Applications of the MLPG Method in Engineering & Sciences: A Review

J. Sladek[1]**, P. Stanak**[1]**, Z-D. Han**[2]**, V. Sladek**[1]**, S.N. Atluri**[2]

Abstract: A review is presented for analysis of problems in engineering & the sciences, with the use of the meshless local Petrov-Galerkin (MLPG) method. The success of the meshless methods lie in the local nature, as well as higher order continuity, of the trial function approximations, high adaptivity and a low cost to prepare input data for numerical analyses, since the creation of a finite element mesh is not required. There is a broad variety of meshless methods available today; however the focus is placed on the MLPG method, in this paper. The MLPG method is a fundamental base for the derivation of many meshless formulations, since the trial and test functions can be chosen from different functional spaces. In the last decade, a broad community of researchers and scientists contributed to the development and implementation of the MLPG method in a wide range of scientific disciplines.

This paper first presents the basics and principles of the MLPG method, the meshless local approximation techniques for trial and test functions, applications to elasticity and elastodynamics, plasticity, fracture and crack analysis, heat transfer and fluid flow, coupled problems involving multiphase materials, and techniques for increasing the accuracy and computational effectiveness. Various applications to 2-D planar problems, axisymmetric problems, plates and shells or 3-D problems are included.

An increased number of published papers in literature in the recent years can be considered as a measure of the growing research activity in the general scope of the MLPG method, and thus, several trends and ideas for future research interest are also outlined.

Keywords: meshless local Petrov-Galerkin (MLPG) method, meshless local approximation schemes for trial and test functions, local weak forms, numerical applications

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[1] Slovak Academy of Sciences, Bratislava, Slovakia.
[2] International Collaboratory for Fundamental Studies in the Engineering Sciences, UCI, Irvine, USA.

1   Introduction

Computer modeling is often viewed as a means of supporting a broad diversity of science and technology areas. Since the development of advanced computers, computational scientists have applied their knowledge as a new branch of science, complementing the experiment and the traditional theory. Numerical models after successful validation, enable advanced design, optimization and control of new products and processes. There are various numerical modeling techniques available these days, among which the finite element method (FEM) is the most popular and widely used, among engineers and scientists. Even though the method is well established, robust and with enormous influence over the past decades, it still suffers from several drawbacks. FEM requires the creation of a geometric mesh consisting of finite elements discretizing the solution domain. In general it is desired that the mesh is as ideal as possible and well-structured, since distorted geometry may have a negative impact on the solution accuracy. Thus a human-labor intensive process of constructing high quality meshes is required. Other problems of finite element mesh may also appear, for example element locking in modeling of thin-walled structures, costly remeshing or element distortion during large deformation analyses. In cases of crack propagation analysis, the crack growth usually does not coincide with the element boundaries and remeshing techniques must be applied to tackle this problem.

In order to reduce the labor of creating the finite element mesh and reduce the computational cost various mesh reduction techniques were researched and developed. Among them the boundary element method (BEM) received the attention of many researchers [Brebbia et al. (1984); Manolis and Beskos (1988); Balas, Sladek and Sladek (1989); Banerjee (1994); Dominguez (1993); Wrobel and Aliabadi (2002)]. The BEM requires the construction of a mesh only on the boundary of the solution domain, thus allowing for a significant reduction in the time consuming mesh generation. The conventional BEM is accurate for many engineering problems, however it requires the availability of fundamental solutions or the Green's functions to the governing partial differential equations (PDE). For the problems including coupled problems with complex constitutive equations or continuously nonhomogeneous material properties, it is extremely difficult or even impossible to obtain appropriate fundamental solutions unless some field variables are introduced into the solution domain [Okada, Rajiyah, Atluri(1988); Okada, Rajiyah, Atluri (1989)]. Therefore, the application of the BEM is restricted to a relatively small class of engineering problems, where fundamental solutions are available.

In order to overcome the drawbacks of the mesh-based methods meshless methods have been developed for solving PDEs in engineering and the sciences. Focusing only on nodes or points instead of elements used in conventional FEM or BEM,

meshless approaches have certain advantages. The computational model is respresented simply by a set of nodes scattered in the solution domain and on the boundary. There is no connection between the nodes as in case of elements or any restriction on their mutual position. The connectivity between nodes may be prescribed during the computation process, thus reducing human labor related to creation of the FE mesh. The problem of remeshing may be simply diminished by adding or removing the nodes as needed, even during the computation and by redefining the connectivity only in locations where the nodes were added. Higher order continuity of physical fields in the solution domain can be preserved, which for example leads to globally continuous stress fields, thus simplifying the post-processing.

The principal feature of the meshless methods is the use of appropriate approximation schemes that can approximate the data specified on the randomly located nodes without use of predefined mesh. Depending mainly on the type of approximation scheme various meshless methods can be specified, starting with smoothed particle hydrodynamics (SPH) [Gingold and Monaghan (1977); Monaghan (1988)], that is considered as an initial idea of meshless modeling. Since the introduction of the diffuse element method [Nayroles, Touzot and Villon (1992)] many other meshless methods originated including element-free Galerkin (EFG) method [Belytschko, Lu and Gu (1994)], reproducing kernel particle methods (RKPM) [Liu, Jun and Zhang (1995)], meshless method using radial basis functions (RBFs) [Kansa (1990); Wedland (1999); Chen (2000)] or partition of unity finite element method (PUFEM) [Babuska and Melenk (1997)]. Several review articles were published that track the application of these methods [Belytschko et al. (1996); Liu et al. (1996); Li and Liu (2002); Nguyen et al. (2008)]. In these above mentioned methods the mesh is not required for the interpolation of the trial and test functions; however the use of shadow elements is often inevitable for the integration in the global weak-form based on the Galerkin approach. Therefore, these methods can not be considered as truly meshless.

