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Wavelet operational matrix method for solving fractional integral and differential equations of Bratu-type

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Abstract: In this paper, a wavelet operational matrix method based on the second kind Chebyshev wavelet is proposed to solve the fractional integral and differential equations of Bratu-type. The second kind Chebyshev wavelet operational matrix of fractional order integration is derived. A truncated second kind Chebyshev wavelet series together with the wavelet operational matrix is utilized to reduce the fractional integral and differential equations of Bratu-type to a system of nonlinear algebraic equations. The convergence and the error analysis of the method are also given. Two examples are included to verify the validity and applicability of the proposed approach.

Keywords: Fractional Bratu-type equations, the second kind Chebyshev wavelet, operational matrix, uniqueness, convergence, error analysis, numerical solution.

1 Introduction

In recent years fractional differential equations and fractional integral equations have found applications in several differential disciplines. A large class of dynamical systems appearing throughout the field of science and engineering, and applied mathematics is described by the integral and differential equations of fractional order [Chen (2007); Bagley and Calico (1991); Rossikhin and Shitikova (1997)]. Owing to the increasing applications, much attention has been given to the solution of fractional integral and differential equations. Many numerical and analytical methods for solving fractional integral and differential equations have been proposed in recent decades. These methods include Laplace Transforms [Podlubny (1999)], Adomian Decomposition Method [Momani (2007)], Variational Iteration Method [Das (2009)], Fractional Differential Transform Method [Arikoglu and Ozkol (2009)], and Fractional Difference Method [Meerschaert and Tadjeran (2006)].

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Bratu's problem [Chui (1997); Davis (1962); Frank-Kamenetski (1955); Hassan and Erturk (2007); Caglar and Caglar (2009)] is also discussed in all kinds of applications, such as chemical reaction theory, the fuel ignition model of the thermal combustion theory and nanotechnology. About Bratu's problem, both mathematicians and physicists have devoted a lot of effort. In Ref. [Syam and Hamdan (2006)], Syam and Hamdan presented the Laplace Adomian decomposition method for solving Bratu's problem. Wazwaz [Wazwaz (2005)] proposed the Adomian decomposition method for solving Bratu's problem. Yigit Aksoy and Mehmt Pakdemirli had solved Bratu-type equation of new perturbation iteration solutions [Aksoy and Pakdemirli (2010)]. Ref. [Boyd (2003)] uses Chebyshev polynomial expansions to solve the Bratu-type equations.

The operational matrices of fractional order integration for the Legendre wavelet [Rehman and Khan (2011)], Chebyshev wavelet [Li (2010)], Haar wavelet [Yi and Chen (2012)], and CAS wavelet [Sawwdi (2011)] have been developed to solve the fractional differential equations. In this paper, our study focuses on fractional integral- differential equations of Bratu-type by using the second kind Chebyshev wavelet operational matrix of fractional order integration with using the block pulse functions. Compared with the methods in the Ref. [Yi and Chen (2012); Chen, Sun, Li and Fu (2013); Wang, Meng, Ma, and Wu (2013); Wei, Chen, Li and Yi (2012)], we give some exact theoretical analysis to support the numerical results of this paper, so our manuscript is well constructed. According to the estimated absolute errors, they are very close to the exact absolute errors, and they can be used for measurement when the exact solution of the equation is not give. Therefore, we say those are better than the method in the Ref. [Yi and Chen (2012)].

Consider the equation as follows:

$$D^{\alpha}u(x)+\lambda\int_{0}^{x}k(x,t)e^{u(t)}dt+g(x)=0, \quad n-1<\alpha\leq n, \quad 0\leq x,t\leq 1 \tag{1}$$

subject to the initial conditions

$$u^{(j)}(0)=b_j \quad , \quad j=0,1,\ldots,n-1 \tag{2}$$

2 Preliminaries and notations

In this section, we will use some definitions and mathematical preliminaries of the fractional calculus theory as follows [Podlubny (1999)].

The Rieman-Liouville fractional integration of order $\alpha>0$ is defined as

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau, \quad n-1<\alpha\leq n \tag{3}$$

where$n$is a positive integer.

The Caputo fractional differential operator D^{α} of order $\alpha > 0$ is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{4}$$

It has the following two basic properties for $n-1 < \alpha \leq n$

$$D^{\alpha} I^{\alpha} f(t) = f(t) \tag{5}$$

and

$$I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0 \tag{6}$$

where $n$is a positive integer.

