

# Solution of Quadratic Integral Equations by the Adomian Decomposition Method

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**Abstract:** Quadratic integral equations are a class of nonlinear integral equations having many important uses in engineering and sciences. In this work we display an efficient application of the Adomian decomposition method to the quadratic integral equations of Volterra type. The analytical approximate solution obtained can be directly inserted into the original equation to verify the accuracy and estimate the error with a computing software. Four numerical examples demonstrate the efficiency of the method.

**Keywords:** Quadratic integral equations, Adomian decomposition method, Adomian polynomials, Nonlinear integral equations.

## 1 Introduction

Integral equations play an important role in functional analysis and in their applications in engineering, mathematical physics, economics and other fields. In particular, quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory (see e.g. [Agarwal, O'Regan, and Wong (1999); Corduneanu (1991); Hu, Khavani, and Zhuang (1989); Wazwaz (2011)]).

For the solutions of general integral equations, various methods have been proposed, including analytical and numerical methods [Corduneanu (1991); Wazwaz (2011); Zou and Li (2010); Liu and Atluri (2009); Kelmanson and Tenwick (2010); Banaś, Caballero, Rocha, and Sadarangani (2005)]. In [El-Sayed, Hashem, and

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Ziada (2010)] the quadratic integral equations of Volterra type in the form of

$$x(t) = a(t) + g(t, x(t)) \int_0^t f(s, x(s)) ds \quad (1)$$

have been investigated by the Picard iterative method and the Adomian decomposition method (ADM). But we find the ADM was not applied appropriately. In this work we present efficient applications of the ADM to the quadratic integral equations in Eq. (1).

The ADM [Adomian and Rach (1983); Adomian (1983, 1994); Wazwaz (2011); Lai, Chen, and Hsu (2008)] is a well-known method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, etc. The ADM provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. The accuracy of the analytic approximate solutions obtained can be verified by direct substitution. Advantages of the ADM over Picard iterated method in resolution were demonstrated in [Rach (1987)]. A key notion is the Adomian polynomials, which are tailored to particular nonlinearities to solve nonlinear operator equations.

The ADM decomposes the solution into a series

$$x(t) = \sum_{n=0}^{\infty} x_n(t), \quad (2)$$

and then decomposes the analytic nonlinear term  $Nx(t)$  into a series

$$Nx(t) = \sum_{n=0}^{\infty} A_n(t), \quad (3)$$

where the  $A_n(t)$ , depending on  $x_0(t), x_1(t), \dots, x_n(t)$ , are called the Adomian polynomials, and are obtained for the nonlinearity  $Nx(t) = F(t, x(t))$  by the formula [Adomian and Rach (1983)]

$$A_n(t) = \frac{1}{n!} \left[ \frac{\partial^n}{\partial \lambda^n} F(t, \sum_{k=0}^{\infty} x_k(t) \lambda^k) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots, \quad (4)$$

where  $\lambda$  is a grouping parameter of convenience.

The first five Adomian polynomials are

$$\begin{aligned}
 A_0(t) &= F(t, x_0), \\
 A_1(t) &= F'(t, x_0)x_1, \\
 A_2(t) &= F'(t, x_0)x_2 + F''(t, x_0)\frac{x_1^2}{2!}, \\
 A_3(t) &= F'(t, x_0)x_3 + F''(t, x_0)x_1x_2 + F^{(3)}(t, x_0)\frac{x_1^3}{3!}, \\
 A_4(t) &= F'(t, x_0)x_4 + F''(t, x_0)\left(\frac{x_2^2}{2!} + x_1x_3\right) + F^{(3)}(t, x_0)\frac{x_1^2x_2}{2!} + F^{(4)}(t, x_0)\frac{x_1^4}{4!},
 \end{aligned}$$

where  $F', F'', F^{(3)}, \dots$  stand for the partial derivatives of  $F$  about its second argument.

Recently new, efficient algorithms for the Adomian polynomials were presented by Duan (2010a,b, 2011). Here we list Duan’s Corollary 3 algorithm in [Duan (2011)] as follows.

$$A_n(t) = \sum_{k=1}^n F^{(k)}(t, x_0)C_n^k, \quad n \geq 1, \tag{5}$$

where

$$C_n^1 = x_n, \quad n \geq 1, \tag{6}$$

$$C_n^k = \frac{1}{n} \sum_{j=0}^{n-k} (j+1)x_{j+1}C_{n-1-j}^{k-1}, \quad 2 \leq k \leq n. \tag{7}$$

The recurrence operations in Eqs. (6) and (7) do not involve the differentiation, but only require the elementary operations of addition and multiplication. So it is eminently convenient for computer algebra systems such as MATHEMATICA, MAPLE or MATLAB to generate the Adomian polynomials. In Appendix A, we present the MATHEMATICA code based on this algorithm.