Need for a truly meshless method lead to development of meshless local Petrov-Galerkin method (MLPG) [Atluri and Zhu (1998a)] and at the same time also local boundary integral equation (LBIE) method [Zhu, Zhang and Atluri (1998); Atluri et al. (2000)]. In both these methods no finite elements are required, neither for interpolation of trial and test functions for the solution variables, nor for the integration of the symmetric or unsymmetric local weak-form of governing equations. All integrals can be easily evaluated over regularly shaped, overlapping domains of arbitrary shape (in general, circles for 2-D problems and spheres for 3-D problems) and their respective boundaries. In each domain only one nodal point is located, thus local sense of the approach is kept. The LBIE method can be considered simply as a special case of the MLPG approach [Atluri, Cho and Kim (1999)].

Unlike in the conventional Galerkin method where the trial and the test functions are chosen from the same space, according the Petrov-Galerkin principle the trial and the test functions are chosen from different functional spaces. The nodal trial function may correspond to any one of moving least-squares (MLS) [Lancaster and Salkauskas (1981)], partition of unity (PU), Shepard function, or RBF types of interpolations [Kansa (1990); Atluri and Shen (2002a)]; and the test function may be totally different, and may correspond to any one of MLS, PU, Shepard function, RBF, a Heaviside step function, a Dirac delta function, the Gaussian weight function of MLS, a special form of the fundamental solution to the differential equation, or any other convenient function, in the support domain of the test function. Atluri and Shen (2002b) have derived six MLPG formulations depending on various test functions applied and marked them MLPG1 – MLPG6. Tab. 1 shows characteristic features of each formulation. MLPG5 formulation using Heaviside unit step function prooved to be promising, fast and robust method as it doesn't involve either domain or singular integrals for generation of stiffness matrix in linear elasicity problems [Atluri and Shen (2002ab); Atluri (2004)]. MLPG5 become very popular and a significant number of scientists is using it in their research.

Table 1: Meshless local Petrov-Galerkin methods as characterized by Atluri and Shen (2002b)

Type of the method	Test function utilized	Type of the integral in the weak form
MLPG1	MLS weight function	domain integral
MLPG2	Dirac delta function	none
MLPG3	Discrete least-squares	domain integral
MLPG4	Fundamental solution	singular boundary integral
MLPG5	Heaviside unit step function	regular boundary integral
MLPG6	same as the trial function	domain integral

Different size of supports as well as shapes of trial and test functions are possible, which makes MLPG method very flexible. The MLPG method, based on a local formulation, can include all the other meshless methods based on global formulation, as special cases [Atluri (2004)]. Various methods have been developed based on the MLPG approach, including the primal MLPG method [Atluri and Zhu (1998)], Local BIE [Zhu, Zhang and Atluri (1998); Atluri, Cho and Kim (1999)] already introduced above; and also the finite volume method [Atluri and Shen (2002a)], the BIE [Atluri, Han and Shen (2003)], the mixed finite volume method [Atluri, Han and Rajendran (2004)], the mixed collocation method [Atluri, Liu and Han (2006a)], the mixed finite difference method [Atluri, Liu and Han

(2006b)], the Galerkin equivalent of the MLPG method [Han & Atluri(2011)], and several others as summarized in [Atluri (2004)].

This review paper attempts to focus primary on the developments of the MLPG metod in various fields of engineering and scientific problems. The paper is organized as follows: Section 2 briefly describes the MLPG weak-form approach and introduces the concept of the MLS meshless interpolation of scattered node data. Section 3 gives detailed review of MLPG application to mechanics of solids. The section is divided to four subsections and describes in detail elasticity and plasticity problems, applications to plates and shells, solutions of the coupled multiphysics problems and fracture analysis problems, respectively. In Section 4 the problems of heat conduction and fluid flow are presented in detail. Advanced numerical techniques for increasing the performance of the MLPG are introduced in Section 5. Finally, applications to some special problems of engineering and sciences are presented and several future research perspectives are suggested in Section 6.

2   MLPG local weak form approach

In order to illustrate the principles of solving problems described by PDEs using the MLPG method, the solution of potential problem described by the Poisson's equation is presented as in [Atluri and Shen (2002b)].

Many of the so-called meshless methods, such as the EFG method, are based on the global weak form over the entire problem domain Ω. In the MLPG, however, a local weak form over a local subdomain Ω_s, which is located entirely inside the global domain Ω is used, as shown in Fig. 1. This is the most distinguishing feature of the MLPG. It is noted that the local sub-domain Ω_s can be of an arbitrary shape.

Let us now consider the linear Poisson's equation given in form

$$\nabla^2 u(\mathbf{x}) = p(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{1}$$

where u is the unknown potential field , p represents a given source function and Ω is the problem domain with outer boundary $\partial\Omega = \Gamma = \Gamma_u \cup \Gamma_q$, where boundary conditions are specified as

$u = \tilde{u} \ on \ \Gamma_u$

$$\frac{\partial u}{\partial n} = q = \tilde{q} \ on \ \Gamma_q \tag{2}$$

where $\tilde{u}$ and $\tilde{q}$ are the prescribed potential and normal flux prescribed on the boundary Γ_u and Γ_q, respectively, and n is the unit outward normal to the boundary Γ.

Several symmetric and unsymmetric weak formulations are available as shown in [Atluri and Shen (2002b)] and also monograph by Atluri (2004). The local weak
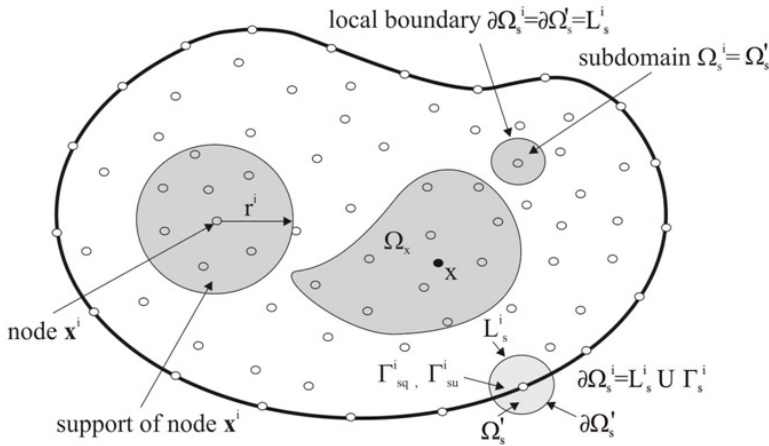
Figure 1: Local boundaries for weak formulation, the domain Ω_x for MLS approximation of the trial function, and support area of weight function around node $\mathbf{x}^i$.

