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# Inverse Nodal Problem for the Differential Operator with a Singularity at Zero

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**Abstract:** In this study, some results are given about Sturm-Liouville operator having a singularity at zero. For this problem, asymptotic form of nodal data and a reconstruction formula for the potential function are given. In addition, a numerical example is established and illustrated the results in some tables and graphics.

**Keywords:** Inverse Nodal Problem, Reconstruction Formula, Nodal Points.

## 1 Introduction

Inverse problems are studied for certain special classes of ordinary differential operators. Typically, in eigenvalue problems, one measures the frequences of a vibrating system and tries to infer some physical properties of the system. An early important result in this direction, which gave vital impetus for the further development of inverse problem theory, was obtained in 1929 [Ambartsumyan (1929)]. Inverse problems of Sturm-Liouville operator have been studied for a long time and found some important applications such as in quantum scattering theory, vibration of a string and other branches of sciences. This problem can be solved by several methods as transformation operator, Weyl Titchmarch function, and nodal points (zeros of eigenfunctions) [Borg (1946); Gasymov and Levitan (1968); Gelfand and

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Levitan (1951); Hochstadt (1973); Levinson (1949); Marchenko (1950); Panakhov and Yılmazer (2012)].

In later years, Hald [Hald (1984)], Pöschel and Trubowitz [Pöschel and Trubowitz (1987)], Rundell and Sack [Rundell and Sack (2001)] and Isaacson and Trubowitz [Isaacson and Trubowitz (1983)] solved inverse Sturm-Liouville problems by using some new methods.

In some interesting works, Hald and McLaughlin [Hald and McLaughlin (1989)], and Browne and Sleeman [Browne and Sleeman (1996)] have taken a new approach to inverse spectral theory for the Sturm-Liouville problem. In this theory, nodal points are used as spectral data. In later years, inverse nodal problems were studied by many authors [Cheng and Law (2006); Koyunbakan (2006); Koyunbakan and Panakhov (2006), (2007); Law, Shen and Yang (1999); McLaughlin (1988); Shieh and Yurko (2008); Yang and X. F. Yang (2010); Yang (2010)].

The inverse spectral theory for Sturm-Liouville problems is most throughly developed for potentials that are real valued and square integrable [Pöschel and Trubowitz (1987); Ralston and Trubowitz (1988)]. Previous extensions of the theory of Trubowitz and his coworkers have included several types of singular Sturm-Liouville problems. Carlson [Carlson (1994)] extended the inverse spectral theory to a new class of problems having the form

$$-y'' + [q(x) + V(x)]\, y = \lambda y, \tag{1.1}$$

with boundary conditions

$$y(0) = y(1) = 0. \tag{1.2}$$

where $q(x) \in L_2[0,1]$ and $V(x)$ is a fixed function, locally integrable on $(0,1)$ and satisfying the estimate [Carlson (1994)]

$$|V(x)| \leq Kx^{-r},$$

for some $K \geq 0$ and $0 \leq r < \frac{3}{2}$. In particular, the case of a Coulomb singularity ($r = 1$) at the origin is included [Topsakal and Amirov (2010)]. Now, we will establish some estimates for the solutions of (1.1) with $0 \leq r < 2$ [Carlson (1994)]. By using variation of parameters, every solution $Y(x,\lambda)$ of equation (1.1) can be written as a solution of integral equation

$$Y(x,\lambda) = A\cos(\omega[1-x]) - B\frac{\sin(\omega[1-x])}{\omega} + \int_x^1 \frac{\sin(\omega[t-x])}{\omega}[V(t)+q(t)]Y(t,\lambda)dt + o\left(\frac{1}{\omega^3}\right). \tag{1.3}$$

where $\omega = \sqrt{\lambda}$. If $0 \le r < 2$, then $Y(x,\lambda)$ is an entire function of $\lambda$ for each $x \in [0,1]$ [Carlson (1994)].