For other algorithms for the Adomian polynomials see [Adomian and Rach (1983, 1992a); Rach (1984, 2008); Wazwaz (2000, 2009); Abdelwahid (2003); Abbaoui, Cherruault, and Seng (1995); Zhu, Chang, and Wu (2005); Biazar, Ilie, and Khoshkenar (2006); Azreg-Aïnou (2009); Duan (2010b,a, 2011); Duan and Guo (2010)].

The convergence of the ADM has been proved by several investigators [Cherruault and Adomian (1993); Abbaoui and Cherruault (1994); Gabet (1994); Rach (2008); Abdelrazec and Pelinovsky (2011)]. For example, Abdelrazec and Pelinovsky (2011) have published a rigorous proof of convergence for the ADM under the aegis of the Cauchy-Kovalevskaya theorem for initial value problems. A key concept is that the Adomian decomposition series is a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series

about the initial solution component function, which permits solution by recursion. For a bibliography and recent developments of the ADM see [Rach (2012); Duan, Rach, Baleanu, and Wazwaz (2012)].

## 2 Resolution by the ADM

We consider the following quadratic integral equations [El-Sayed, Hashem, and Ziada (2010); Banaś, Caballero, Rocha, and Sadarangani (2005)]

$$x(t) = a(t) + g(t, x(t)) \int_0^t f(s, x(s)) ds, \quad (8)$$

where  $a(t)$  is a specified continuous function, and  $f$  and  $g$  have partial derivatives of arbitrary order with respect to their second arguments.

The ADM decomposes the solution into a series  $x(t) = \sum_{n=0}^{\infty} x_n(t)$ , then decomposes the nonlinear functions  $f(t, x(t))$  and  $g(t, x(t))$  into the series of the Adomian polynomials

$$f(t, x(t)) = \sum_{n=0}^{\infty} B_n(t), \quad g(t, x(t)) = \sum_{n=0}^{\infty} C_n(t), \quad (9)$$

where

$$B_n(t) = \frac{1}{n!} \left[ \frac{\partial^n}{\partial \lambda^n} f(t, \sum_{k=0}^{\infty} x_k(t) \lambda^k) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots, \quad (10)$$

and  $C_n(t)$  is defined in a similar manner. Thus the nonlinearity in Eq. (8) is decomposed into

$$Nx(t) = g(t, x(t)) \int_0^t f(s, x(s)) ds = \sum_{n=0}^{\infty} A_n(t), \quad (11)$$

where the Adomian polynomials are

$$A_n(t) = \sum_{k=0}^n C_{n-k}(t) \int_0^t B_k(s) ds. \quad (12)$$

Here we present the rational expression for the Adomian polynomials compared with that in [El-Sayed, Hashem, and Ziada (2010)].

Especially, if  $g(t, x(t))$  is linear in  $x(t)$ , i.e.  $g(t, x(t)) = b(t) + c(t)x(t)$ , then the  $C_n(t)$  would be

$$C_0(t) = b(t) + c(t)x_0(t), \quad C_n(t) = c(t)x_n(t), \quad n \geq 1. \quad (13)$$

The Adomian recursion scheme for the solution components is

$$x_0(t) = a(t), \tag{14}$$

$$x_{n+1}(t) = A_n(t), n \geq 0. \tag{15}$$

We can use several modified recursion schemes for different computational advantages, such as the modified recursion schemes proposed in [Wazwaz (1999); Wazwaz and El-Sayed (2001); Duan (2010a); Duan and Rach (2011); Duan, Rach, and Wang (2013)]. We note that in the modified recursion schemes in [Duan (2010a); Duan and Rach (2011); Duan, Rach, and Wang (2013)], a parameter was introduced in order to extend the effective region of convergence of the decomposition series solution.

The  $n$ -term approximation for the solution is

$$\varphi_n(t) = \sum_{k=0}^{n-1} x_k(t). \tag{16}$$

In the sequel we consider four numeric examples. Examples 1 and 2 have exact analytic solutions, while Examples 3 and 4 do not have exact analytic solutions. We emphasize that if we use the Picard iterative method in Examples 3 and 4, the integration could not be calculated analytically.

In the case of absence of the exact solution, such as in Examples 3 and 4 below, the accuracy of the analytic approximate solutions  $\varphi_n(t)$  can be readily verified by direct substitution for considering the absolute error remainder function

$$|ER_n(t)| = \left| \varphi_n(t) - a(t) - g(t, \varphi_n(t)) \int_0^t f(s, \varphi_n(s)) ds \right| \tag{17}$$

by using MATHEMATICA.

**Example 1.** Consider the quadratic integral equation

$$x(t) = t^2 - \frac{t^{10}}{35} + \frac{t}{5} x(t) \int_0^t s^2 x^2(s) ds, 0 \leq x \leq 1. \tag{18}$$

The equation has the exact solution  $x^*(t) = t^2$ .