form (LWF) of Eq. (1) is obtained by integrating all terms over the local subdomain Ω_s. Applying Gauss divergence theorem to the LWF one obtains

$$\int_{L_S} qw^* d\Gamma + \int_{\Gamma_{su}} qw^* d\Gamma + \int_{\Gamma_{sq}} \tilde{q}w^* d\Gamma - \int_{\Omega_s} \left(u_{,i}w^*_{,i} + pw^* \right) d\Omega - \alpha \int_{\Gamma_u} (u - \tilde{u}) w^* d\Gamma = 0$$

(3)

where w^* is the test function, α is the penalty parameter used to impose essntial boundary conditions, L_s is the part of the local boundary where no boundary conditions are prescribed, Γ_{su} and Γ_{sq} are parts of local boundary with prescribed essential and natural boundary conditions. In general, $\Gamma_s = \Gamma_{su} \cup \Gamma_{sq}$ and $\partial \Omega_s = L_s \cup \Gamma_s$. For a subdomain located entirely inside the global domain Ω, there is no intersection between $\partial \Omega_s$ and Γ, thus $L_s = \partial \Omega_s$ and the integrals over Γ_{su} and Γ_{sq} are vanishing. It is possible further simplify Eq. (3) by choosing a test function w^* such that it vanishes over L_s, which is, usually, a circle (for an internal node) in a 2-D problem, or the circular arc (for a node on the global boundary Γ) [Atluri and Zhu (1998)]. If the Heaviside unit step function is used as the test function

$$w^*(\mathbf{x}) = \begin{cases} 1 \text{ at } \mathbf{x} \in (\Omega_s \cup \partial \Omega_s) \\ 0 \text{ at } \mathbf{x} \notin (\Omega_s \cup \partial \Omega_s) \end{cases}$$

(4)

in the local weak form (3), we obtain well known form MLPG5 [Atluri and Shen (2002b)] . Finally, applying test function (4) into LWF (3) gives local integral

equation

$$\int_{L_S} q d\Gamma + \int_{\Gamma_{su}} q d\Gamma + \int_{\Gamma_{sq}} \tilde{q} d\Gamma - \int_{\Omega_s} p d\Omega - \alpha \int_{\Gamma_u} (u - \tilde{u}) d\Gamma = 0 \qquad (5)$$

Note that the local weak form in Eq. (3) doesn't depend on the size and shape of Ω_s, therefore simple regular shapes can be chosen to ease the implementation of the method.

At this point suitable approximation scheme must be chosen to approximate a trial function over an arbitrary solution domain using only values at finite number of nodal points located inside the domain and on its boundary.

2.1 MLS approximation scheme

Moving least squares (MLS) approximation is widely used and considered superior for various problems [Atluri, Kim and Cho (1999); Belytschko et al. (1996); Han and Atluri (2003); Atluri (2004)]. In the following, the method is briefly introduced to illustrate its basic concept and applicability for meshless formulations.

Let us consider a sub-domain Ω_x of the problem domain Ω in the neighbourhood of a point **x** for the definition of the MLS approximation of the trial function around **x** (Fig. 1). To approximate the distribution of the trial function u in Ω_x over a number of randomly located nodes $\{\mathbf{x}^a\}$, $a = 1, 2, ...n$, the MLS approximant $u^h(\mathbf{x})$ of $u(\mathbf{x})$ is defined by

$$u^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\tilde{\mathbf{a}}(\mathbf{x}), \forall \mathbf{x} \in \Omega_x \qquad (6)$$

where $\mathbf{p}^T(\mathbf{x}) = \left[p^1(\mathbf{x}), p^2(\mathbf{x}), ..., p^m(\mathbf{x})\right]$ is a complete monomial basis of order m, and $\tilde{\mathbf{a}}(\mathbf{x}) = \left[\mathbf{a}^1(\mathbf{x}), \mathbf{a}^2(\mathbf{x}), ..., \mathbf{a}^m(\mathbf{x})\right]^T$ is composed of vectors $\mathbf{a}^j(\mathbf{x}) = \left[a_1^j(\mathbf{x}), a_2^j(\mathbf{x}), a_3^j(\mathbf{x})\right]^T$ which are functions of the spatial co-ordinates $\mathbf{x} = [x_1, x_2]^T$ for a 2-D problem.

The coefficient vector $\tilde{\mathbf{a}}(\mathbf{x})$ is determined by minimizing a weighted discrete L_2-norm defined as

$$J(\mathbf{x}) = \sum_{a=1}^{n} v^a(\mathbf{x}) \left[\mathbf{p}^T(\mathbf{x}^a)\tilde{\mathbf{a}}(\mathbf{x}) - \hat{u}^a\right]^2, \qquad (7)$$

where $v^a(\mathbf{x}) > 0$ is the weight function associated with the node a and the square power is considered in the sense of scalar product. Recall that n is the number of nodes in Ω_x for which the weight function $v^a(\mathbf{x}) > 0$ and $\hat{u}^a$ are the fictitious nodal values, but not the nodal values of the unknown trial function $u^h(\mathbf{x})$, in general. The stationarity of J in Eq. (7) with respect to $\tilde{\mathbf{a}}(\mathbf{x})$ leads to

$$\mathbf{A}(\mathbf{x})\tilde{\mathbf{a}}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\hat{\mathbf{u}} = 0, \qquad (8)$$

where

$$\hat{\mathbf{u}} = \left[\hat{u}^1, \ \hat{u}^2, \ ..., \ \hat{u}^n \right]^T$$

$$\mathbf{A}(\mathbf{x}) = \sum_{a=1}^{n} v^a(\mathbf{x})\mathbf{p}(\mathbf{x}^a)\mathbf{p}^T(\mathbf{x}^a),$$

$$\mathbf{B}(\mathbf{x}) = \left[v^1(\mathbf{x})\mathbf{p}(\mathbf{x}^1), v^2(\mathbf{x})\mathbf{p}(\mathbf{x}^2), ..., v^n(\mathbf{x})\mathbf{p}(\mathbf{x}^n) \right] . \tag{9}$$

