 CMES, vol.92, no.1, pp.103-121, 2013

A Regularized Method of Fundamental Solutions Without Desingularization

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Abstract: Some regularized versions of the Method of Fundamental Solutions are investigated. The problem of singularity of the applied method is circumvented in various ways using truncated or modified fundamental solutions, or higher order fundamental solutions which are continuous at the origin. For pure Dirichlet problems, these techniques seem to be applicable without special additional tools. In the presence of Neumann boundary condition, however, they need some desingularization techniques to eliminate the appearing strong singularity. Using fundamental solutions concentrated to lines instead of points, the desingularization can be omitted. The method is illustrated via numerical examples.

Keywords: Method of Fundamental Solutions, regularization, desingularization.

1 Introduction

The popularity of the Method of Fundamental Solutions (MFS, [Alves, Chen and Sarler (2002)]) is based on its simplicity, accuracy and the fact that it is a truly meshless method: it requires neither domain, nor boundary mesh generation. What is needed is a set of points scattered along the boundary without any mesh structure. Originally, the fundamental solution of the partial differential equation to be solved (and/or its derivatives) is shifted to some *source points* located *outside* of the domain of the problem and the approximate solution is sought as a linear combination of the shifted fundamental solutions. Since the fundamental solution of the familiar second-order partial differential equations with constant coefficients are radial function, the MFS can be considered a special radial basis function (RBF-) technique. To compute the unknown coefficients of the linear combination, the boundary conditions are taken into account at some *boundary collocation points*. This results in an linear algebraic system, which is extremely ill-conditioned, especially when the source points are far from the boundary. On the other hand, if they are located too close to the boundary, logarithmic type numerical singularities

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appear in the approximate solution. The presence of Neumann boundary condition makes the situation even worse since it results in stronger singularities at the vicinity of the source points.

The problem of singularity has been treated in various ways. In the boundary knot method (BKM, see [Chen (2002); Chen, Shen, Shen and Yuan (2005)]), nonsingular general solutions are utilized so that the source and collocation points are allowed to coincide without generating singularities. The technique is also an RBF-type method, but still produces highly ill-conditioned matrices. In the desingularization techniques [Young, Chen and Lee (2005); Chen and Wang (2010); Sarler (2008); Sarler (2008); Sarler (2009)], the singular terms are recalculated by using simple analytic solutions of the original problem. Another possibility is to approximate the original second-order problem by a (singularly perturbed) fourth-order one. The new problem can then be solved by MFS, However, since the fundamental solution is now *continuous* at the origin, the problem of singularity does not appear. Unfortunately, this approach fails in the presence of Neumann boundary conditions, therefore it also requires some desingularization technique.

In this paper, we outline several regularization methods which avoid the problem of singularity in the case of pure Dirichlet problems. In the presence of Neumann boundary condition, they have to be coupled with desingularization. After that, another type of MFS is introduced based on special fundamental solutions concentrated to lines (or planes in 3D) instead of single points. These fundamental solutions exhibit much weaker singularity than the traditional fundamental solutions; in particular, they are continuous everywhere. This results is methods (which are *not* of RBF-type any more) that need no desingularization even in the presence of Neumann boundary conditions. The resulting algebraic system remains ill-conditioned (or moderately ill-conditioned). However, the use of the external source points is avoided. The presented methods are illustrated by numerical examples concerning the Laplace as well as the modified Helmholtz equation.

2 The traditional MFS

Consider the following model problem, defined in a bounded 2D domain Ω:

$$(\Delta - \lambda^2 I)u = 0 \quad \text{in } \Omega, \qquad u|_{\Gamma_1} = u_0, \quad \frac{\partial u}{\partial n}|_{\Gamma_2} = v_0 \tag{1}$$

If the Helmholtz parameter λ equals to 0, we have the familiar Laplace equation. Here the boundary Γ is decomposed to the union of the Dirichlet boundary Γ_D and the Neumann boundary Γ_N (the latter may be empty). It is well known that the inhomogeneous equation $\Delta - \lambda^2 I)u = f$ can be converted to the Problem (1) by using the Method of Particular Solutions, is a particular solution (without requiring

any boundary condition) has been computed. This can be performed by using FFT or an RBF-technique completely independently of the solution of the homogeneous problem (1).