3 The second kind Chebyshev wavelet and operational matrix of fractional integration

3.1 The second kind Chebyshev wavelet [Wang and Fan (2012)]

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi(t)$called the mother wavelet.

Definition 1. The following functions

$$\psi_{ab}(t) = |a|^{-\frac{1}{2}} \psi(\frac{t-b}{a}), \; a, b \in R, a \neq 0 \tag{7}$$

form a family of continuous wavelets.

If we restrict the parameters $a$and $b$to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1, b_0 > 0$, where $n$and$k$are positive integers, the family of discrete wavelets are defined as

$$\psi_{kn}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0) \tag{8}$$

where ψ_{kn} form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, ψ_{kn} form an orthogonal basis.

The second Chebyshev wavelet $\psi_{nm}(t) = \psi(k,n,m,t)$involve four arguments, where $n = 1, 2, \cdots, 2^{k-1}$,$k$is assumed any positive integer,$m$is the degree of the second kind Chebyshev polynomials and $t$is the normalized time. They are defined on the interval $[0,1)$ as

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{U}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0, & otherwise, \end{cases} \tag{9}$$

where

$$\tilde{U}_m(t) = \sqrt{\frac{2}{\pi}} U_m(t) \tag{10}$$

and $m = 0, 1, \cdots, M-1$.In Eq.(10), the coefficients are used for orthonormality. $U_m(t)$ is the second kind Chebyshev polynomials of degree m with respect to the weight function $\omega(t) = \sqrt{1-t^2}$ on interval $[-1, 1]$ and satisfying the following recursive formula

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \cdots. \tag{11}$$

We should note that in dealing with the second kind Chebyshev wavelet, the weight function $\tilde{\omega}(t) = \omega(2t-1)$ has to be dilated and translated as

$$\omega_n(t) = \omega(2^k t - 2n + 1) \tag{12}$$

3.2 Function approximation

A function $f(t)$ defined on the interval $[0, 1)$may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t) \tag{13}$$

where $c_{nm} = (f(t), \psi_{nm}(t))_{\omega_n} = \int_0^1 \omega_n(t) \psi_{nm}(t) f(t) dt$.
If the infinite series in Eq.(13) is truncated, then Eq.(13) can be written as

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t) \tag{14}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \left[c_{10}, c_{11}, \cdots, c_{1(M-1)}, c_{20}, \cdots, c_{2(M-1)}, \cdots, c_{2^{k-1}0}, \cdots, c_{2^{k-1}(M-1)}\right]^T,$$

$$\Psi(t) = \left[\psi_{10}, \psi_{11}, \cdots, \psi_{1(M-1)}, \psi_{20}, \cdots, \psi_{2(M-1)}, \cdots, \psi_{2^{k-1}0}, \cdots, \psi_{2^{k-1}(M-1)}\right]^T \tag{15}$$

Any function $\int_0^x \psi\left(x'\right) dx' = P\psi(x)$ may be also approximated as

$$k(x,t) \approx \Psi^T(x) K \Psi(t) \tag{16}$$

where $\int_0^x \psi\left(x'\right) dx' = P\psi(x)$ is a $2^{k-1}M \times 2^{k-1}M$ matrix.

3.3 The second kind Chebyshev wavelet product operational matrix

The following property of the product of two the second kind Chebyshev wavelet vector functions will also be used as

$$\Psi(t)\Psi^T(t)C \approx \tilde{C}\Psi(t) \tag{17}$$

where $\tilde{C}$ is $2^{k-1}M \times 2^{k-1}M$ product operation matrix. For $k = 2$, $M = 3$ we have

$$\tilde{C} = 2\sqrt{\frac{2}{\pi}} \begin{bmatrix} \tilde{C}_1 & O \\ O & \tilde{C}_2 \end{bmatrix} \tag{18}$$

where $\tilde{C}_i$ $(i = 1,\ 2)$ are 3×3 matrices given by

$$\tilde{C}_i = \begin{bmatrix} c_{i0} & c_{i1} & c_{i2} \\ c_{i1} & c_{i0} + c_{i2} & c_{i1} \\ c_{i2} & c_{i1} & c_{i0} + c_{i2} \end{bmatrix} \tag{19}$$

3.4 The operational matrix of fractional integration

Taking the collocation points $t_i = \frac{2i-1}{2^k M}$, $i = 1, 2, \cdots, 2^{k-1}M$, then the second kind Chebyshev wavelet matrix $\Psi_{m\times m}$ will be defined as

$$\Psi_{m\times m} = [\Psi(t_1), \Psi(t_2), \cdots, \Psi(t_m)] \tag{20}$$

where $m = 2^{k-1}M$.