The solution of eq. (8) for $\tilde{\mathbf{a}}(\mathbf{x})$ and the subsequent substitution into Eq. (6) lead to the following expression

$$u^h(\mathbf{x}) = \mathbf{\Phi}^T(\mathbf{x}) \cdot \hat{\mathbf{u}} = \sum_{a=1}^{n} \phi^a(\mathbf{x})\hat{u}^a \tag{10}$$

where

$$\mathbf{\Phi}^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) . \tag{11}$$

In eq. (10), $\phi^a(\mathbf{x})$ is usually referred to as the shape function of the MLS approximation corresponding to the nodal point $\mathbf{x}^a$. From Eqs. (9) and (11), it can be seen that $\phi^a(\mathbf{x}) = 0$ when $v^a(\mathbf{x}) = 0$. In practical applications, $v^a(\mathbf{x})$ is often chosen in such a way that it is non-zero over the support of the nodal point $\mathbf{x}_i$. The support of the nodal point $\mathbf{x}^a$ is usually taken to be a circle of the radius r_i centred at $\mathbf{x}^a$ (see Fig. 1). The radius r_i is an important parameter of the MLS approximation because it determines the range of the interaction (coupling) between the degrees of freedom defined at considered nodes.

Wide range of MLS weight function is available, depending on the level of continuity required by the analysed problem. A 4th-order spline-type weight function [Atluri (2004)] is defined as

$$v^a(\mathbf{x}) = \begin{cases} 1 - 6\left(\frac{d^a}{r^a}\right)^2 + 8\left(\frac{d^a}{r^a}\right)^3 - 3\left(\frac{d^a}{r^a}\right)^4 & 0 \leq d^a \leq r^a \\ 0 & d^a \geq r^a \end{cases}, \tag{12}$$

where $d^a = \|\mathbf{x} - \mathbf{x}^a\|$ and r^a is the radius of the circular support domain. With Eq. (12), the C^1-continuity of the weight function is ensured over the entire domain. The continuity of the MLS approximation is given by the minimum between the continuity of the basis functions and that of the weight function. The size of the support r^a should be large enough to cover a sufficient number of nodes in the domain of definition to ensure the regularity of the matrix $\mathbf{A}$. The value of n is determined by the number of nodes lying in the support domain with radius r^a.

The partial derivatives of the MLS shape functions are obtained as [Atluri (2004)]

$$\phi_{,k}^a = \sum_{j=1}^{m} \left[p_{,k}^j (\mathbf{A}^{-1}\mathbf{B})^{ja} + p^j (\mathbf{A}^{-1}\mathbf{B}_{,k} + \mathbf{A}_{,k}^{-1}\mathbf{B})^{ja} \right] \quad , \tag{13}$$

wherein $\mathbf{A}_{,k}^{-1} = \left(\mathbf{A}^{-1}\right)_{,k}$ represents the derivative of the inverse of $\mathbf{A}$ with respect to x_k, which is given by $\mathbf{A}_{,k}^{-1} = -\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1}$.

The directional derivatives of $u(\mathbf{x})$ are approximated in terms of the same nodal values as

$$u_{,k}(\mathbf{x}) = \sum_{a=1}^{n} \phi_{,k}^a(\mathbf{x})\hat{u}^a \quad . \tag{14}$$

The MLS approximation of primary field value (10) and its derivative (14) can be also applied to enforce boundary conditions through collocation approach on appropriate boundary nodes, thus the use of penalty method or Lagrange multipliers can be avoided.

Finally eqs. (10) and (14) are inserted into LIE (5) in order to approximate the unknown potential field and its flux and obtain discretized local integral equation in form

$$\int_{L_S} \sum_{a=1}^{n} \phi_{,i}^a n_i(\mathbf{x})\hat{u}^a d\Gamma + \int_{\Gamma_{su}} \sum_{a=1}^{n} \phi_{,i}^a n_i(\mathbf{x})\hat{u}^a d\Gamma + \int_{\Gamma_{sq}} \tilde{q} d\Gamma - \int_{\Omega_s} p d\Omega$$
$$- \alpha \int_{\Gamma_u} \left(\sum_{a=1}^{n} \phi^a(\mathbf{x})\hat{u}^a - \tilde{u} \right) d\Gamma = 0 \tag{15}$$

Note that after performing evaluation of all integrals and solution of obtained system of algebraic equations the fictituous values of potential at nodal points are obtained, therefore Eq. (10) must be applied to recover the actual values of potential u.

3 Solids

Mechanics of solids covers a broad range of interesting topics. Significant number of researchers conducting their research in the field of numerical methods is focused especially to the analysis of physical behavior of various solids. In the following subsections the application of the MLPG method to several key topics of solid mechanics is presented.

3.1 Elasticity and plasticity

Since the introduction of MLPG to the scientific community, the method has been applied to analyze broad range of engineering problems in mechanics. Ability of

the MLPG to tackle linear and also nonlinear problems was soon recognized [Atluri and Zhu (1998a), (1998b)]. The analysis of beams by MLPG with use of various applied test functions and trial function approximations has been presented afterwards [Atluri, Cho and Kim (1999); Raju and Phillips (2003)]. The researchers also applied MLPG for elastostatics [Atluri and Zhu (2000)], elastodynamics [Batra and Ching (2002)] or solutions of non-hyper-singular traction and displacement boundary integral equations [Han and Atluri (2003)]. Recent advances in the development of the MLPG method are described in the monograph by Atluri and Sladek (2009).