**Lemma 1.1.** [Carlson (1994)] For $0 \le r < 2$, equation (1.1) has a unique solution $Y_1(x,\lambda)$ (or $Y_2(x,\lambda)$) whose derivative extends continously to $x = 0$ and satisfies

$$Y_1(0,\lambda) = 0, \quad Y'_1(0,\lambda) = 1, \tag{1.4}$$

$$Y_2(0,\lambda) = 1, \quad Y'_2(0,\lambda) = 0. \tag{1.5}$$

Equation (1.1) has following solutions with the initial conditions (1.4) and (1.5), respectively

$$Y_1(x,\lambda) = \frac{\sin(\omega x)}{\omega} + \int_0^x \frac{\sin(\omega[x-t])}{\omega}[V(t)+q(t)]Y_1(t,\lambda)dt + o\left(\frac{1}{\omega^3}\right), \tag{1.6}$$

$$Y_2(x,\lambda) = \cos(\omega x) + \int_0^x \frac{\sin(\omega[x-t])}{\omega}[V(t)+q(t)]Y_2(t,\lambda)dt + o\left(\frac{1}{\omega^3}\right). \tag{1.7}$$

Let $\lambda_0 < \lambda_1 < ... \to \infty$ be the eigenvalues of the problem (1.1), (1.5) and $0 < x_1^n < ... < x_j^n < 1$, $j = 1,2,...,n-1$, nodal points of the $n-$th eigenfunction. Let $\lambda_n$ be the $n-$th eigenvalue and $x_j^n$ be $j-$th nodal point of the $n-$th eigenfunction $Y_n$. Also let $I_j^n = \left[x_j^n,\, x_{j+1}^n\right]$ be the $j-$th nodal domain of the $n-$th eigenfunction and let $l_j^n = \left|I_j^n\right| = x_{j+1}^n - x_j^n$ be the associated nodal length. $j_n(x)$ be the largest index $j$ such that $0 \le x_j^{(n)} < x$.

**Lemma 1.2.** [Law, Shen and Yang (1999)] Suppose that $f \in L^1[0,1]$. Then for almost every $x \in [0,1]$, with $j = j_n(x)$,

$$\lim_{n\to\infty} \frac{\omega_n}{\pi} \int_{x_j^n}^{x_{j+1}^n} f(t)dt = f(x).$$

## 2 Main Results

In this section, our purpose is to develop asymptotic expressions for the points $x_j^n$ and $l_j^n (j=1,2,...,n-1,\ n=1,2,...)$ at which $Y_2(x,\lambda)$, the eigenfunction corresponding to the eigenvalue $\omega_n$ of the problem (1.1), (1.5), vanishes and to give a reconstruction formula for the potential function $q(x)$ of the problem (1.1), (1.5).

**Theorem 2.1.** We consider the equation

$$-Y''+[q(x)+V(x)]Y=\lambda Y, \tag{2.1}$$

with the initial conditions

$$Y(0,\lambda)=1, \quad Y'(0,\lambda)=0. \tag{2.2}$$

Then, the nodal points of the problem (2.1)-(2.2) are

$$x_j^n=\frac{\left(j-\frac{1}{2}\right)\pi}{\omega_n}+\frac{1}{\omega_n^2}\int\limits_0^{x_j^n}\cos(\omega_n t)\,[q(t)+V(t)]\,Y(t,\omega_n^2)dt+o\left(\frac{1}{\omega_n^3}\right), \tag{2.3}$$

and the nodal length is

$$l_j^n=\frac{\pi}{\omega_n}+\frac{1}{\omega_n^2}\int\limits_{x_j^n}^{x_{j+1}^n}\cos(\omega_n t)\,[q(t)+V(t)]\,Y(t,\omega_n^2)dt+o\left(\frac{1}{\omega_n^3}\right). \tag{2.4}$$