We define an m-set of Block Pulse Functions (BPFs) [Wang and Fan (2012)], the set of these functions over the interval $[0, T)$ is defined as

$$\varphi_i(t) = \begin{cases} 1, & (i-1)\frac{T}{m} \le t < i\frac{T}{m} \\ 0, & otherwise \end{cases} \tag{21}$$

where $i = 1, 2, \cdots, m$, $\frac{T}{m} = h$.

The BPFs have two useful properties which will be used further.

i) Disjointedness

$$\varphi_i(t)\varphi_j(t) = \begin{cases} 0, & i \ne j, \\ \varphi_i(t), & i = j, \end{cases} \qquad t \in [0, T), \qquad i, j = 1, 2, \cdots, m \tag{22}$$

ii) Orthogonality

$$\int_0^1 \varphi_i(t)\varphi_j(t)dt = \begin{cases} 0, & i \ne j, \\ h, & i = j, \end{cases} \qquad t \in [0, T), \qquad i, j = 1, 2, \cdots, m \tag{23}$$

Letting $\Phi(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t)]$, Kilicman and Al Zhour had given the BPFs operational matrix of fractional integration F_α as

$$I^\alpha \Phi(t) \approx F_\alpha \Phi(t) \tag{24}$$

where

$$F_\alpha = \left(\frac{1}{m}\right)^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{25}$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Now we derive the second kind Chebyshev wavelet operational matrix of the fractional integration. Let

$$I^\alpha \Psi(t) \approx P^\alpha \Psi(t) \tag{26}$$

P^α is called the second kind Chebyshev wavelet operational matrix of the fractional integration.

When $T = 1$, the relationship between the second kind Chebyshev wavelet and the BPFs is

$$\Psi(t) = \Psi_{m\times m} \Phi(t) \tag{27}$$

Then we have

$$I^\alpha \Psi(t) \approx I^\alpha \Psi_{m\times m} \Phi(t) = \Psi_{m\times m} I^\alpha \Phi(t) = \Psi_{m\times m} F_\alpha \Phi(t) \tag{28}$$

From Eq.(26) and Eq.(27), we obtain

$$I^\alpha \Psi(t) \approx P^\alpha \Psi(t) = P^\alpha \Psi_{m\times m} \Phi(t) \tag{29}$$

Then the second kind Chebyshev wavelet operational matrix of the fractional integration is given by

$$P^\alpha = \Psi_{m\times m} F_\alpha \Psi_{m\times m}^{-1} \tag{30}$$

For instance, $M = 3$, $k = 2$ and $\alpha = 0.5$, we have

$$P^{0.5} = \begin{bmatrix} 0.1513 & -0.2077 & -0.1558 & -3.7364 & -1.5403 & -0.0746 \\ 0.2077 & 0.5841 & 0.2077 & 1.8244 & 0.1826 & 0.0033 \\ -0.1212 & -0.1615 & 0.1860 & -0.7450 & -0.2871 & -0.0096 \\ 0 & 0 & 0 & 0.1513 & -0.2077 & -0.1558 \\ 0 & 0 & 0 & 0.2077 & 0.5841 & 0.2077 \\ 0 & 0 & 0 & -0.1212 & -0.1615 & 0.1860 \end{bmatrix}.$$