Han and Atluri (2004a) introduced MLPG approach for solution of 3-D elastostatic and 3-D elastodynamical problems [Han and Atluri (2004b)] that reported the ability of MLPG to solve high-speed shock wave propagation problems and proved certain advantages over FEM. Problems of singularities and material discontinuities for 3-D elasticity were investigated by Li at al., (2003). They combined MLPG5 formulation together with MLPG2 (with Dirac's delta function as test function) to treat definition of boundary conditions and material discontinuitities. Meshless analysis of solids considering anisotropic elasticity was studied by Sladek, Sladek and Atluri, (2004a). Gu and Liu (2001a) employed MLPG for free and forced vibration analysis of various solids. Application of the MLPG for strain-gradient theory problems was presented in [Tang, Shen and Atluri (2003a)]. Micromechanical analysis of textile-reinforced composites based on the representative volume element (RVE) of the fiber and matrix was presented in [Dang and Sankar (2008)]. Advantages of the MLPG in discretization of the RVE with material discontinuity over FEM are presented therein.

New procedure of analyzing transient elastodynamic problems was proposed by Soares, Sladek and Sladek (2009). The authors combined Newmark algorithm with the time-domain Green's matrices of the elastodynamic problem in order to generate a recursive relationship for evaluation of displacements and velocities at each time-step. Non-linear dynamic analyses were also considered [Soares, Sladek and Sladek (2010)].

Meshless approaches proved to be successful in modeling of functionally graded materials (FGM). These materials can be characterized as multiphase composites with phase volume fractions gradually varying in space, in a pre-determined profile [Suresh and Mortensen (1998)]. Difference between FGM material and standard laminate composite can be observed in Fig. 2, where no distinct interface between FGM constituents is observed. Since no finite elements are required in meshless methods, continuous variation of material properties of FGMs can be prescribed to nodal points instead of elements as in FEM, thus higher accuracy can be expected. Standard BEM formulations cannot be used for analysis as fundamental solutions

for general functionally graded materials are not available in 2-D and 3-D elasticity. Elastodynamic behavior of continuously non-homogeneous solids was presented by Sladek, Sladek and Zhang (2003a). Laplace transform technique has been applied to treat time dependency of unknown quantities. Sladek et al. (2006a) successfully modeled linear viscoelastic solids with continuously non-homogeneous material properties. 3-D elastic analysis of anisotropic functionally graded solids by MLPG was presented in [Sladek, Sladek and Solek (2009)].
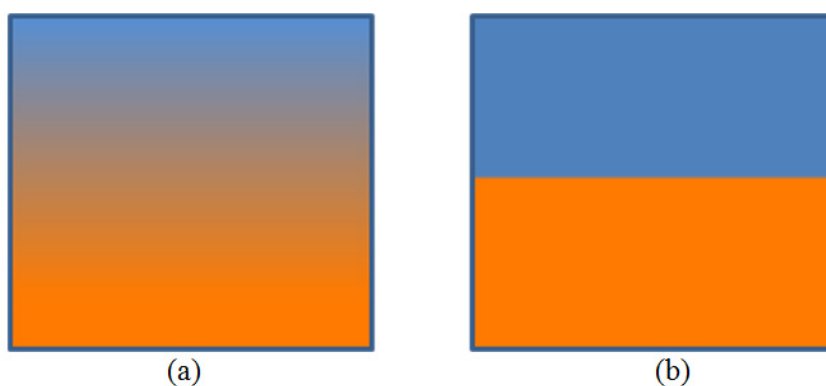

Figure 2: Difference between two-phase functionally graded material (a) and two-layer composite

Influence of transient thermal load on elastic response of solids was analyzed by Sladek et al. (2009a). Coupled thermoelastic theories were also considered [Sladek et al. (2006b); Hosseini, Sladek and Sladek (2011)]. Hosseini et al. (2011) combined the MLPG method with Monte-Carlo simulation to treat stochastic distribution of FGM material properties in thermoelastic transient analysis of thick hollow cylinder. Akbari et al. (2010) performed meshless analysis based on the MLPG approach for thermoelastic wave propagation in 2-D FGM domain.

For certain range of engineering problems the linear elasticity is no longer applicable, thus engineers and scientists must also turn to various non-linear models including also plasticity effects. Han, Rajendran and Atluri (2005) have presented the MLPG approach for solution of non-linear problems including large deformations and rotations. Non-linear elasto-plastic analysis of 3-D solids was recently studied in [Razaei Mojdehi, Darvizeh and Basti (2012)]. Von Mises yield criterion was used as a yield function to determine whether the material has yielded. Hybrid MLPG method was developed by Heaney, Augarde and Deeks (2010) for analysis of elasto-plastic problems in geomechanics. Their formulation permits also for the inclusion of the infinite boundaries.

Because of the total elimination of the mesh, MLPG method prooved to be a promising method for solving high-speed contact, impact and penetration problems with severe material-distortion [Han et al. (2006)]. MLPG solution approach for impact response and ballistic penetration of ceramic materials was developed by Liu et al., (2006). Fig. 3 shows final deformation and fragmentation of ceramic plate after impact of a projectile.
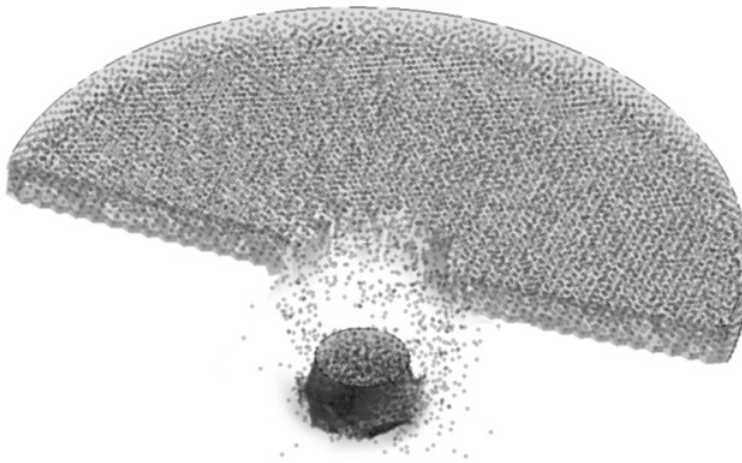

Figure 3: Ballistic impact penetration of ceramic plate analyzed by MLPG method [Liu et al. (2006)].