The MFS produces an approximate solution of the original problem (1) in the following form:

$$u(x) \sim \sum_{j=1}^{N} \alpha_j \Phi(x - \tilde{x}_j), \tag{2}$$

where $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N$ are external source points, and Φ is a fundamental solution of the operator $(\Delta - \lambda^2 I)$:

$$\Phi(r) = \begin{cases} \frac{1}{2\pi}\log(r) & \text{if } \lambda = 0 \\ \\ \frac{1}{2\pi}K_0(\lambda r) & \text{if } \lambda > 0 \end{cases} \tag{3}$$

where K_0 denotes the usual Bessel function. Φ always has a logarithmic type singularity at the origin: this is the reason that the source points shold be located outside of the domain Ω.

The unknown coefficients $\alpha_1, ..., \alpha_N$ in Equation (2) can be computed by the boundary conditions:

$$\begin{aligned} &\sum_{j=1}^{N} \alpha_j \Phi(x_k - \tilde{x}_j) = u_0(x_k) \qquad (x_k \in \Gamma_D) \\ &\sum_{j=1}^{N} \alpha_j \frac{\partial \Phi}{\partial n_k}(x_k - \tilde{x}_j) = v_0(x_k) \qquad (x_k \in \Gamma_N) \end{aligned} \tag{4}$$

where $x_1, ..., x_N \in \Gamma$ are the boundary collocation points, and n_k is the outward normal unit vector at x_k. The number of the collocation points is not necessarily equals to the number of the source points. If the two numbers are different, (4) is to be solved e.g. in the sense of least squares. For simplicity, we assume that the numbers of the source and collocation points are equal.

In fact, the form (2) gives us a solution which is automatically extended outside of Ω. From numerical points of view, it is highly ill-posed problem, even if this extension exists. If the solution is smooth enough, the method produces extremely exact approximate solution, but the system (4) is generally strongly ill-conditioned. The further the source points are located from the boundary, the greater the condition number. As an illustrative example, consider the simple 2D Laplace equation with the test solution

$$u(x, y) = \cos \pi x \cdot \sinh \pi y \tag{5}$$

Here we used the more familiar notations x, y for the spatial variables. The domain of the problem is the unit circle, and we assumed pure Dirichlet boundary condition. Table 1 shows the relative L_2-errors of the approximate solution with different numbers of source points (N) and different distances from the boundary (d), while Table 2 shows the corresponding condition numbers. The source as well as the collocation points are equally spaced along the circle with radius $(1+d)$ and the unit circle, respectively.

Table 1: Traditional MFS, relative L_2-errors (%), Test solution: (5). N is the number of sources, d is their distance from the boundary. Pure Dirichlet boundary condition.

$d \setminus N$	16	32	64	128	256
1/8	9.83	0.764	0.007	2.0E–6	2.9E–13
1/4	4.53	0.054	1.9E–5	5.9E–12	2.3E–13
1/2	1.77	0.001	1.4E–9	1.4E–13	6.6E–13
1	0.39	4.9E–5	8.0E–13	8.0E–12	6.1E–12
2	0.35	1.4E–5	3.1E–11	8.1E–11	3.0E–10

Table 2: Traditional MFS, condition numbers. Test solution: (5). N is the number of sources,d is their distance from the boundary. Pure Dirichlet boundary condition.

$d \setminus N$	16	32	64	128	256
1/8	9	46	616	5.3E+4	2.0E+8
1/4	19	227	1.6E+4	4.0E+7	1.2E+14
1/2	83	4.2E+3	5.5E+6	4.8E+12	$>$ 1.0E+16
1	1.4E+3	7.2E+5	9.5E+10	$>$ 1.0E+16	$>$ 1.0E+16
2	5.7E+4	7.5E+8	$>$ 1.0E+16	$>$ 1.0E+16	$>$ 1.0E+16

3 Regularization techniques and desingularization

Consider again the model problem (1). The following regularization techniques are based on replacing the fundamental solution with another function that has no singularity at the origin. This makes it possible to avoid the problem of singularity, when pure Dirichlet boundary condition is applied. In the presence of Neumann boundaries, however, special desingularization tools are required.

From now on, the source and the boundary collocation points are supposed to coincide: they are denoted by $x_1, ..., x_N \in \Gamma$.