**Proof:** We will use the solution (1.7) to get asymptotic formulas for nodal data. From (1.7), we obtain

$$Y(x,\lambda)=\cos(\omega x)+\int\limits_0^x\frac{\sin(\omega[x-t])}{\omega}\,[V(t)+q(t)]\,Y(t,\lambda)dt+o\left(\frac{1}{\omega^3}\right),$$

$$\begin{aligned}Y(x,\lambda)=\cos(\omega x)&+\frac{\sin(\omega x)}{\omega}\int\limits_0^x\cos(\omega t)\,[V(t)+q(t)]\,Y(t,\lambda)dt\\&-\frac{\cos(\omega x)}{\omega}\int\limits_0^x\sin(\omega t)\,[V(t)+q(t)]\,Y(t,\lambda)dt+o\left(\frac{1}{\omega^3}\right).\end{aligned}$$

If $Y(x,\lambda)=0$, then as long as $\sin(\omega x)$ is not close to zero, we get

$$\cot(\omega x)+\frac{1}{\omega}\int_0^x \cos(\omega t)\left[V(t)+q(t)\right]Y(t,\lambda)dt$$

$$-\frac{\cot(\omega x)}{\omega}\int_0^x \sin(\omega t)\left[V(t)+q(t)\right]Y(t,\lambda)dt+o\left(\frac{1}{\omega^3}\right)=0.$$

Now, we take $\omega=\omega_n$ and $x=x_j^n$. Since the Taylor's expansion for the arccotangent function is given by

$$\operatorname{arccot} x=\left(j-\frac{1}{2}\right)\pi-\sum_{k=0}^{\infty}\frac{(-1)^{2k}x^{2k+1}}{2k+1}, \text{ for some integers } j,$$

then

$$x_j^n=\frac{\left(j-\frac{1}{2}\right)\pi}{\omega_n}+\frac{1}{\omega_n^2}\int_0^{x_j^n}\cos(\omega_n t)\left[q(t)+V(t)\right]Y(t,\omega_n^2)dt+o\left(\frac{1}{\omega_n^3}\right).$$

The nodal length is

$$l_j^n=x_{j+1}^n-x_j^n=\frac{\pi}{\omega_n}+\frac{1}{\omega_n^2}\int_{x_j^n}^{x_{j+1}^n}\cos(\omega_n t)\left[q(t)+V(t)\right]Y(t,\omega_n^2)dt+o\left(\frac{1}{\omega_n^3}\right).$$

**Theorem 2.2.** We consider the equation

$$-Y''+\left[q(x)+V(x)\right]Y=\lambda Y, \tag{2.5}$$

with the initial conditions

$$Y(0,\lambda)=0, \quad Y'(0,\lambda)=1. \tag{2.6}$$

Then, the nodal points of the problem (2.5)-(2.6) are

$$x_j^n=\frac{j\pi}{\omega_n}+\int_0^{x_j^n}\sin(\omega_n t)\left[q(t)+V(t)\right]Y(t,\omega_n^2)dt+o\left(\frac{1}{\omega_n^3}\right),$$

and the nodal length is

$$l_j^n = \frac{\pi}{\omega_n} + \int_{x_j^n}^{x_{j+1}^n} \sin(\omega_n t)\,[q(t)+V(t)]\,Y(t,\omega_n^2)dt + o\left(\frac{1}{\omega_n^3}\right).$$

**Proof:** Proof is similar to Theorem 2.1.

**Theorem 2.3.** Assume that $q \in L^1[0,1]$, then

$$q(x) = \lim_{n\to\infty}\left[2w_n^2\left(\frac{w_n l_j^n}{\pi} - 1\right) - V(x)\right].$$

for almost everywhere $x \in (0,1)$ with $j = j_n(x)$.

**Proof:** From (2.4), we obtain that

$$l_j^n = \frac{\pi}{\omega_n} + \frac{1}{\omega_n^2}\int_{x_j^n}^{x_{j+1}^n} \cos^2(\omega_n t)\,[q(t)+V(t)]\,dt + o\left(\frac{1}{\omega_n^3}\right),$$

and

$$2w_n^2\left(\frac{w_n l_j^n}{\pi} - 1\right) = \frac{w_n}{\pi}\int_{x_j^n}^{x_{j+1}^n} [q(t)+V(t)]\,dt + \frac{w_n}{\pi}\int_{x_j^n}^{x_{j+1}^n} \cos 2(\omega_n t)\,[q(t)+V(t)]\,dt + o\left(\frac{1}{\omega_n}\right).$$