We decompose the solution into  $x(t) = \sum_{n=0}^{\infty} x_n(t)$ , and nonlinearity into

$$Nx(t) = x(t) \int_0^t s^2 x^2(s) ds = \sum_{n=0}^{\infty} A_n(t), \tag{19}$$

where

$$A_n(t) = \sum_{k=0}^n x_{n-k}(t) \int_0^t s^2 B_k(s) ds, B_k(t) = \sum_{i=0}^k x_i(t) x_{k-i}(t). \tag{20}$$

By the Adomian recursion scheme

$$x_0(t) = t^2 - \frac{t^{10}}{35}, \tag{21}$$

$$x_{n+1}(t) = \frac{t}{5} A_n(t), n \geq 0, \tag{22}$$

we obtain

$$\begin{aligned} x_1(t) &= \frac{t^{10}}{35} - \frac{29t^{18}}{18375} + \frac{61t^{26}}{2113125} - \frac{t^{34}}{4930625}, \\ x_2(t) &= \frac{29t^{18}}{18375} - \frac{1927t^{26}}{14791875} + \frac{150434t^{34}}{34391109375} - \frac{400574t^{42}}{5215984921875} \\ &\quad + \frac{387353t^{50}}{545659325859375} - \frac{93t^{58}}{32646284453125}, \\ &\dots \end{aligned}$$

Hence the approximate sequence of the solution is

$$\begin{aligned} \varphi_1(t) &= t^2 - \frac{t^{10}}{35}, \\ \varphi_2(t) &= t^2 - \frac{29t^{18}}{18375} + \frac{61t^{26}}{2113125} - \frac{t^{34}}{4930625}, \\ \varphi_3(t) &= t^2 - \frac{4t^{26}}{39445} + \frac{143459t^{34}}{34391109375} - \frac{400574t^{42}}{5215984921875} \\ &\quad + \frac{387353t^{50}}{545659325859375} - \frac{93t^{58}}{32646284453125}, \\ &\dots \end{aligned}$$

We note that the terms  $\mp \frac{t^{10}}{35}$  appearing in  $x_0(t)$  and  $x_1(t)$  are called the noise terms [Adomian and Rach (1986, 1992b); Wazwaz (1997)]. The exact solution  $u^*(t)$  is obtained by cancelling the noise term  $-\frac{t^{10}}{35}$  in  $x_0(t)$ .

Table 1: The maximal error parameters  $ME_n$  (Example 1).

$n$	1	2	3	4	5	6	7	8
$ME_n$	0.0285714	0.00154957	0.0000973117	$6.53177 \times 10^{-6}$	$4.55482 \times 10^{-7}$	$3.25589 \times 10^{-8}$	$2.36821 \times 10^{-9}$	$1.74487 \times 10^{-10}$

In order to examine the convergence we calculate the maximal error parameters

$$ME_n = \max_{0 \leq t \leq 1} |\varphi_n(t) - t^2|. \tag{23}$$

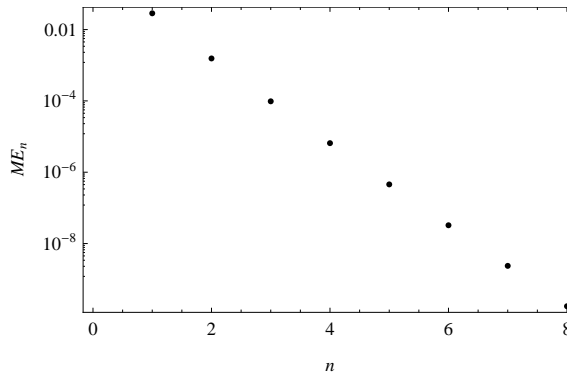


Figure 1: The logarithmic plots of the values of  $ME_1$  through  $ME_8$  (Example 1).

The values of  $ME_n$  for  $n = 1, 2, \dots, 8$  are listed in Table 1. The logarithmic plots of the values of  $ME_1$  through  $ME_8$  are displayed in Fig. 1, which demonstrates an approximately exponential rate of convergence for the decomposition series.

If we use Wazwaz’s modified recursion scheme [Wazwaz (1999)]

$$x_0 = t^2, \tag{24}$$

$$x_1 = -\frac{t^{10}}{35} + \frac{t}{5}A_0, \tag{25}$$

$$x_{n+1} = \frac{t}{5}A_n(t), \quad n \geq 1, \tag{26}$$

we obtain  $A_0 = x_0(t) \int_0^t s^2 x_0^2(s) ds = \frac{t^9}{7}$ , and  $x_1(t) = 0$ . Hence  $x_n(t) = 0$  for all  $n \geq 1$ . The exact solution is obtained  $u^*(t) = t^2$ .

**Example 2.** Consider the quadratic integral equation

$$x(t) = t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} + \frac{t^3}{10} x^2(t) \int_0^t (s+1)x^3(s) ds, \quad 0 \leq x \leq 1. \tag{27}$$

The equation has the exact solution  $x^*(t) = t^3$ .