4 Numerical Algorithm of the problem (1)

Consider the nonlinear fractional integral-differential equations of Bratu-type:

$$D^{\alpha}u(x)+\lambda\int_{0}^{x}k(x,t)e^{u(t)}dt+g(x)=0,$$

$$u^{(j)}(0)=b_j,\quad 0\le x,\quad t\le 1,\quad n-1<\alpha\le n,$$

λ is arbitrary parameter, $k(x,t)\in L^2([0,1]\times[0,1])$, $g(x)$ is a known function.
Let $D^{\alpha}u(x)\approx C^T\Psi(x),\ g(x)\approx G^T\Psi(x),\ k(x,t)\approx\Psi^T(x)K\Psi(t),\ e^{u(t)}\approx\sum\limits_{i=0}^{N}\frac{[u(t)]^i}{i!}$, then

$$\begin{aligned} u(x)&\approx C^TP^{\alpha}\Psi(x)+\sum_{k=0}^{n-1}u^{(k)}(0)\tfrac{x^k}{k!}=C^TP^{\alpha}\Psi(x)+Q\Psi(x)\\ &=\left(C^TP^{\alpha}+Q\right)\Psi(x)\end{aligned}\tag{31}$$

where coefficient Q is known and can be obtained by using the initial conditions. Substituting Eq.(27) into Eq.(31), we get

$$u(x)\approx\left(C^TP^{\alpha}+Q\right)\Psi(x)=\left(C^TP^{\alpha}+Q\right)\Psi_{m\times m}\Phi(x)\tag{32}$$

Define $\left(C^TP^{\alpha}+Q\right)\Psi_{m\times m}=[a_1,a_2,\dots,a_m]=A$, then $u(x)\approx A\Phi(x)$.
Applying the properties of BPFs, we have

$$[u(x)]^i\approx[a_1^i,a_2^i,\dots,a_m^i]\Phi(x)=A_i\Phi(x)\tag{33}$$

Substituting the above expanded forms into Eq.(1), we can obtain

$$\begin{aligned} &D^{\alpha}u(x)+\lambda\textstyle\int_0^x k(x,t)e^{u(t)}dt+g(x)\\ &\approx C^T\Psi(x)+\lambda\textstyle\int_0^x\Psi^T(x)K\Psi(t)\sum\limits_{j=0}^{N}\frac{[u(t)]^j}{j!}dt+G^T\Psi(x)\\ &\approx C^T\Psi(x)+\lambda\Psi^T(x)K\textstyle\int_0^x\Psi_{m\times m}\Phi(t)\sum\limits_{j=0}^{N}\frac{\Phi^T(t)A_j^T}{j!}dt+G^T\Psi(x)\\ &\approx C^T\Psi(x)+\lambda\Psi^T(x)K\Psi_{m\times m}\sum\limits_{j=0}^{N}\frac{1}{j!}\textstyle\int_0^x\widetilde{A_j^T}\Phi(t)dt+G^T\Psi(x)\\ &\approx C^T\Psi(x)+\lambda\Psi^T(x)K\Psi_{m\times m}\sum\limits_{j=0}^{N}\frac{1}{j!}\widetilde{A_j^T}F_1\Phi(x)+G^T\Psi(x)=0\end{aligned}\tag{34}$$

where $\widetilde{A_j^T}$ is the product operational matrix of A_j^T.

Putting the collocation points$\{x_i\}_{i=1}^{2^{k-1}M}$into Eq. (34), Eq.(34) will be

$$C^T\Psi(x_i)+\lambda\Psi^T(x_i)K\Psi_{m\times m}\sum_{j=0}^{N}\frac{1}{j!}\widetilde{A_j^T}F_1\Phi(x_i)+G^T\Psi(x_i)=0 \tag{35}$$

Solving the nonlinear algebraic equations Eq.(35) by using the Newton iteration method, we get the vectorC^T, and then we obtain the approximate solution $u(x)=\left(C^TP^\alpha+Q\right)\Psi(x)$.

5 Existence of uniqueness and convergence

Theorem1. (Uniqueness theorem) Eq. (1) has a unique solution whenever $0<\beta<1$, where $\beta{=}\frac{\lambda ML}{\Gamma(\alpha)\alpha(\alpha+1)}$, $\quad M=\sup\limits_{0\le x,t\le 1}|k(x,t)|<\infty$.

Proof. Suppose $t\in[0,1]$, then $u(t)$ is bounded. Therefore the nonlinear term $e^{u(t)}$ in Eq. (1) is Lipschitz continuous with$|e^u-e^{u_*}|\le L|u-u_*|$, $L>0$.