By virtue of Lemma 1.2., this yields

$$\lim_{n\to\infty}\frac{\omega_n}{\pi}\int_{x_j^n}^{x_{j+1}^n} [q(t)+V(t)]\,dt = q(x)+V(x), \text{ for almost every } x \in (0,1).$$

Then

$$\lim_{n\to\infty} 2w_n^2\left(\frac{w_n l_j^n}{\pi} - 1\right) = q(x) + V(x) + \frac{w_n}{\pi}\int_{x_j^n}^{x_{j+1}^n} \cos 2(\omega_n t) q(t)dt$$
$$+ \frac{w_n}{\pi}\int_{x_j^n}^{x_{j+1}^n} \cos 2(\omega_n t) V(t)dt + o\left(\frac{1}{\omega_n}\right).$$

It remains to show that for almost every $x \in (0,1)$,

$$H_n(x) = \frac{w_n}{\pi} \int\limits_{x_j^n}^{x_{j+1}^n} \cos 2(\omega_n t) q(t) dt, \quad K_n(x) = \frac{w_n}{\pi} \int\limits_{x_j^n}^{x_{j+1}^n} \cos 2(\omega_n t) V(t) dt,$$

tend to zero as $n \to \infty$. The rest of proof is the same as Law's method [Law, Shen and Yang (1999)].

Now, we will give a uniqueness theorem. It says that the potential function $q(x)$ for a Sturm-Liouville operator with singularity at zero is uniquely determined by a dense set of nodal points.

**Theorem 2.4.** Suppose that $q$ is integrable. Then, $q - \int\limits_0^1 q$ is uniquely determined by any dense set of nodal points.

**Proof:** Assume that we have two problems of the type (1.1),(1.5) with potential functions $q, \widetilde{q}$. Let the nodal points $x_j^n, \widetilde{x}_j^n$ satisfying $x_j^n = \widetilde{x}_j^n$ form a dense set in $[0,1]$. We take solutions (1.1),(1.5) as $Y_n$ for $q$ and $\widetilde{Y}_n$ for $\widetilde{q}$. It follows from (2.1) that

$$\left(Y_n^{\prime} \widetilde{Y}_n - Y_n \widetilde{Y}_n^{\prime}\right)^{\prime} = \left[q - \widetilde{q} + (\widetilde{\lambda} - \lambda)\right] Y_n \widetilde{Y}_n. \tag{2.7}$$

Let $x_j^n = \widetilde{x}_j^n$ . To show that $q = \widetilde{q}$ , we integrate both sides of (2.7) from 0 to $x_j^n$ and using initial conditions (1.5). Then we obtain

$$\left(Y_n^{\prime} \widetilde{Y}_n - Y_n \widetilde{Y}_n^{\prime}\right) \Big|_0^{x_j^n} = \int\limits_0^{x_j^n} \left[q - \widetilde{q} + (\widetilde{\lambda} - \lambda)\right] Y_n \widetilde{Y}_n dx,$$

and

$$0 = \int\limits_0^{x_j^n} \left[q - \widetilde{q} + (\widetilde{\lambda} - \lambda)\right] Y_n \widetilde{Y}_n dx.$$

We take a sequence $x_j^n$ accumulating at an arbitrary $x \in [0,1]$. Hence,

$$0 = \int\limits_0^x \left(q - \widetilde{q} - \int\limits_0^1 (q - \widetilde{q}) ds\right) Y_n \widetilde{Y}_n dx,$$

and this holds for all $x$. We can therefore conclude that $q-\int_0^1 q$ is uniquely determined by a dense set of nodes.

## 3 Numerical Example

In this section, we shall give a numerical example about exact eigenfunction, nodal parameters and reconstruction of potential function for following Sturm-Liouville problem by using computer algebra system-Mathematica.