We decompose the solution into  $x(t) = \sum_{n=0}^{\infty} x_n(t)$ , and decompose the nonlinearity

$$Nx(t) = x^2(t) \int_0^t (s+1)x^3(s) ds$$

into

$$Nx(t) = \sum_{n=0}^{\infty} A_n(t), \quad A_n(t) = \sum_{k=0}^n C_{n-k}(t) \int_0^t (s+1) B_k(s) ds, \tag{28}$$

where

$$C_k(t) = \sum_{i=0}^k x_i(t)x_{k-i}(t), B_k(t) = \sum_{j=0}^k \sum_{i=0}^j x_i(t)x_{j-i}(t)x_{k-j}(t). \tag{29}$$

By the Adomian recursion scheme

$$x_0(t) = t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}, \tag{30}$$

$$x_{n+1}(t) = \frac{t^3}{10} A_n(t), n \geq 0, \tag{31}$$

we calculate that the solution components

$$x_1(t) = \frac{t^{19}}{100} + \frac{t^{20}}{110} - \frac{41t^{35}}{130000} - \frac{19t^{36}}{33000} - \frac{89t^{37}}{338800} + \frac{183t^{51}}{45500000} + \dots - \frac{t^{105}}{9985162000000},$$

...

and the approximate sequence of the solution

$$\begin{aligned} \varphi_1(t) &= t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}, \\ \varphi_2(t) &= t^3 - 0.000315385t^{35} - 0.000575758t^{36} - 0.000262692t^{37} + 4.02198 \times 10^{-6}t^{51} \\ &\quad + 0.0000110338t^{52} + 0.0000100873t^{53} + 3.07324 \times 10^{-6}t^{54} - 2.75483 \times 10^{-8}t^{67} \\ &\quad - 1.00819 \times 10^{-7}t^{68} - 1.38333 \times 10^{-7}t^{69} - 8.43403 \times 10^{-8}t^{70} \\ &\quad - 1.92791 \times 10^{-8}t^{71} + 1.05911 \times 10^{-10}t^{83} + 4.84184 \times 10^{-10}t^{84} \\ &\quad + 8.85249 \times 10^{-10}t^{85} + 8.09136 \times 10^{-10}t^{86} + 3.69726 \times 10^{-10}t^{87} \\ &\quad + 6.75669 \times 10^{-11}t^{88} - 1.72414 \times 10^{-13}t^{99} - 9.45221 \times 10^{-13}t^{100} \\ &\quad - 2.15888 \times 10^{-12}t^{101} - 2.62948 \times 10^{-12}t^{102} - 1.80128 \times 10^{-12}t^{103} \\ &\quad - 6.58026 \times 10^{-13}t^{104} - 1.00149 \times 10^{-13}t^{105}, \\ \varphi_3(t) &= t^3 - 0.0000125824t^{51} - 0.0000345005t^{52} - 0.0000315262t^{53} - 9.60073 \times 10^{-6}t^{54} \\ &\quad + 3.69878 \times 10^{-7}t^{67} + \dots, \\ &\dots \end{aligned}$$

We note that the exact solution  $u^*(t) = t^3$  is obtained by cancelling the noise term  $-\frac{t^{19}}{100} - \frac{t^{20}}{110}$  in  $x_0(t)$ .

Table 2: The maximal error parameters  $ME_n$  (Example 2).

$n$	1	2	3	4	5	6	7	8
$ME_n$	0.0190909	0.00112599	0.0000833805	$6.83848 \times 10^{-6}$	$5.94711 \times 10^{-7}$	$5.37331 \times 10^{-8}$	$4.98753 \times 10^{-9}$	$4.72338 \times 10^{-10}$

We check the convergence by calculating the maximal error parameters

$$ME_n = \max_{0 \leq t \leq 1} |\varphi_n(t) - t^3|. \tag{32}$$



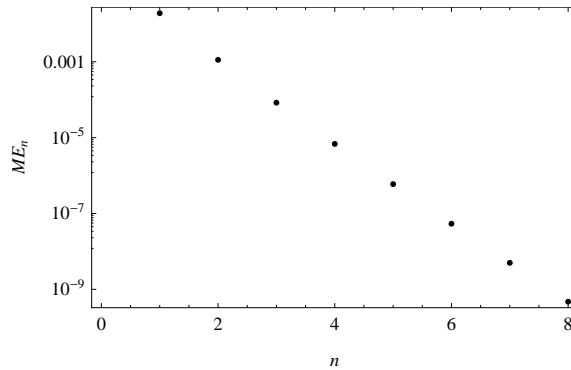


Figure 2: The logarithmic plots of the values of  $ME_1$  through  $ME_8$  (Example 2).

The values of  $ME_n$  for  $n = 1, 2, \dots, 8$  are listed in Table 2. The logarithmic plots of the values of  $ME_1$  through  $ME_8$  are displayed in Fig. 2, which demonstrates an approximately exponential rate of convergence for the decomposition series.

If we use Wazwaz’s modified recursion scheme [Wazwaz (1999)]

$$x_0 = t^3, \tag{33}$$

$$x_1 = -\frac{t^{19}}{100} - \frac{t^{20}}{110} + \frac{t^3}{10}A_0, \tag{34}$$

$$x_{n+1} = \frac{t^3}{10}A_n(t), \quad n \geq 1, \tag{35}$$

we obtain  $A_0 = x_0^2(t) \int_0^t (s+1)x_0^3(s)ds = \frac{t^{17}}{11} + \frac{t^{16}}{10}$ , and  $x_1(t) = 0$ . So  $x_n(t) = 0$  for all  $n \geq 1$ . The exact solution is obtained  $u^*(t) = t^3$ .