Regularization by using truncated fundamental solutions. Now instead of the fundamental solution (3), the following truncated functions are used:

$$\Phi(r) = \begin{cases} \frac{1}{2\pi} K_0(\lambda r) & \text{if } r > \frac{1}{c} \\ \\ \frac{1}{2\pi} K_0(\lambda \cdot \frac{1}{c}) & \text{if } r \leq \frac{1}{c} \end{cases} \tag{6}$$

In the case of the Laplace equation (i.e. when $\lambda = 0$), the following simpler form can be applied:

$$\Phi(r) = \begin{cases} \frac{1}{2\pi} \log(c \cdot r) & \text{if } r > \frac{1}{c} \\ \\ 0 & \text{if } r \leq \frac{1}{c} \end{cases} \tag{7}$$

Here c denotes a positive scaling constant that should be defined carefully. The value $\frac{1}{c}$ should remain under the characteristic distance of the boundary collocation points. Thus, the effect of truncation is restricted to a narrow vicinity of the boundary collocation points $x_1, ..., x_N$ This results in a small distortion of the domain as well as the Dirichlet boundary data, therefore it can be expected that the method generates only small errors et least in case of pure Dirichlet boundary conditions. Table (3) shows the relative L_2-errors of the approximate solution with different numbers of boundary collocation points (N) and different values of the scaling parameter c. Here the model problem (1) is solved in the unit circle supplied with pure Dirichlet boundary conditions. The boundary collocation points are equally spaced along Γ. The Helmholtz constant is set to the value $\lambda := 1$. The test solution of (1) is

$$u(x,y) = \cos x \cdot \sinh(\sqrt{1+\lambda^2} \cdot y) \tag{8}$$

and the Dirichlet boundary condition is defined in a consistent way. Table (4) shows the corresponding condition numbers. The optimal value of $\frac{1}{c}$ seems to be proportional to the characteristic distance of the boundary collocation points (as expected), i.e. the optimal value of c is proportional to the number of boundary collocation points. In this case, the condition numbers remain moderate and much less that in the case of the traditional MFS. The price of this property is the fact that the exactness is lower that in the case of the traditional MFS, even if the optimal scaling constant is applied.

Table 3: Method of truncated fundamental solutions, relative L_2-errors (%). Test solution: (8). N is the number of sources, c is the scaling constant. Pure Dirichlet boundary condition.

$c \setminus N$	16	32	64	128	256
8	13.01	13.24	11.99	11.99	11.87
16	**3.53**	6.67	6.69	6.14	6.14
32	12.37	**1.19**	3.34	3.34	3.07
64	21.14	6.34	**0.207**	1.65	1.65
128	28.39	11.67	3.19	**0.084**	0.827
256	34.43	16.44	6.15	1.63	**0.008**

Table 4: Method of truncated fundamental solutions, condition numbers. Test solution: (8). N is the number of sources, c is the scaling constant. Pure Dirichlet boundary condition.

$c \setminus N$	16	32	64	128	256
8	11	1.8E+3	3.7E+3	1.2E+6	2.5E+6
16	6	23	1.1E+4	2.3E+4	2.1E+6
32	4	12	48	7.7E+4	1.5E+5
64	3	8	24	97	5.2E+5
128	3	6	16	49	195
256	2	5	12	33	98

Desingularization. For Neumann and mixed problems, the above method does not work due to the strong singularity of the normal derivative of the function $\Phi(x-x_k)$ at the point x_k, i.e. the diagonal elements of the system matrix of (4). The normal derivative of u at $x_k \in \Gamma_N$ is expressed as

$$\frac{\partial u}{\partial n}(x_k) = \sum_{j \neq k} \alpha_j B_{kj} + \alpha_k \cdot B_{kk}$$

where for $j \neq k$:

$$B_{kj} := \frac{\partial \Phi}{\partial n}(x_k - x_j)$$

but the diagonal entry B_{kk} cannot be calculated in a similar way due to the singularity of the derivative of Φ. In the desingularization techniques [Young, Chen and

Lee (2005); Chen and Wang (2010); Sarler (2008); Sarler (2009)], these diagonal entries are calculated in the following way. Let w be a known, smooth solution of the original problem (1). Expressing w in the same form:

$$w(x) \sim \sum_{j=1}^{N} \beta_j \Phi(x - x_j), \tag{9}$$

the coefficients β_j can be computed from the Dirichlet data. Having computed $\beta_1, ..., \beta_N$, the Neumann data are expressed in a similar form as earlier:

$$\frac{\partial w}{\partial n}(x_k) = \sum_{j \neq k} \beta_j B_{kj} + \beta_k \cdot B_{kk},$$

from which the diagonal term B_{kk} can be computed:

$$B_{kk} = \frac{1}{\beta_k} \left(\frac{\partial w}{\partial n}(x_k) - \sum_{j \neq k} \beta_j B_{kj} \right) \tag{10}$$

and the system (4) of the original problem now has the form:

$$\begin{aligned} &\sum_{j=1}^{N} \alpha_j A_{kj} = u_0(x_k) \quad (x_k \in \Gamma_D) \\ &\sum_{j=1}^{N} \beta_j B_{kj} = v_0(x_k) \quad (x_k \in \Gamma_N) \end{aligned} \tag{11}$$

where for $j \neq k$:

$$A_{kj} = \Phi(x_k - x_j), \qquad B_{kj} = \frac{\partial \Phi}{\partial n_k}(x_k - x_j)$$

and B_{kk} is defined by (10).

In the case of the simple Laplace equation, w can be chosen to be e.g. an identically constant function. In this case, $\frac{\partial w}{\partial n}$ is identically zero, which simplifies the calculation. In the more general modified Helmholtz problem, a possible definition of w can be $w(x,y) := e^{\lambda x}$ or $w(r) := I_0(\lambda r)$, where I_0 is the familiar Bessel function.

Remarks:

- Using several predefined solutions, it is possible to define not only the diagonal terms B_{kk} but their neighboring entries as well, which further reduces the effect of singularity of the function $\frac{\partial \Phi}{\partial n}(x - x_k)$.

- It should be pointed out that using any type of desingularization, it is sufficient to compute the values of the solution *along the Neumann boundary* Γ_N *only*. Once the values along the whole boundary have been computed, the values inside the domain Ω can be reconstructed by solving *a pure Dirichlet problem*, which is often a simpler task from computational point of view, as mentioned in the Introduction. Moreover, applying quadtree-based multilevel techniques, the problem of the use of large, ill-conditioned matrices can be completely avoided [Gáspár (2002); Gáspár (2004)].

To illustrate the method, consider again the model problem (1) with the test solution (8), where the Helmholtz constant is set to the value $\lambda := 1$. Now the boundary Γ is decomposed into two half-circles (Dirichlet and Neumann boundaries). The boundary collocation points are equally spaced along the boundary. Table 5 shows the relative L_2-errors on the boundary Γ (cf. the previous Remark) with different numbers of boundary collocation points (N) and different scaling constants c. It can clearly be seen that the optimal value of c is again inversely proportional to the characteristic distance of the boundary collocation points.

Table 5: Method of truncated fundamental solutions combined with desingularization. Relative L_2-errors (%) on the boundary. Test solution: (8). N is the number of sources, c is the scaling constant. Mixed boundary conditions.

$c \setminus N$	16	32	64	128	256
8	23.28	23.28	20.81	21.91	21.97
16	**0.491**	12.48	12.39	11.16	11.48
32	31.32	**0.063**	6.469	6.431	5.820
64	64.31	15.90	**0.008**	3.294	3.282
128	224.1	35.95	7.183	**0.001**	1.663
256	273.8	212.8	15.57	3.466	**1.4E–4**

Regularization by using fourth-order fundamental solutions. Now instead of the fundamental solution (3), the following RBFs are used (see [Gáspár (2008)] for details):

$$\Phi(r) := -\frac{1}{2\pi}\frac{1}{c^2-\lambda^2}\left(K_0(cr)-K_0(\lambda r)\right) \tag{12}$$

if $\lambda > 0$. In case of the Laplace equation (i.e. when $\lambda = 0$):

$$\Phi(r) := -\frac{1}{2\pi c^2}\left(K_0(cr) + \log(cr)\right) \tag{13}$$

Here c is again a scaling parameter. Note that these functions are *continuous* at the origin (the singularities cancel out). Using the series expansion of the function K_0, it can easily be deduced, that for the function defined by (12), the equality

$$\Phi(0) = -\frac{1}{2\pi}\frac{-\log c + \log\lambda}{c^2 - \lambda^2}$$

holds. A similar expression can be deduced also for the function (13):

$$\Phi(0) = -\frac{1}{2\pi c^2}\left(\log 2 - \gamma\right),$$

where γ denotes the Euler constant: $\gamma = 0.5772156....$

The functions (12) and (13) are the fundamental solutions of the fourth-order partial differential operator $\Delta(\Delta - c^2 I)$ and $(\Delta - \lambda^2 I)(\Delta - c^2 I)$, respectively. Consequently, the approximate solution

$$\sum_{j=1}^{N} \alpha_j \Phi(x - x_j) \tag{14}$$

solves *exactly* the fourth-order problems

$$\Delta\left(I - \frac{1}{c^2}\Delta\right)u = 0, \qquad \text{and} \qquad (\Delta - \lambda^2 I)\left(I - \frac{1}{c^2}\Delta\right)u = 0,$$

respectively. By a careful choice of the scaling parameter, the error of the approximate solution can be minimized (see [Gáspár (2004); Gáspár (2008)] for details). The parameter c should be again inversely proportional to the characteristic distance of the boundary collocation points.