**Example 3.1.** Let consider following initial value problem for the special case of $q(x)=x$ and $V(x)=\frac{1}{\sqrt{x}}$

$$-Y''(x,\lambda)+(x+\frac{1}{\sqrt{x}})Y(x,\lambda)=\lambda Y(x,\lambda),\ Y(0,\lambda)=1,Y'(0,\lambda)=0. \tag{3.1}$$

We can obtain the eigenfunction of this problem as

$$Y(x,\lambda)=\cos(\omega_n x)-\frac{\cos(\omega_n x)}{\omega_n}\int_0^x \sin^2(\omega_n t)\left[t+\frac{1}{\sqrt{t}}\right]dt+o\left(\frac{1}{\omega_n^3}\right),$$

or

$$\begin{aligned}Y(x,10)=\cos(10\pi x)+\frac{\cos(10\pi x)}{8000\pi^3}&\left(-1-200\pi^2(4\sqrt{x}+x^2)+\cos(20\pi x)\right.\\&\left.+20\pi\left(2.\sqrt{10}\pi FresnelC(2.\sqrt{10x}\right)+x\sin(20\pi x)\right)\end{aligned}$$

where $n=10$, $FresnelC(z)=\int_0^z \cos\left(\frac{1}{2}\pi x^2\right)dx$, $z=2\sqrt{10x}$. And nodal datas are as following

$$x_j^n=\frac{\left(j-\frac{1}{2}\right)\pi}{\omega_n}+\frac{1}{\omega_n^2}\int_0^{x_j^n}\cos^2(\omega_n t)\left[t+\frac{1}{\sqrt{t}}\right]dt+o\left(\frac{1}{\omega_n^3}\right).$$

$$l_j^n=\frac{\pi}{\omega_n}+\frac{1}{\omega_n^2}\int_{x_j^n}^{x_{j+1}^n}\cos^2(\omega_n t)\left[t+\frac{1}{\sqrt{t}}\right]dt+o\left(\frac{1}{\omega_n^3}\right).$$

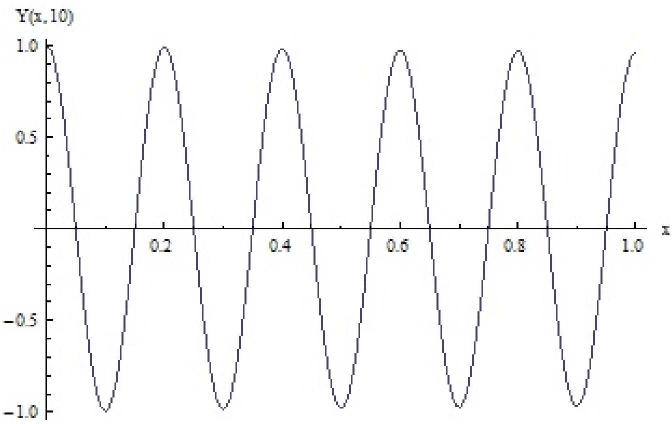


Figure 1: The graph of eigenfunction $Y(x,\lambda_n)$ for the problem (3.1) where n=10.

In Fig. 1., we illustrate the graph of eigenfunction $Y(x,\lambda_n)$ for the problem (3.1) where $n = 10$ and $0 \le x \le 1$.

The detailed results for nodal points and nodal lengths are shown in Tab. 1. and Tab. 2. Especially, Tab. 1. shows that these results are accurate and explicit. We can see that the conditions of oscillation theorem are provided. In Tab. 1., it can be seen that nodal points make an oscillation between 0 and 1.

Table 1: Nodal points of the problem (3.1) for $j = \overline{1,10}$ and $n = \overline{1,10}$