**Example 3.** Consider the quadratic integral equation

$$x(t) = t^3 + \frac{t^2}{4}(1+x(t)) + \frac{1}{4}(1+x(t)) \int_0^t \cos \frac{x^2(s)}{1+x^2(s)} ds, \quad 0 \leq t \leq 1. \tag{36}$$

The equation does not have an exact analytic solution.

Using the decomposition of the solution  $x(t) = \sum_{n=0}^{\infty} x_n(t)$ , the nonlinearity

$$Nx(t) = (1+x(t)) \int_0^t \cos \frac{x^2(s)}{1+x^2(s)} ds$$

is decomposed into

$$Nx(t) = \sum_{n=0}^{\infty} A_n(t), \quad A_n(t) = \sum_{k=0}^n C_{n-k}(t) \int_0^t B_k(s) ds, \tag{37}$$

where

$$C_0(t) = 1 + x_0(t), C_k(t) = x_k(t), k \geq 1,$$

and the Adomian polynomials for  $f(x(t)) = \cos \frac{x^2(t)}{1+x^2(t)}$  are

$$\begin{aligned} B_0(t) &= \cos \frac{x_0^2(t)}{1+x_0^2(t)}, \\ B_1(t) &= -\frac{2x_0(t)x_1(t)}{(x_0^2(t)+1)^2} \sin \frac{x_0^2(t)}{x_0^2(t)+1}, \\ B_2(t) &= \frac{-2x_2(t)x_0^3(t) + 3x_1^2(t)x_0^2(t) - 2x_2(t)x_0(t) - x_1^2(t)}{(x_0^2(t)+1)^3} \sin \frac{x_0^2(t)}{x_0^2(t)+1} \\ &\quad - \frac{2x_0^2(t)x_1^2(t)}{(x_0^2(t)+1)^4} \cos \frac{x_0^2(t)}{x_0^2(t)+1}, \\ &\dots \end{aligned}$$

By the parametrized recursion scheme [Duan (2010a); Duan and Rach (2011); Duan, Rach, and Wang (2013)]

$$x_0 = c, \tag{38}$$

$$x_1 = -c + t^3 + \frac{t^2}{4}(1 + x_0(t)) + \frac{1}{4}A_0, \tag{39}$$

$$x_{n+1} = \frac{t^2}{4}x_n(t) + \frac{1}{4}A_n(t), n \geq 1, \tag{40}$$

we obtain the approximate sequence of the solution such as for  $c = 0.5$ ,

$$\begin{aligned} \varphi_2(t) &= 0.367525t + 0.375t^2 + t^3, \\ \varphi_3(t) &= 0.268857t + 0.331288t^2 + 1.1778t^3 + 0.326846t^4 + 0.25t^5, \\ \varphi_4(t) &= 0.239709t + 0.333811t^2 + 1.1557t^3 + 0.388937t^4 + 0.346312t^5 \\ &\quad + 0.123425t^6 + 0.0424382t^7, \\ &\dots \end{aligned}$$

The curves of the approximate solutions  $\varphi_n(t)$  for  $n = 2, 3, 4, 5, 6$  are plotted in Fig. 3, where  $\varphi_5(t)$  and  $\varphi_6(t)$  overlap.

In order to examine the convergence we consider the absolute error remainder function

$$|ER_n(t)| = \left| \varphi_n(t) - t^3 - \frac{t^2}{4}(1 + \varphi_n(t)) - \frac{1}{4}(1 + \varphi_n(t)) \int_0^t \cos \frac{(\varphi_n(s))^2}{1 + (\varphi_n(s))^2} ds \right|.$$

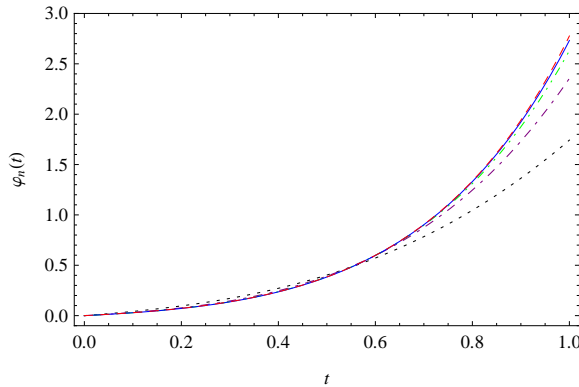


Figure 3: Curves of the approximate solutions  $\varphi_n(t)$  for  $n = 2$  (dot line),  $n = 3$  (dot-dash line),  $n = 4$  (dot-dot-dash line),  $n = 5$  (solid line) and  $n = 6$  (dash line) (Example 3).

Table 3: The absolute error remainder function  $|ER_n(t)|$  (Example 3).