The approach is mathematically elegant; however, the Neumann boundary conditions still require desingularization as described above. First, consider the problem (1) with an arbitrary predefined, smooth solution $w \neq 0$. Expressing w in the RBF-form:

$$w(x) \sim \sum_{j=1}^{N} \beta_j \Phi(x - x_j),$$

the coefficients β_j can be calculated. Now expressing $\frac{\partial w}{\partial n}$, we have:

$$\frac{\partial w}{\partial n} \sim \sum_{j \neq k} \beta_j B_{kj} + \beta_k B_{kk},$$

where $B_{kj} := \frac{\partial \Phi}{\partial n_k}(x_k - x_j$ for $j \neq k$. From here, the diagonal term B_{kk} can be calculated for all boundary points x_k which are located on the Neumann boundary Γ_N. For Dirichlet boundary points, no desingularization is needed.

Remark: The regularization based on higher order fundamental solutions can be applied for more general problems as well. In [Gáspár (2009)], the technique is applied to steady Stokes problem. Note, however, that for Stokes problems, the usual boundary condition is of Dirichlet type, so that desingularization is not needed.

As an example, consider again the model problem (1) with the test solution (8), where the Helmholtz constant is set to the value $\lambda := 1$. The boundary Γ is again decomposed into tho half-circles as Dirichlet and Neumann parts. The boundary collocation points are equally spaced along the boundary. Table 6 shows the relative L_2-errors on the boundary Γ with different numbers of boundary collocation points (N) and different scaling constants c. It can be seen again that the optimal value of c is inversely proportional to the characteristic distance of the boundary collocation points.

Table 6: MFS with fourth-order fundamental solution combined with desingularization. Relative L_2-errors (%) on the boundary. Test solution: (8). N is the number of sources, c is the scaling constant. Mixed boundary conditions.

$c \setminus N$	16	32	64	128	256
8	28.35	31.32	31.51	31.40	31.31
16	**4.307**	15.41	17.08	17.24	17.22
32	87.78	**2.252**	8.039	8.974	9.082
64	54.11	12.78	**1.138**	4.112	4.612
128	168.1	150.4	5.918	**0.570**	2.081
256	412.4	103.4	14.03	2.873	**0.285**

Regularization by using nearly fundamental solutions. This technique can be applied to potential problems only. Instead of the fundamental solutions (3), the

following function is applied (which is already not of RBF-type):

$$\Phi_j(x) := \log \frac{r_1 + r_2 + \sqrt{(r_1+r_2)^2 - a^2}}{a}, \tag{15}$$

Here $a > 0$ is a scaling constant and

$$\begin{aligned} r_1 &:= \sqrt{\left(x - \tfrac{a}{2}e_j^{(x)}\right)^2 + \left(y - \tfrac{a}{2}e_j^{(y)}\right)^2}, \\ r_2 &:= \sqrt{\left(x + \tfrac{a}{2}e_j^{(x)}\right)^2 + \left(y + \tfrac{a}{2}e_j^{(y)}\right)^2} \end{aligned} \tag{16}$$

where $e_j = (e_j^{(x)}, e_j^{(y)})$ is the tangential unit vector at the boundary point x_j. Clearly, r_1 and r_2 are the distances between the point (x,y) and the endpoints of the segment $\Gamma_j := \left[-\frac{a}{2}e_j, \frac{a}{2}e_j\right]$. (Here we used the more familiar notations x, y for the spatial variables.) It is easy to see, that Φ_j is continuous everywhere, vanishes along the segment Γ_j and harmonic outside. Thus the shifted function $\Phi(x - x_j)$ vanishes on a small straight segment which contains the boundary collocation point x_j and has tangential direction. The approximate solution of (1) can be sought in the following nearly RBF-form:

$$u(x) \sim \sum_{j=1}^{N} \alpha_j \Phi_j(x - x_j)$$

Since all Φ_j's are continuous, no singularity problem arises in case of pure Dirichlet problem. In the presence of Neumann boundaries, the method still needs desingularization.