Let$u$andu^*be two different solutions of Eq. (1) , then we can get

$$D^\alpha u(x)=-\lambda\int_0^x k(x,t)e^{u(t)}dt-g(x) \tag{36}$$

$$D^\alpha u_*(x)=-\lambda\int_0^x k(x,t)e^{u_*(t)}dt-g(x) \tag{37}$$

Using Rieman-Liouville fractional integration, we have

$$\begin{aligned}I^\alpha D^\alpha u(x)=&-\frac{1}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}g(\xi)d\xi\\&-\frac{\lambda}{\Gamma(\alpha)}\int_0^x\int_0^\xi(x-\xi)^{\alpha-1}k(\xi,t)e^{u(t)}dtd\xi\end{aligned} \tag{38}$$

$$\begin{aligned}I^\alpha D^\alpha u_*(x)=&-\frac{1}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}g(\xi)d\xi\\&-\frac{\lambda}{\Gamma(\alpha)}\int_0^x\int_0^\xi(x-\xi)^{\alpha-1}k(\xi,t)e^{u_*(t)}dtd\xi\end{aligned} \tag{39}$$

Because $I^\alpha D^\alpha x(t)=x(t)-\sum\limits_{k=0}^{n-1}\frac{t^k}{k!}x^{(k)}(0^+)$, so Eq. (38) and Eq. (39) can transform as

$$\begin{aligned}u(x)-\sum_{j=0}^{n-1}\frac{x^j}{j!}u^{(j)}(0)=&-\frac{1}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}g(\xi)d\xi\\&-\frac{\lambda}{\Gamma(\alpha)}\int_0^x\int_0^\xi(x-\xi)^{\alpha-1}k(\xi,t)e^{u(t)}dtd\xi\end{aligned} \tag{40}$$

$$u_*(x) - \sum_{j=0}^{n-1} \frac{x^j}{j!} u_*^{(j)}(0) = -\frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} g(\xi) d\xi \\ - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x-\xi)^{\alpha-1} k(\xi,t) e^{u_*(t)} dt d\xi. \tag{41}$$

Then we have

$$\begin{aligned}
|u-u_*| &= \left| -\tfrac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x-\xi)^{\alpha-1} k(\xi,t) e^{u(t)} dt d\xi + \tfrac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x-\xi)^{\alpha-1} k(\xi,t) e^{u_*(t)} dt d\xi \right| \\
&= \left| -\tfrac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x-\xi)^{\alpha-1} k(\xi,t) \left[e^{u(t)} - e^{u_*(t)} \right] dt d\xi \right| \\
&\le \tfrac{\lambda}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x-\xi)^{\alpha-1} k(\xi,t) \left| e^{u(t)} - e^{u_*(t)} \right| dt d\xi \\
&\le \tfrac{\lambda ML}{\Gamma(\alpha)} \int_0^x \int_0^\xi (x-\xi)^{\alpha-1} |u-u_*| \, dt d\xi \\
&= \tfrac{\lambda ML}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} |u-u_*| \, \xi d\xi \\
&= \tfrac{\lambda ML}{\Gamma(\alpha)} |u-u_*| \int_0^x \xi (x-\xi)^{\alpha-1} d\xi \\
&= \tfrac{\lambda ML}{\Gamma(\alpha)} |u-u_*| \left(\tfrac{x^{\alpha+1}}{\alpha} - \tfrac{x^{\alpha+1}}{\alpha+1} \right) \\
&= \tfrac{\lambda ML}{\Gamma(\alpha)} |u-u_*| x^{\alpha+1} \tfrac{1}{\alpha(\alpha+1)} \\
&\le \tfrac{\lambda ML}{\Gamma(\alpha)\alpha(\alpha+1)} |u-u_*|
\end{aligned}$$

Therefore $|u-u_*| \left(1 - \frac{\lambda ML}{\Gamma(\alpha)\alpha(\alpha+1)} \right) \le 0$.

This implies that $|u-u_*|(1-\beta) \le 0$, where $\beta = \frac{\lambda ML}{\Gamma(\alpha)\alpha(\alpha+1)}$.

As $0 < \beta < 1$, $|u-u_*| = 0$, imply $u = u_*$, so we can prove Eq.(1) has the uniqueness solution.