$t$	0	0.2	0.4	0.6	0.8	1
$ ER_3(t) $	0	0.00442553	0.00832103	0.00146290	0.0622715	0.259762
$ ER_4(t) $	0	0.000995395	0.000130127	0.000642902	0.0195373	0.112419
$ ER_5(t) $	0	0.000951998	0.00113826	0.00115901	0.00715229	0.0582594
$ ER_6(t) $	0	0.000108734	0.000192676	0.000501003	0.00302914	0.0348328
$ ER_7(t) $	0	0.000329753	0.000274385	0.0000755925	0.00109902	0.0114960
$ ER_8(t) $	0	0.000110965	0.000183536	0.000174717	0.000208731	0.00388898

(41)

We calculate  $|ER_n(t)|$  for  $n = 3, 4, \dots, 8$  at  $t = 0, 0.2, 0.4, 0.6, 0.8, 1$ , respectively, and list the results in Table 3.

**Example 4.** Consider the quadratic integral equation

$$x(t) = e^{-t} + \frac{1}{2}t^2 e^{-t} x(t) \int_0^t e^{-s} \ln(1 + sx(s)) ds, \quad 0 \leq t \leq 2. \tag{42}$$

The equation does not have an exact analytic solution.

We decompose the solution into  $x(t) = \sum_{n=0}^{\infty} x_n(t)$ , and the nonlinearity

$$Nx(t) = x(t) \int_0^t e^{-s} \ln(1 + sx(s)) ds$$

into

$$Nx(t) = \sum_{n=0}^{\infty} A_n(t), A_n(t) = \sum_{k=0}^n x_{n-k}(t) \int_0^t e^{-s} B_k(s) ds, \tag{43}$$

where the Adomian polynomials for  $f(x(t)) = \ln(1 + tx(t))$  are

$$\begin{aligned} B_0(t) &= \ln(1 + tx_0(t)), \\ B_1(t) &= \frac{tx_1(t)}{tx_0(t) + 1}, \\ B_2(t) &= \frac{t(2(tx_0(t) + 1)x_2(t) - tx_1^2(t))}{2(tx_0(t) + 1)^2}, \\ B_3(t) &= \frac{t(t^2x_1^3(t) - 3t(tx_0(t) + 1)x_2(t)x_1(t) + 3(tx_0(t) + 1)^2x_3(t))}{3(tx_0(t) + 1)^3}, \\ &\dots \end{aligned}$$

From the recursion scheme

$$x_0 = 0, \tag{44}$$

$$x_1 = e^{-t} + \frac{1}{2}t^2e^{-t}A_0, \tag{45}$$

$$x_{n+1} = \frac{1}{2}t^2e^{-t}A_n(t), n \geq 1, \tag{46}$$

we obtain the approximate sequence of the solution

$$\begin{aligned} \varphi_2(t) &= \varphi_3(t) = e^{-t}, \\ \varphi_4(t) &= e^{-t} + \frac{1}{8}e^{-2t}t^2 + e^{-4t} \left( -\frac{t^2}{8} - \frac{t^3}{4} \right), \\ \varphi_5(t) &= e^{-t} + \frac{23}{216}e^{-2t}t^2 + e^{-4t} \left( -\frac{t^2}{8} - \frac{t^3}{4} \right) + e^{-5t} \left( \frac{t^2}{54} + \frac{t^3}{18} + \frac{t^4}{12} \right), \\ &\dots \end{aligned}$$

Table 4: The absolute error remainder function  $|ER_n(t)|$  (Example 4).

$t$	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$ ER_3(t) $	0	0.000195686	0.00157691	0.00410976	0.00680731	0.00884484	0.00989530	0.0100149	0.00944188	0.00844876	0.00726669
$ ER_4(t) $	0	0.0000105471	0.000136179	0.000419007	0.000717199	0.000894629	0.000921634	0.000842118	0.000715015	0.000582861	0.000466660
$ ER_5(t) $	0	$9.26944 \times 10^{-7}$	0.0000237930	0.000116835	0.000292902	0.000507293	0.000695039	0.000810830	0.000842408	0.000802853	0.000716466
$ ER_6(t) $	0	$9.29067 \times 10^{-8}$	$4.10764 \times 10^{-6}$	0.0000260772	0.0000743638	0.000136215	0.000189122	0.000218407	0.000222298	0.000207160	0.000181365

The curves of the approximate solutions  $\varphi_n(t)$  for  $n = 3, 4, 5$  are plotted in Fig. 4, where  $\varphi_4(t)$  and  $\varphi_5(t)$  overlap.