Many references can be identified for application of the LBIE/MLPG4 approach with test functions defined as modified fundamental solution to analyzed differential equation. Implementation of LBIE approach for linear elasticity [Atluri et al. (2000)], non-linear problems [Zhou, Zhang and Atluri (1999)] elasticity with non-homogeneous material properties [Sladek, Sladek and Atluri (2000)], elastodynamics [Sladek, Sladek and Van Keer (2003)], thermoelasticity [Sladek, Sladek and Atluri (2001)], frequency domain elastic problems [Sellountos and Polyzos (2003)], 2-D elastostatics [Sellountos, Vavourakis and Polyzos (2005)], non-singular elasticity [Vavourakis and Polyzos (2007)] or incompressible and nearly incompressible elastostatic problems [Vavourakis and Polyzos (2008)] can be observed. Sellountos, Sequeira and Polyzos (2009) compared elastodynamic LBIE formulations with RBF approximation for both transient and steady-state Fourier transform domains. Problems involving 3-D axially symmetric FGM solids were treated in [Sladek, Sladek and Zhang (2008a)]. Recently, new and simple LBIE scheme using RBFs has been introduced for linear elasticity in [Sellountos, Poly-

zos and Atluri (2012)]. In their approach stresses are evaluated without derivatives of local RBFs via LBIE valid for stresses.

3.2 *Plates and shells*

Plate and shell structures are widely used in broad range of applications including aerospace, automotive, maritime or civil structures. There are several theories for description of behavior of plates and shells depending mainly on the thickness of the structure. Classical thin plate Kirchhoff-Love theory assumes certain non-physical simplifications related mainly to the omission of the shear deformation and rotary inertia, which become pronounced for increased plate thickness. The effects of rotary inertia and shear deformation are taken into account in the first order shear deformation theory (FSDT) also known as Reissner-Mindlin plate theory [Mindlin (1951)] and some higher order theories [Reddy (1997)].

Meshless methods are widely applied for the plate and shell analyses. Several review articles can be found focusing mainly on the element-free Galerkin method and reproducing kernel particle method [Liu et al. (1996); Li and Liu (2002); Liew, Zhao and Ferreira (2011)]. Sladek, Sladek and Mang (2002) used LBIE formulations for analysis of thin Kirchhoff plate. They overcome the high-order derivatives by decomposing the original 4th order governing partial differential equation (PDE) into two PDEs of the second order. The strong formulation for solution of general thin plate bending problems has been developed by Sladek, Sladek and Sator (2013) utilizing the combination of the decomposition technique with meshless approximations for field variables. Long and Atluri (2002) first used MLPG method with MLS approximation for solving the thin plate bending problem described by a standard Kirchhoff formulation of plate equation. Gu and Liu (2001b) applied MLPG formulation for static and free vibration analysis of thin plates. Application of Reissner-Mindlin theory for MLPG analysis of moderately thick plates under dynamic load was performed in [Sladek et al. (2007a); Sladek et al. (2007b)]. Original 3-D thick plate problem is reduced to a 2-D problem. This approach was subsequently developed also for viscoelastic plates [Sladek, Sladek and Zhang (2008b)] and plates under thermal load [Sladek et al. (2008a)]. Sladek et al., (2010a) used the MLPG method to solve laminated plates described by the Reissner-Mindlin theory. They obtained expressions for bending moment and shear force by integration through the laminated plate thickness for considered constitutive equations in each lamina. Geometric nonlinear MLPG analysis was presented by Baltacioglu and Civalek (2010) for anisotropic composite plates resting on the nonlinear two-parameter foundation. Von Karman equation has been applied for derivation of governing equation of bending of thick rectangular plate. Wen and Aliabadi (2012) applied local integral equation method for FGM plates described

by Reissner theory. In their approach they used analytical approach for closed form evaluation of domain and boundary integrals.

Higher order plate theory has been also successfully applied for thermoelastic and dynamic analysis of FGM plates [Quian and Batra (2004); Qian, Batra and Chen (2004a), (2004b)]. 3-D analysis of thick plates using MLPG method has been performed by Soric et al. (2004). They have used two sets of nodes on upper and lower plate surfaces with linear interpolation over the thickness for the in-plane displacements and hierarchical quadratic interpolation for the transversal displacements in order to eliminate the thickness locking phenomenon. Kinematics of 3-D solid instead of the conventional plate assumptions has been also used by Li et al. (2005) for the locking-free analysis of thick plates. Mixed MLPG approach was applied by Jarak and Soric (2008) for 3-D analysis of rectangular plates. Use of mixed approach is leading to independent MLS approximations of strains and nodal displacements that also helps to eliminate shear locking effect.

A concept of 3-D solid was also adopted by Jarak, Soric and Hoster (2007) for the analysis of shell structures. Soric and Jarak (2010) used mixed meshless approach for the analysis of shell-like structures. In their approach certain strain and stress components are approximated independently, however after some manipulations the global system of equations yields only nodal displacements as unknowns. Reissner-Mindlin theory for shell analysis by the MLPG and LBIE is presented in [Sladek et al. (2006c); (2007c)]. Sladek et al. (2008b) considered orthotropic and FGM shells under transient load assuming Laplace transform technique for time domain solution. Thermal analysis of shear deformable shallow shells with FGM properties is conducted in [Sladek et al. (2008c)]. Meshless modeling of laminated shells by a higher-order theory and multiquadric RBFs was presented in [Ferreira, Roque and Jorge (2006)].