Theorem2. (Convergence theorem) The series solution (16) of problem (1) using the second kind Chebyshev wavelet method converges toward $u(x)$

Proof Suppose $L^2(R)$ be the Hilbert space and $\psi_{k,n}(t) = |a|^{-\frac{1}{2}} \psi(a_0^k t - nb_0)$, where $\psi_{k,n}(t)$ form a basis of $L^2(R)$. When $a_0 = 2$ and $b_0 = 1$, $\psi_{k,n}(t)$ form an orthonormal basis.

For $k = 1$, let $u(x) = \sum_{i=0}^{M-1} c_{1i} \psi_{1i}(x)$, where $c_{1i} = \langle u(x), \psi_{1i}(x) \rangle$ and $\langle \cdot\,,\,\cdot \rangle$represents the inner product, we have $u(x) = \sum_{i=1}^{n} \langle u(x), \psi_{1i}(x) \rangle \psi_{1i}(x)$.

Let's denote $\psi_{1i}(x)$ as $\psi(x)$, $\gamma_j = \langle u(x), \psi(x) \rangle$.

Define the sequence of partial sums $\{S_n\}$ of $\gamma_j \psi(x_j)$, let S_n and S_m be arbitrary partial sums with $n \ge m$. We will prove that $\{S_n\}$ is a Cauchy sequence in Hilbert space.

Let $S_n = \sum\limits_{j=1}^{n} \gamma_j \psi(x_j)$, then

$$\begin{aligned}\langle u(x), S_n\rangle &= \left\langle u(x), \sum_{j=1}^{n} \gamma_j \psi(x_j)\right\rangle \\ &= \sum_{j=1}^{n} \overline{\gamma_j} \left\langle u(x), \psi(x_j)\right\rangle \\ &= \sum_{j=1}^{n} \overline{\gamma_j}\gamma_j \\ &= \sum_{j=1}^{n} \left|\gamma_j\right|^2.\end{aligned}$$

We will assert that $\|S_n - S_m\|^2 = \sum\limits_{j=m+1}^{n} \left|\gamma_j\right|^2$ for $n > m$.

Now

$$\begin{aligned}\left\|\sum_{j=m+1}^{n} \gamma_j \psi(x_j)\right\|^2 &= \left\langle \sum_{i=m+1}^{n} \gamma_i \psi(x_i), \sum_{j=m+1}^{n} \gamma_j \psi(x_j)\right\rangle \\ &= \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \gamma_i \overline{\gamma_j} \left\langle \psi(x_i), \psi(x_j)\right\rangle \\ &= \sum_{j=m+1}^{n} \gamma_j \overline{\gamma_j} \\ &= \sum_{j=m+1}^{n} \left|\gamma_j\right|^2,\end{aligned}$$

i.e. $\|S_n - S_m\|^2 = \sum\limits_{j=m+1}^{n} \left|\gamma_j\right|^2$ for $n > m$.

According to Bessel's inequality, we have $\sum\limits_{j=1}^{\infty} \left|\gamma_j\right|^2$ is convergent and $\|S_n - S_m\|^2 \to 0$, as $n, m \to \infty$, i.e. $\|S_n - S_m\| \to 0$ and $\{S_n\}$ is a Cauchy sequence and it converges to say S. We claim that $u(x) = S$.

In fact

$$\begin{aligned}\left\langle S - u(x), \psi(x_j)\right\rangle &= \left\langle S, \psi(x_j)\right\rangle - \left\langle u(x), \psi(x_j)\right\rangle \\ &= \left\langle \lim_{n\to\infty} S_n, \psi(x_j)\right\rangle - \gamma_j \\ &= \lim_{n\to\infty} \left\langle S_n, \psi(x_j)\right\rangle - \gamma_j \\ &= \gamma_j - \gamma_j \\ &= 0\ .\end{aligned}$$

Hence, $u(x) = S$ and $\sum\limits_{j=1}^{n} \gamma_j \psi(x_j)$ converges to $u(x)$.

This completes the proof.

6 Error Analysis

Let's consider $e_{k,M,\alpha}(x) = u(x) - u_{k,M,\alpha}(x)$ as the error function of the approximate solution $u_{k,M,\alpha}(x)$ for $u(x)$, where $u(x)$ is the exact solution of Eq. (1).