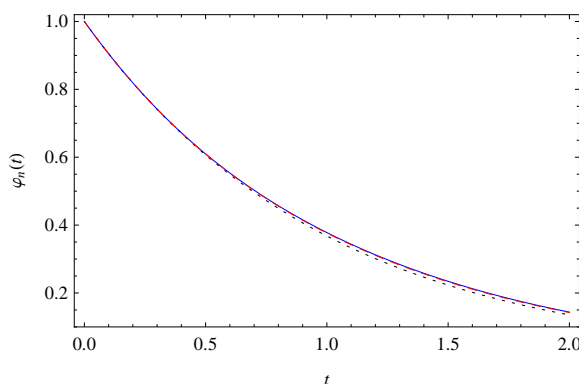


Figure 4: Curves of the approximate solutions  $\varphi_n(t)$  for  $n = 3$  (dot line),  $n = 4$  (solid line) and  $n = 5$  (dash line) (Example 4).

Furthermore, we check the absolute error remainder function

$$|ER_n(t)| = \left| \varphi_n(t) - e^{-t} - \frac{1}{2}t^2 e^{-t} \varphi_n(t) \int_0^t e^{-s} \ln(1 + s\varphi_n(s)) ds \right|.$$

We calculate  $|ER_n(t)|$  for  $n = 3, 4, 5, 6$  at  $t = 0, 0.2, 0.4, 0.6, \dots, 2$ , respectively, and list the results in Table 4.

### 3 Conclusions

We demonstrate an efficient application of the ADM to the quadratic integral equations. The ADM gives a sequence of analytical approximate solutions, which can be verified by direct substitution. This is convenient for the error estimation and parameter analysis for engineering problems. Four numerical examples demonstrate the efficiency of the method and the fast convergent rate of the decomposition solutions.

If we use the Picard iterative method in Examples 3 and 4 we would encounter integrations which can not be calculated analytically. The numeric examples display that the ADM is more efficient for the analytic approximate solutions of the nonlinear equations than the Picard iterative method.

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## References

- Abbaoui, K.; Cherruault, Y.** (1994): Convergence of Adomian's method applied to nonlinear equations. *Math. Comput. Modelling*, vol. 20, no. 9, pp. 60–73.
- Abbaoui, K.; Cherruault, Y.; Seng, V.** (1995): Practical formulae for the calculus of multivariable Adomian polynomials. *Math. Comput. Modelling*, vol. 22, pp. 89–93.
- Abdelrazec, A.; Pelinovsky, D.** (2011): Convergence of the Adomian decomposition method for initial-value problems. *Numer. Methods Partial Differential Equations*, vol. 27, pp. 749–766.
- Abdelwahid, F.** (2003): A mathematical model of Adomian polynomials. *Appl. Math. Comput.*, vol. 141, pp. 447–453.
- Adomian, G.** (1983): *Stochastic Systems*. Academic, New York.
- Adomian, G.** (1994): *Solving Frontier Problems of Physics: The Decomposition Method*. Kluwer Academic, Dordrecht.
- Adomian, G.; Rach, R.** (1983): Inversion of nonlinear stochastic operators. *J. Math. Anal. Appl.*, vol. 91, pp. 39–46.
- Adomian, G.; Rach, R.** (1986): The noisy convergence phenomena in decomposition method solutions. *J. Comput. Appl. Math.*, vol. 15, pp. 379–381.
- Adomian, G.; Rach, R.** (1992a): Generalization of Adomian polynomials to functions of several variables. *Comput. Math. Appl.*, vol. 24, pp. 11–24.
- Adomian, G.; Rach, R.** (1992b): Noise terms in decomposition solution series. *Comput. Math. Appl.*, vol. 24, pp. 61–64.
- Agarwal, R. P.; O'Regan, D.; Wong, P. J. Y.** (1999): *Positive Solutions of Differential, Difference and Integral Equations*. Kluwer, Dordrecht.
- Azreg-Aïnou, M.** (2009): A developed new algorithm for evaluating Adomian polynomials. *CMES-Comput. Model. Eng. Sci.*, vol. 42, pp. 1–18.
- Banaś, J.; Caballero, J.; Rocha, J.; Sadarangani, K.** (2005): Monotonic solutions of a class of quadratic integral equations of volterra type. *Comput. Math. Appl.*, vol. 49, pp. 943–952.
- Biazar, J.; Ilie, M.; Khoshkenar, A.** (2006): An improvement to an alternate algorithm for computing Adomian polynomials in special cases. *Appl. Math. Comput.*, vol. 173, pp. 582–592.
- Cherruault, Y.; Adomian, G.** (1993): Decomposition methods: a new proof of convergence. *Math. Comput. Modelling*, vol. 18, pp. 103–106.

