	-1.4545E-02	-1.5049E-02
$\phi_2 u^*$	General	-5.1030E-08	-3.9583E-07	-4.0356E-05	-6.7932E-05	-7.0701E-05	-7.0970E-05
	Conventional	-1.5179E-06	3.1624E-03	4.2508E-02	2.1993E-01	3.1620E-01	3.2782E-01
$\phi_3 u^*$	General	2.3404E-07	-9.6050E-07	-3.9180E-04	-1.3210E-03	-1.5231E-03	-1.5434E-03
	Conventional	-6.9787E-06	1.0358E-04	6.6305E-04	4.0438E-03	5.5950E-03	5.7729E-03

position of the projection point. On the contrary, the accuracy of results using the conventional distance transformation becomes very poor.

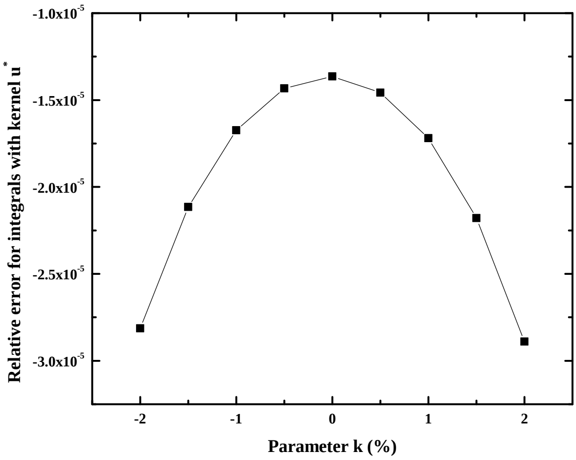

Figure 6: Relative errors for integrals with kernel u^*

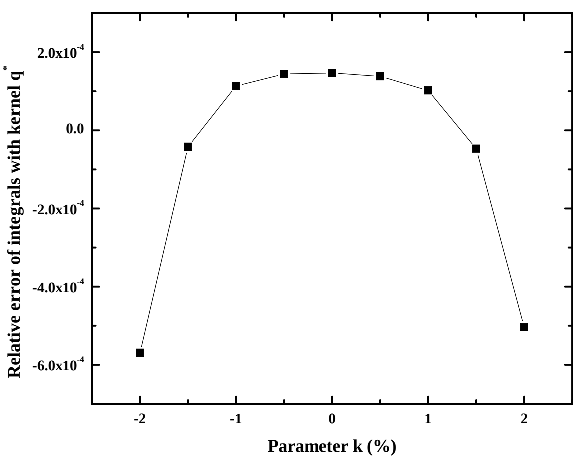

Figure 7: Relative errors for integrals with kernel q^*

Remark 3. For the case of curved boundary element with $|c| < 1$, acceptable results can be obtained when there is a little offset, namely, the accuracy of the presented method is less sensitive to the position of the projection point. This property is particularly beneficial when the projection point cannot be accurately calculated.

6 Conclusions

In this paper, the drawbacks of the conventional distance technique, such as the sensitivity to the position of projection point, are investigated by numerical tests. A general distance transformation technique is developed to remove or weaken the limitations of projection point, which is based on a more general definition of the projection point.

The presented method has been verified through several numerical examples with different kernel functions and relative distances. The results demonstrate that the projection point is completely unessential for straight liner elements. When the conventional projection point is outside of the curved boundary element, the conventional projection point is unessential. If the conventional projection point is located inside the element interval, the presented method is less sensitive to the position of the projection point than the conventional distance transformation. The extended form to deal with hyper-singular integrals will be researched in the later work.

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