3.3 *Coupled multiphysics problems*

Some advanced materials combine superior mechanical properties, as well as incorporate inherent capability to sense and adapt their static and dynamic response. Governing equations for these materials involve several physical fields that are mutually coupled, thus change in one field induces some change also in another ones. Typical example is a piezoelectric material. In piezoelectric materials the elastic and electric fields are coupled as

$$\sigma_{ij}(\mathbf{x},t) = C_{ijkl}\varepsilon_{kl}(\mathbf{x},t) - e_{kij}E_k(\mathbf{x},t) \tag{16}$$

$$D_i(\mathbf{x},t) = e_{ikl}\varepsilon_{kl}(\mathbf{x},t) + h_{ik}E_k(\mathbf{x},t) \tag{17}$$

where D_i is a vector of electric displacements, C_{ijkl}, e_{kij} and h_{ik} represent elastic, piezoelectric and dielectric material constants, respectively. The electric field vec-

tor E_k is defined as a negative gradient of electric potential. It is clearly observed that piezoelectric constant e_{kij} couples both fields. Governing equations for general piezoelectric body under quasi-electrostatic assumption include the equation of motion for elastic displacements u_i and the first Maxwell's equation of electrostatics for the vector of electric displacements D_i.

Finite element method dominated the numerical analysis of piezoelectric materials [Benjeddou (2000)], however with the start of the new millenium the meshless methods were also applied. Ohs and Aluru (2001) used meshless point collocation method for analysis of microelectromechanical systems (MEMS), Wu, Chiu and Wang (2008a) applied RKPM for piezoelectric multilayered plates. MLPG was applied for plane piezoelectricity [Sladek et al. (2006d)] and also to thermo-piezoelectricity [Sladek et al. (2007d)] where pyroelectric coefficients has been also considered. The MLS scheme is adopted for approximation of all physical fields. Transient dynamic problems of 3-D axisymmetric piezoelectric solids with continuously non-homogeneous material properties were analyzed by Sladek et al., (2008d). Advantage of axial symmetry in reduction of original 3-D problem into 2-D one was also utilized in analysis of piezoelectric FGM circular plates under thermal and mechanical load [Sladek et al. (2013)]. Obtained results showed that gradation of thermal expansion coefficients has larger influence on resulting mechanical deflection and electric potential than the gradation of mechanical and electrical parameters. Piezoelectric materials are often used in plate-like shapes, therefore analysis of piezoelectric plates is increasingly important. Several solution approaches for piezoelectric plates and shells can be found in review article by Wu, Chiu and Wang (2008b). Piezoelectric plates are usually poled in thickness (vertical) direction, thus 3-D analysis should be expected. Sladek et al. (2010b) used special approach to reduce this 3-D problem to 2-D. They obtained plate equations for piezoelectric material from variational equation of electroelasticity by means of appropriate expansion of the mechanical displacements and electric potentials in powers of thickness coordinate. Computational cost is increased because more unknowns must be computed for each node. Laminated plates with piezoelectric layers based on the Reissner-Mindlin theory were also analyzed [Sladek et al. (2012a)]. Proposed technique however assumes small thickness of the piezoelectric layer which is sufficient if the layer acts as a sensor or actuator.

Magneto-electro-elasticity is also a coupled field phenomenon closely related to piezoelectricity. As the name implies, the magnetic field is also considered to be mutually coupled together with the electrical and elastic ones. Constitutive equations for magneto-electro-elastic material can be considered as an extension of eqs. (16), (17) and are specified as

$$\sigma_{ij}(\mathbf{x},t) = C_{ijkl}\varepsilon_{kl}(\mathbf{x},t) - e_{kij}E_k(\mathbf{x},t) - d_{kij}H_k(\mathbf{x},t) \tag{18}$$

$$D_i(\mathbf{x},t) = e_{ikl}\varepsilon_{kl}(\mathbf{x},t) + h_{ik}E_k(\mathbf{x},t) + \alpha_{ik}H_k(\mathbf{x},t) \tag{19}$$

$$B_i(\mathbf{x},t) = d_{ikl}\varepsilon_{kl}(\mathbf{x},t) + \alpha_{ki}E_k(\mathbf{x},t) + \gamma_{ik}H_k(\mathbf{x},t) \tag{20}$$

where B_i is magnetic induction vector, symbols d_{ikl}, α_{ik} and γ_{ik} represent piezomagnetic constants, magnetoelectric constants and magnetic permeabilities, respectively. Note that piezoelectric constant e_{ikl}, piezomagnetic d_{ikl} and magnetoelectric constant α_{ik} provide mutual coupling.

Such materials belong to group of intelligent or smart materials. Smart materials are characterized by ability of converting energy form one form into another by response to an external impulse, thus for the magneto-electro-elastic material it is conversion of mechanical energy to electric and magnetic energy and vice-versa. Transient analysis of magneto-electro-elastic 2-D problems with non-homogeneous material properties was given by Sladek et al. (2008e). MLPG was also applied for the solution of problem including layered composites made of piezoelectric and piezomagnetic layers [Sladek et al. (2012b)]. In certain sensory applications, it is desirable to have high values of the magnetoelectric coefficient α_{ik}. As shown in [Sladek et al. (2012c)] the total magnetoelectric coefficient of the structure composed of piezoelectric and piezomagnetic layers can be enhanced if optimal gradation of material properties is prescribed. Electromagnetic wave propagation can be somehow linked to magneto-electro-elasticity as the set of Maxwell equations is applied in both cases. Soares (2009a) presented MLPG modeling of electromagnetic wave propagation problems.