Therefore, $u_{k,M,\alpha}(x)$ satisfies the following problem

$$D^{\alpha}u_{k,M,\alpha}(x) + \lambda \int_0^x k(x,t)e^{u_{k,M,\alpha}(t)}dt + g(x) + R_{k,M}(x) = 0,$$

$0 \le x, \quad t \le 1, \quad n-1 < \alpha \le n,$

where $R_{k,M}(x)$ is the residual function.

$$R_{k,M}(x) = -D^{\alpha}u_{k,M,\alpha}(x) - \lambda \int_0^x k(x,t)e^{u_{k,M,\alpha}(t)}dt - g(x). \tag{42}$$

We proceed to find an approximation $e^{u_{k,M,\alpha}(t)}$ to the error function $e^{u(t)}$ in the same way as we did before for the solution of the problem (1). Subtracting Eq. (42) from Eq. (1), the error function $e^{u(t)}$ satisfies the problem

$$D^{\alpha}\left[u(x) - u_{k,M,\alpha}(x)\right] + \lambda \int_0^x k(x,t)\left[e^{u(t)} - e^{u_{k,M,\alpha}(t)}\right]dt - R_{k,M}(x) = 0 \tag{43}$$

namely

$$D^{\alpha}e_{k,M,\alpha}(x) + \lambda \int_0^x k(x,t)\sum_{i=0}^{N}\frac{[u(t)]^i - [u_{k,M,\alpha}(t)]^i}{i!}dt - R_{k,M}(x) = 0. \tag{44}$$

It should be noted that in order to construct the approximate $e^{u_{k,M,\alpha}(t)}$ to $e^{u(t)}$, only Eq. (44) needs to be recalculated in the same way as we did before for the solution of Eq. (1).

7 Numerical examples

Example1. Consider the following equation

$$D^{0.5}u(x) + \int_0^x e^{u(t)}dt + g(x) = 0 \ ,$$

subject to the initial conditions $u(0) = 1$, $g(x) = -1.1284x^{0.5} - e^1(e^x - 1)$. The exact solution is $u(x) = x+1$.

Take $N = 2^{k-1}M$, the absolute errors for different $k,M,N$are shown in **Table 1**. Using the error estimation in section 6, we can also obtain the estimated absolute errors which are shown in **Table 2**.

Table 1: Absolute errors for$M = 3$and different values ofk,N

x	$k=2,N=6$	$k=3,N=12$	$k=4,N=24$	$k=5,N=48$	$k=6,N=96$
0.1	2.2875e-003	8.1865e-004	5.7142e-004	9.1786e-005	7.1786e-006
0.2	1.8938e-003	7.7151e-004	5.0586e-004	8.2651e-005	6.1650e-006
0.3	1.5116e-003	7.4436e-004	4.5134e-004	7.5165e-005	5.6126e-006
0.4	1.1344e-003	6.9015e-004	3.9816e-004	7.0019e-005	4.7125e-006
0.5	9.8256e-004	6.2983e-004	3.0167e-004	6.3286e-005	4.0187e-006
0.6	9.4160e-004	5.5743e-004	2.2438e-004	5.8132e-005	3.1865e-006
0.7	9.0197e-004	4.9917e-004	1.1285e-004	4.6729e-005	2.6159e-006
0.8	8.8176e-004	4.2215e-004	8.8512e-005	4.1101e-005	1.3346e-006
0.9	8.2835e-004	3.3462e-004	8.1021e-005	2.9621e-005	8.8761e-007

Table 2: Estimated absolute errors for$M = 3$and different values of k,N

x	$k=2,N=6$	$k=3,N=12$	$k=4,N=24$	$k=5,N=48$	$k=6,N=96$
0.1	2.5162e-003	8.4698e-004	6.1153e-004	9.4165e-005	7.4126e-006
0.2	2.2128e-003	8.0128e-004	5.3184e-004	8.6012e-005	6.5815e-006
0.3	1.9764e-003	7.6127e-004	4.9476e-004	7.9159e-005	6.1012e-006
0.4	1.4166e-003	7.2186e-004	4.3571e-004	7.4651e-005	5.2170e-006
0.5	1.0136e-003	6.6235e-004	3.4553e-004	6.6025e-005	4.4175e-006
0.6	9.7543e-004	5.8341e-004	2.6348e-004	6.1145e-005	3.6547e-006
0.7	9.3562e-004	5.2874e-004	1.4542e-004	5.7109e-005	3.0178e-006
0.8	9.1549e-004	4.5189e-004	9.2436e-005	4.5162e-005	1.7219e-006
0.9	8.5658e-004	3.7264e-004	8.7178e-005	3.2420e-005	9.8978e-007

From the **Table 1**, we can conclude that the numerical solutions are more and more close to the exact solution when k,N increase. We observe from **Table 1** and **Table 2** that the actual and estimated absolute errors are almost the same. Hence, the estimated absolute errors can be used for measurement when the exact solution of any problem is unknown.