As an illustration, consider again the model problem (1) in the unit circle with the harmonic test solution (5) (the Helmholtz constant is now zero). The boundary Γ is decomposed into two half-circles as Dirichlet and Neumann parts. The boundary collocation points are equally spaced along the boundary. Table 7 shows the relative L_2-errors on the boundary Γ with different numbers of boundary collocation points (N) and different scaling constants c: here, in order to be consistent with the previous examples, c is defined by $c := \frac{2}{a}$. The optimal value of c is still inversely proportional to the characteristic distance of the boundary collocation points.

Note that the technique can be generalized to 3D potential problems as well. Now the functions (15) can be defined with the help of the function

$$\Phi(x,y,z) := \frac{2}{\pi} \arcsin \frac{a}{\sqrt{z^2 + \left(r - \frac{a}{2}\right)^2} + \sqrt{z^2 + \left(r + \frac{a}{2}\right)^2}}, \tag{17}$$

Table 7: The use of potentials concentrated on straight segments conbined with desingularization. Relative L_2-errors (%) on the boundary. Test solution: (5). N is the number of sources, c is the scaling constant. Mixed boundary conditions.

$c \setminus N$	16	32	64	128	256
8	**3.568**	6.287	0.974	0.939	0.350
16	22.42	**0.945**	2.757	0.358	0.413
32	41.82	10.14	**0.488**	1.252	0.163
64	61.30	19.71	4.989	**0.265**	0.590
128	80.80	29.38	9.754	2.489	**0.140**
256	100.2	39.08	14.58	4.863	1.245

where $r := \sqrt{x^2+y^2}$. The function Φ identically equals to 1 on the closed disc of the xy-plane centered at the origin with radius $\frac{a}{2}$ and harmonic outside [Atkinson, Young and Brezovich (1983)].

4 Regularization techniques without desingularization

All the previous techniques presented in the previous section need desingularization due to the fact that they are based on nearly fundamental solutions which are continuous at the origin, but their normal derivative still exhibits numerical singularity. This can be avoided if more smooth functions Φ are applied. Typically this is the case in the boundary knot method (BKM, see [Chen (2002); Chen, Shen, Shen and Yuan (2005)]), where nonsingular general solutions are used, which are still radial functions. Here we outline a different approach which is based on fundamental solutions concentrated to straight lines/planes. More precisely, let L be a partial differential operator and let n be a unit vector, Consider a function Φ_n satisfying the equation

$$L\Phi_n = \delta_n \tag{18}$$

where δ_n denotes the Dirac distribution concentrated to the line/plane which passes the origin and has the normal vector n. An approximate solution of the equation $Lu = 0$ is sought in the form:

$$u(x) \sim \sum_{j=1}^{N} \alpha_j \Phi_{n_j}(x - x_j) \tag{19}$$

where n_j is the normal direction at the boundary point x_j. In other words, the approximate solution is sought as a linear combination of the above fundamental solutions concentrated to the *tangents* of Ω at the boundary collocation points,

provided for the time being that the domain Ω is strictly convex. The coefficients $\alpha_1, ..., \alpha_N$ can be computed by enforcing the boundary conditions:

$$\begin{aligned} &\textstyle\sum_{j=1}^{N} \alpha_j \Phi_{n_j}(x_k - x_j) = u_0(x_k) \qquad (x_k \in \Gamma_D) \\ &\textstyle\sum_{j=1}^{N} \alpha_j \frac{\partial \Phi_{n_j}}{\partial n_k}(x_k - x_j) = v_0(x_k) \qquad (x_k \in \Gamma_N) \end{aligned} \tag{20}$$

The main advantage of the approach is that the above fundamental solutions are smoother and have much simpler form than the traditional fundamental solutions (concentrated to the origin). In the case of the modified Helmholtz operator $(\Delta - \lambda^2 I)$, the fundamental solution is as follows

$$\Phi_n(x,y) = \frac{1}{2\lambda} e^{-\lambda |y|} \tag{21}$$

provided that the normal vector is $n = (0,1)$. Otherwise, the fundamental solution can be obtained by rotating the coordinate system, i.e.

$$\Phi_n(x,y) = \frac{1}{2\lambda} e^{-\lambda |x \cdot n^{(x)} + y \cdot n^{(y)}|}, \tag{22}$$

if $n = \left(n^{(x)}, n^{(y)}\right)$.