| $x_j^n$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ | $n=9$ | $n=10$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $j=1$ | 0.602 | 0.267 | 0.173 | 0.128 | 0.101 | 0.084 | 0.072 | 0.063 | 0.055 | 0.050 |
| $j=2$ | | 0.779 | 0.510 | 0.379 | 0.302 | 0.251 | 0.215 | 0.188 | 0.167 | 0.150 |
| $j=3$ | | | 0.847 | 0.631 | 0.503 | 0.418 | 0.358 | 0.313 | 0.278 | 0.250 |
| $j=4$ | | | | 0.882 | 0.704 | 0.586 | 0.501 | 0.438 | 0.389 | 0.350 |
| $j=5$ | | | | | 0.905 | 0.753 | 0.644 | 0.563 | 0.501 | 0.450 |
| $j=6$ | | | | | | 0.920 | 0.788 | 0.689 | 0.612 | 0.550 |
| $j=7$ | | | | | | | 0.931 | 0.814 | 0.723 | 0.651 |
| $j=8$ | | | | | | | | 0.939 | 0.834 | 0.751 |
| $j=9$ | | | | | | | | | 0.946 | 0.851 |
| $j=10$ | | | | | | | | | | 0.951 |

Table 2: Nodal lengths of the problem (3.1) for $j=\overline{1,10}$ and $n=\overline{1,10}$

| $l_j^n$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ | $n=9$ | $n=10$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $j=1$ | 1.1019 | 0.5122 | 0.3372 | 0.2518 | 0.2009 | 0.1672 | 0.1432 | 0.1252 | 0.1113 | 0.1001 |
| $j=2$ | 1.1372 | 0.5126 | 0.3368 | 0.2515 | 0.2008 | 0.1671 | 0.1431 | 0.1252 | 0.1112 | 0.1001 |
| $j=3$ | 1.1812 | 0.5146 | 0.3370 | 0.2515 | 0.2007 | 0.1671 | 0.1431 | 0.1251 | 0.1112 | 0.1001 |
| $j=4$ | 1.2279 | 0.5171 | 0.3374 | 0.2515 | 0.2007 | 0.1671 | 0.1431 | 0.1251 | 0.1112 | 0.1001 |
| $j=5$ | 1.2759 | 0.5198 | 0.3379 | 0.2516 | 0.2008 | 0.1671 | 0.1431 | 0.1251 | 0.1112 | 0.1000 |
| $j=6$ | 1.3246 | 0.5226 | 0.3384 | 0.2518 | 0.2008 | 0.1671 | 0.1431 | 0.1251 | 0.1112 | 0.1000 |
| $j=7$ | 1.3737 | 0.5255 | 0.3389 | 0.2519 | 0.2009 | 0.1671 | 0.1431 | 0.1251 | 0.1112 | 0.1000 |
| $j=8$ | 1.4232 | 0.5284 | 0.3394 | 0.2521 | 0.2009 | 0.1671 | 0.1431 | 0.1251 | 0.1112 | 0.1000 |
| $j=9$ | 1.4728 | 0.5314 | 0.3400 | 0.2523 | 0.2010 | 0.1672 | 0.1431 | 0.1252 | 0.1112 | 0.1000 |
| $j=10$ | 1.5226 | 0.5344 | 0.3406 | 0.2524 | 0.2010 | 0.1672 | 0.1431 | 0.1252 | 0.1112 | 0.1001 |

Conversely, if we consider $V(x)=\dfrac{1}{\sqrt{x}}$, and use asymptotic formula of $l_j^n$ as in example 3.1., we get

$$\begin{aligned}
&\lim_{n\to\infty}\left[2w_n^2\left(\frac{w_n l_j^n}{\pi}-1\right)-V(x)\right]\\
&=\lim_{n\to\infty}\left[2w_n^2\left(\frac{w_n}{\pi}\left\{\frac{\pi}{\omega_n}+\frac{1}{\omega_n^2}\int_{x_j^n}^{x_{j+1}^n}\cos^2(\omega_n t)\left[t+\frac{1}{\sqrt{t}}\right]dt\right\}-1\right)-V(x)\right]\\
&=x\\
&=q(x)
\end{aligned}$$

Thus, we have reconstructed the potential function $q(x)$ by using nodal data in example 3.1.

## 4 Conclusion

In this work, we have estimated nodal points and nodal lengths for the Sturm-Liouville operator with singularity at zero. We give a reconstruction formula for the potential function $q$ from the nodal data. Furthermore, by using these new parameters, we have shown that potential function of the Sturm-liouville operator with singularity at zero can be established uniquely. And finally, we give a numerical example for nodal parameters and reconstruction formula for $q$.

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