- Corduneanu, C.** (1991): *Integral Equations and Applications*. Cambridge University Press, Cambridge, MA.
- Duan, J. S.** (2010a): Recurrence triangle for Adomian polynomials. *Appl. Math. Comput.*, vol. 216, pp. 1235–1241.
- Duan, J. S.** (2010b): An efficient algorithm for the multivariable Adomian polynomials. *Appl. Math. Comput.*, vol. 217, pp. 2456–2467.
- Duan, J. S.** (2011): Convenient analytic recurrence algorithms for the Adomian polynomials. *Appl. Math. Comput.*, vol. 217, pp. 6337–6348.
- Duan, J. S.; Guo, A. P.** (2010): Reduced polynomials and their generation in Adomian decomposition methods. *CMES-Comput. Model. Eng. Sci.*, vol. 60, pp. 139–150.
- Duan, J. S.; Rach, R.** (2011): A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations. *Appl. Math. Comput.*, vol. 218, pp. 4090–4118.
- Duan, J. S.; Rach, R.; Baleanu, D.; Wazwaz, A. M.** (2012): A review of the Adomian decomposition method and its applications to fractional differential equations. *Commun. Frac. Calc.*, vol. 3, pp. 73–99.
- Duan, J. S.; Rach, R.; Wang, Z.** (2013): On the effective region of convergence of the decomposition series solution. *J. Algorithms Comput. Tech.*, vol. 7, no. 2, pp. 227–247.
- El-Sayed, A. M. A.; Hashem, H. H. G.; Ziada, E. A. A.** (2010): Picard and Adomian methods for quadratic integral equation. *Comput. Appl. Math.*, vol. 29, pp. 447–463.
- Gabet, L.** (1994): The theoretical foundation of the Adomian method. *Comput. Math. Appl.*, vol. 27, pp. 41–52.
- Hu, S.; Khavani, M.; Zhuang, W.** (1989): Integral equations arising in the kinetic theory of gases. *Appl. Analysis*, vol. 34, pp. 261–266.
- Kelmanson, M. A.; Tenwick, M. C.** (2010): Error reduction in Gauss-Jacobi-Nyström quadrature for Fredholm integral equations of the second kind. *CMES-Comput. Model. Eng. Sci.*, vol. 55, pp. 191–210.
- Lai, H. Y.; Chen, C. K.; Hsu, J. C.** (2008): Free vibration of non-uniform Euler-Bernoulli beams by the Adomian modified decomposition method. *CMES-Comput. Model. Eng. Sci.*, vol. 34, pp. 87–116.
- Liu, C. S.; Atluri, S. N.** (2009): A fictitious time integration method for the numerical solution of the Fredholm integral equation and for numerical differentia-

tion of noisy data, and its relation to the filter theory. *CMES-Comput. Model. Eng. Sci.*, vol. 41, pp. 243–261.

**Rach, R.** (1984): A convenient computational form for the Adomian polynomials. *J. Math. Anal. Appl.*, vol. 102, pp. 415–419.

**Rach, R.** (1987): On the Adomian (decomposition) method and comparisons with Picard's method. *J. Math. Anal. Appl.*, vol. 128, pp. 480–483.

**Rach, R.** (2008): A new definition of the Adomian polynomials. *Kybernetes*, vol. 37, pp. 910–955.

**Rach, R.** (2012): A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011. *Kybernetes*, vol. 41, pp. 1087–1148.

**Wazwaz, A. M.** (1997): Necessary conditions for the appearance of noise terms in decomposition series solutions. *Appl. Math. Comput.*, vol. 81, pp. 265 – 274.

**Wazwaz, A. M.** (1999): A reliable modification of Adomian decomposition method. *Appl. Math. Comput.*, vol. 102, pp. 77–86.

**Wazwaz, A. M.** (2000): A new algorithm for calculating Adomian polynomials for nonlinear operators. *Appl. Math. Comput.*, vol. 111, pp. 53–69.

**Wazwaz, A. M.** (2009): *Partial Differential Equations and Solitary Waves Theory*. Higher Education Press, Beijing, and Springer-Verlag, Berlin.

**Wazwaz, A. M.** (2011): *Linear and Nonlinear Integral Equations: Methods and Applications*. Higher Education Press, Beijing, and Springer-Verlag, Berlin.

**Wazwaz, A. M.; El-Sayed, S. M.** (2001): A new modification of the Adomian decomposition method for linear and nonlinear operators. *Appl. Math. Comput.*, vol. 122, pp. 393–405.

**Zhu, Y.; Chang, Q.; Wu, S.** (2005): A new algorithm for calculating Adomian polynomials. *Appl. Math. Comput.*, vol. 169, pp. 402–416.

**Zou, H.; Li, H.** (2010): A novel meshless method for solving the second kind of Fredholm integral equations. *CMES-Comput. Model. Eng. Sci.*, vol. 67, pp. 55–77.

#### **Appendix A: MATHEMATICA code for the Adomian polynomials based on Duan's Corollary 3 algorithm [Duan (2011)]**

```
Adomian[f_, M_] := Module[{c, n, k, j, der},
Table[c[n, k], {n, 1, M}, {k, 1, n}];
der=Table[D[f[Subscript[u,0]], {Subscript[u,0],k}], {k,1,M}];
A[0]=f[Subscript[u,0]];
For[n=1, n<=M, n++, c[n,1]=Subscript[u,n];
```



```
For[k=2, k<=n, k++,  
c[n,k]=Expand[1/n*Sum[(j+1)*Subscript[u,j+1]*c[n-1-j,k-1],  
{j,0,n-k}]]];  
A[n]=Take[der,n].Table[c[n,k], {k,1,n}];  
Table[A[n], {n,0,M}]
```