Equation (21) is not valid, if the Helmholtz parameter λ equals to 0. In this case (i.e. in the case of the Laplace operator), the above fundamental solution has the form:

$$\Phi(x,y) = \frac{1}{2}|y|,$$

if $n = (0,1)$. Otherwise, if $n = \left(n^{(x)}, n^{(y)}\right)$, we have:

$$\Phi(x,y) = \frac{1}{2} \cdot \left| x \cdot n^{(x)} + y \cdot n^{(y)} \right|$$

However, now the functions $\Phi_{n_j}(x - x_j)$ are not linearly independent. In this case, the functions defined by (15) should be used. The scaling parameter a – in contrast to the previous subsection – should be defined to be much higher. The parameter a should be in the same order of magnitude than the diameter of Ω. As expected, the condition numbers become very high when the number of boundary collocation points increase. However, due to the lack of singularity, no desingularization is needed.

To illustrate the method, consider the following test solutions in the unit circle:

$$u(x,y) := \cos \pi x \cdot \sinh \pi y \tag{23}$$

and

$$u(x,y) := \cos x \cdot \sinh(\sqrt{1+\lambda^2} \cdot y) \tag{24}$$

The function (23) is harmonic, while function (24) satisfies the modified Helmholtz equation. It this example, the parameter λ is defined as $\lambda := 2$. In both cases, mixed boundary conditions are prescribed on the half-circles Γ_D and Γ_N, respectively. The boundary collocation points are equally spaced along Γ. Table 8 shows the relative L_2-errors belonging to the test solution (23) on the boundary Γ with different numbers of boundary collocation points (N) and scaling parameter a. Table 9 shows the corresponding condition numbers. In this case, there is no optimal value for the parameter a: the higher the parameter a, the more exact the approximate solution, however, the larger the condition number. For the test solution (24), Table 10 shows the relative L_2-errors of the method on the boundary, and also the corresponding condition numbers with different numbers of boundary points N.

Table 8: The use of potentials concentrated on straight segments without desingularization. Relative L_2-errors on the boundary. Test solution: (23). N is the number of sources, a is the scaling constant. Mixed boundary conditions

$a \backslash N$	16	32	64	128	256
2	16.1	0.013	1.6E–7	9.0E–10	2.7E–8
4	0.47	5.5E–5	6.8E–9	7.1E–8	2.7E–7
8	11.1	2.9E–6	6.6E–6	3.8E–4	4.7E–3

Table 9: The use of potentials concentrated on straight segments without desingularization. Condition numbers. Test solution: (5). N is the number of sources, a is the scaling constant. Mixed boundary conditions

$a \backslash N$	16	32	64	128	256
2	5.1E+3	6.8E+5	1.7E+8	2.1E+13	$>$ 1.0E+16
4	1.1E+4	2.3E+8	1.3E+14	8.3E+15	$>$ 1.0E+16
8	3.3E+7	1.2E+11	5.1E+14	3.4E+15	$>$ 1.0E+16

Table 10: The use of potentials concentrated on straight segments without desingularization. Relative L_2-errors on the boundary and condition numbers. Test solution: (24). N is the number of sources, a is the scaling constant. Mixed boundary conditions

N	16	32	64	128	256
Rel. L_2-error	0.010	3.3E–8	1.0E–12	1.2E–11	3.6E–12
Cond. number	588	2.5E+8	$>$ 1.0E+16	$>$ 1.0E+16	$>$ 1.0E+16

Remark: If the domain is not strictly convex, then the above approaches fail since (19) produces a function which does not satisfy the differential equation everywhere (since it is not differentiable along the corresponding tangents belonging to the boundary collocation points). However, the following smooth functions are still applicable:

$$\Phi(x,y) = \sinh \lambda y \tag{25}$$

for the modified Helmholtz operator, and:

$$\Phi_j(x) := \operatorname{sign} y \cdot \log \frac{r_1 + r_2 + \sqrt{(r_1+r_2)^2 - a^2}}{a}, \tag{26}$$

where $a > 0$ is the scaling constant and

$$r_1 := \sqrt{\left(x - \frac{a}{2}\right)^2 + y^2}, \qquad r_2 := \sqrt{\left(x + \frac{a}{2}\right)^2 + y^2} \tag{27}$$

for the Laplace equation, when the normal vector of the line is $n = (0,1)$. (In the general case, the functions are derived by rotating the coordinate system.) Note that the function (26) is smooth everywhere with the exception of the half-lines $(-\infty, -\frac{a}{2}]$ and $[\frac{a}{2}, \infty)$. However, if the parameter a is large enough, then the functions $\Phi_{n_j}(x - x_j)$ remain harmonic in the whole domain ω.