Example 2. Consider the following fractional differential equation

$$D^{\alpha}u(x)+3\int_0^x (x-t)e^{u(t)}dt+g(x)=0, \quad 0<x,t<1$$

such that the initial conditions $u(0) = 0$, $g(x) = -\frac{1}{2}x^3 - \frac{1}{x}$. The exact solution of the problem for $\alpha = 1$ is given by $u(x) = \ln x$. The comparison of numerical results for $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 1$ and the exact solution for $\alpha = 1$ are shown in **Fig. 1**. From the **Fig. 1**, we can see clearly that the numerical solutions are in very good agreement with the exact solution when $\alpha = 1$.

However, a closer look at the **Fig. 1** reveals that the numerical solutions for $\alpha = 1$

deviate a little from the exact solution for x in $[0.1, 0.2]$. This is an indication of instability on part of the second kind Chebyshev wavelet method, in contrast to other methods.

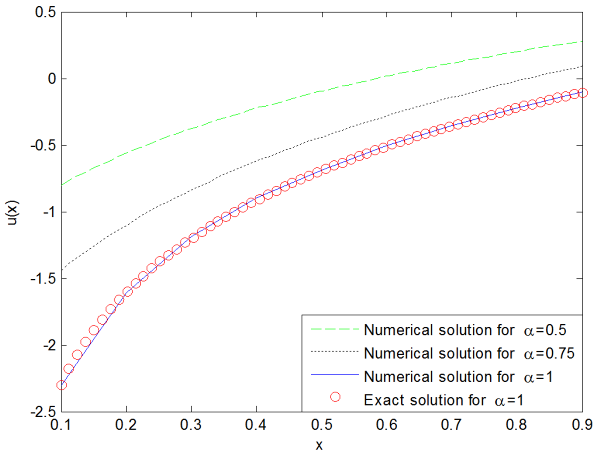

Figure 1: Numerical solution and exact solution of$\alpha = 1$

Table 3: The absolute errors for different$N$

x	Present method		Method in [Wazwaz (2005)]	
	$k = 3, N = 12$	$k = 4, N = 24$	$N = 12$	$N = 24$
0.1	5.235824e-005	8.214678e-006	6.126552e-005	8.241752e-006
0.2	4.814652e-005	8.024885e-006	6.521678e-005	9.712684e-006
0.3	4.643165e-005	7.747140e-006	7.017520e-005	1.127595e-005
0.4	3.543645e-005	7.421648e-006	8.434743e-005	1.576124e-005
0.5	2.814260e-005	7.112765e-006	8.621648e-005	2.312878e-005
0.6	1.754389e-005	6.512749e-006	9.517855e-005	2.926857e-005
0.7	9.326556e-006	6.220874e-006	1.328752e-004	3.512786e-005
0.8	8.486315e-006	5.721658e-006	1.821547e-004	4.012735e-005
0.9	8.054321e-006	5.357824e-006	2.012765e-004	4.714378e-005

In **Table 3**, we list the results obtained by the second kind Chebyshev wavelet method proposed in this paper together with Adomian Decomposition method [Wazwaz (2005)] results. Compared to the Adomian Decomposition method, taking advantage of above method can greatly reduce the computation. Moreover, the method in this paper is easier for implementation.

8 Conclusion

The purpose of this paper is to develop an effective and accurate method for solving the fractional integral and differential equations of Bratu-type. We derive the second kind Chebyshev wavelet operational matrix of fractional order integration and use the wavelet basis together with operational matrix to reduce the fractional integral and differential equations to a system of nonlinear algebraic equations. The sufficient condition that guarantees a unique solution to be the given problem is obtained. The convergence and error analysis of the method is also given. According to the numerical example results, it is observed that the method is a good approximation for the fractional Bratu-type problem.

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