A mixed strategy for mixed problems. The above technique can be combined with the previous regularization methods, since the approximation corresponding to the Dirichlet and Neumann boundaries need not be indentical. A natural idea is to use (6) - (7) or (12) - (13) or (15) along the Dirichlet boundary and (15) (with sufficiently large scaling parameter a) or (21) along the Neumann boundary. In this case the approximate solution has the form:

$$u(x) \sim \sum_{x_j \in \Gamma_D} \alpha_j \Phi_j(x - x_j) + \sum_{x_j \in \Gamma_N} \alpha_j \Psi_j(x - x_j)$$

where the functions Φ_j denote the regularized fundamental solutions (or their approximations as presented previously), and Ψ_j denote a fundamental solution concentrated to the tangent (or its harmonic approximation). Thus, the ill-conditioned part of the problem is restricted to the Neumann boundary.

As an example, consider the harmonic test function

$$\Phi(x,y) = x^2 - y^2 \tag{28}$$

in the unit circle, where mixed boundary condition is prescribed along the half-circles Γ_D and Γ_N, respectively. The boundary collocation points are equally spaced along Γ. Along Γ_D, the truncated fundamental solution (7) is applied, while along Γ_N, the nearly fundamental solution (15) is applied with the scaling parameter $a := 1$. Table 11 shows the relative L_2-errors on the boundary with different numbers of boundary collocation points (N) and with different values of the scaling parameter c (note that c has effect only on basis functions corresponding to the Dirichlet boundary). Table 12 shows the corresponding condition numbers. Observe that the optimal value of c is still proportional to the characteristic distance of the boundary collocation points; however, the condition numbers remain moderate.

Remark: Without going details we note that the ill-conditioned Neumann subproblem can be solved separately by using the Schur complement of the system matrix. Thus, the ill-conditioned character can be restricted to the Neumann boundary.

Table 11: Mixed strategy, relative L_2-errors on the boundary. Test solution: (28). N is the number of sources, a is the scaling constant. Mixed boundary conditions

$c \setminus N$	16	32	64	128	256
16	**5.53**	7.69	4.05	6.43	16.8
32	4.76	**2.78**	2.44	2.63	13.4
64	4.89	4.43	**0.10**	1.18	1.38
128	5.43	10.8	2.59	**0.01**	1.66
256	6.01	16.3	5.13	1.20	**0.002**

5 Summary and conclusions

Some regularized versions of the MFS have been presented based on truncated or higher order fundamental solutions. For the Laplace equation, a non-RBF-type

Table 12: Mixed strategy, condition numbers. Test solution: (28). N is the number of sources, a is the scaling constant. Mixed boundary conditions

$a \setminus N$	16	32	64	128	256
16	70.9	3.5E+3	1.9E+5	3.3E+6	5.4E+11
32	58.1	3.0E+3	6.2E+3	1.4E+7	3.2E+12
64	53.9	2.8E+3	6.2E+3	1.4E+7	1.1E+12
128	51.9	2.6E+3	6.3E+3	1.5E+7	1.2E+12
256	50.9	2.6E+3	6.4E+3	1.5E+7	1.2E+12

method was also outlined based on harmonic functions vanishing on a straight segment. These basis functions are continuous, produce no singularity, so that they can be applied to the pure Dirichlet problems without generating singularities provided that the scaling parameter is properly defined (the optimal value is inversely proportional to the characteristic distance of the boundary collocation points). The accuracy of these methods are less than that of the traditional MFS, but they lead much better conditioned linear systems. However, in the presence of Neumann boundary condition, these methods still need some desingularization technique. In order to avoid desingularization, the use of fundamental solutions concentrated to straight lines has been proposed. Using these functions as basis functions, no desingularization is needed. In the case of mixed boundary condition, this approximation should be applied along the Neumann boundary only, which results in an acceptable compromise between the ill-conditioned character of the discretized problem and the accuracy of the method.

Acknowledgement: The research was partly supported by the European Union (co-financed by the European Social Fund) under the project TÁMOP-4.2.2.A-11/1/KONV-2012-0012.

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