There are several other coupled field problems where MLPG method was successfully applied. Analysis of porous media includes coupling of solid skeleton displacements and interstitial fluid pore pressures. The solution of pore-dynamic problems by the MLPG method was modeled [Soares (2010)] considering elastic and elasto-plastic materials. Parallelization of the MLPG was performed by Bergamashi, Martinez and Pini (2009) for axisymmetric poroelastic problems. The parallel code was based on a concurrent construction of the stiffness matrix by the processors and on a parallel preconditioned iterative method for the solution of the resulting linear system, that is offering high parallel efficiency. Soares et al. (2012) solved poroelastic problems by modified MLPG formulations. They used Taylor series expansions of unknown physical fields and solved related integrals analytically [Sladek and Sladek (2010)] what they termed as a "modified methodology". Iterative procedure is proposed by Soares (2011) for uncoupling of coupled equations of poroelasticity leading to smaller and better conditioned system of equations. Linear and nonlinear models can be analysed because nonlinear relations can be carried out along the iterative steps, adding no extra computational cost to the analysis. Similar iterative approach can be applied also for coupling of

two interacting physical models. An iterative time-domain algorithm for acoustic-elastodynamic coupled analysis was applied in [Soares (2009b)] considering meshless local Petrov-Galerkin formulations. In such approach fluid and solid system is analysed independently (as an uncoupled model) and successive renewal of the variables at the common interfaces is performed, until convergence is achieved.

3.4 *Fracture analysis*

Assessment of structural parts for their ability to function well under presence of cracks in the material is important for preserving their safe operation. Numerous techniques were proposed for fracture analysis of cracked structures [Anderson (2005)] among which numerical computer analysis is currently extensively used FEM, BEM and meshless methods. With the introduction of quarter-point singular elements the FEM dominated the numerical fracture analysis. However modeling of crack growth in FEM is cumbersome, as costly remeshing is required in order to match the geometry of the moving crack. Significant mesh refinement is also required at the crack tip. Extended finite element method (X-FEM) was developed on the basis of finite element approximation enriched by special solutions based on the concept of partition of unity [Babuska and Melenk (1997)] to alleviate the aforementioned problems [Yazid, Abdelkader and Abdelmadjid (2009)]. The alternating methods were developed to introduce more accurate fracture solutions into the finite element solutions without remeshing. Various fracture solutions were introduced for the alternating approach, including the analytical solutions [Wang, Brust, Atluri (1997abc)] and the symmetric Galerkin BEM (SGBEM) [Nikshikov, Park, Atluri (2001); Han, Atluri(2002,2003a); Dong, Atluri (2012)]. A comprehensive comparison between the X-FEM and the SGBEM-FEM alternating method was reported by [Dong, Atluri(2013bc)].

Meshless methods owing to their inherent nature can easily simulate crack propagation [Li and Liu (2004); Hagihara et al. (2007)]. Kim and Atluri (2000) introduced into the MLPG the concept of primary and secondary nodes. Secondary nodes can be easily added and/or moved without change of primary nodes in places where improved accuracy of the solution is required, as in the case of crack growth.

Ching and Batra (2001) applied the MLPG method for determination of crack tip fields in linear elastostatics. They enriched the polynomial basis functions with appropriate functions to describe singular deformation fields near a crack tip and used the diffraction criterion to determine J-integrals and stress intensity factors. The authors then analyzed transient deformations in the crack and notch tip of linear elastic plate [Batra and Ching (2002)]. They observed that the variation of shear stress with distance r ahead of the notch tip exhibits a boundary layer effect, while outside of this region stresses exhibit the $1/\sqrt{r}$ singularity. Applications of MLPG

methods for dynamic fracture problems can be found in works by Gao, Liu and Liu (2006) and Liu, Long and Li (2006). Long, Liu and Li (2008) analyzed elasto-plastic fracture problems in 2-D solids with use of MLPG5 formulations. Cracks occurring on the interface of two dissimilar materials are analyzed by Sladek et al. (2009b). Ideas of structural health monitoring were considered in meshless analysis of vibrations of cracked beams by Andreaus, Batra and Porfiri (2005). Cracks were modeled as torsion springs with use of the Lagrange multipliers.

The advantages of meshless methods in modeling of the continuously non-homogeneous materials mentioned in previous sections can be also exploited for the fracture analysis. Liu, Long and Li (2008) calculated stress intensity factors for the mixed-mode problems in the isotropic FGM material. Crack analysis of 3-D axisymmetric FGM bodies was presented in [Sladek et al., (2005a)] and the approaches for the meshless analysis of cracked continuously nonhomogeneous bodies were summarized by Sladek, Sladek and Zhang (2008c)].

In recent years the researchers have paid increased attention to fracture analysis of coupled problems of piezoelectricity and magnetoelectroelasticity. Coupling of elastic, electric and/or magnetic fields leads to necessity to investigate not only stress intensity factors (SIF) but also electrical displacement intensity factors (EDIF) and/or magnetic induction intensity factors (MIIF) at the crack tip vicinity. Intensity factors for cracks in piezoelectric and magnetoelectroelastic solids are mostly evaluated from the asymptotic expansion of the physical fields in the cracktip vicinity [Garcia-Sanchez et al. (2007)], and for 2-D problem are given as

$$
\begin{pmatrix} K_{II} \\ K_I \\ K_D \\ K_B \end{pmatrix} = \sqrt{\frac{\pi}{2r}} \left[Re\,(\Pi)^{-1} \right] \begin{pmatrix} u_1 \\ u_3 \\ \psi \\ \mu \end{pmatrix}
\tag{21}
$$

where K_{II}, K_I, K_D, K_B are mode II and mode I SIFs, EDIF and MIIF, respectively, Π is matrix determined by material properties [Garcia-Sanchez et al. (2007)] and u_1, u_3, ψ, μ are displacements in directions x_1, x_3, electric and magnetic potentials, respectively.

Cracked continuously nonhomogeneous piezoelectric solids were analyzed in [Sladek et al. (2007e)]. Interface crack problems in the composite made of two dissimilar piezoelectric materials were considered by Sladek et al. (2012d). In certain cases it is impossible to obtain or measure quantities inside the cracked specimen, only on its outer boundary. Inverse fracture problems are applied in these situations. Sladek et al. (2009c) applied MLPG for the inverse problem of fracture in piezoelectric solids. In such a case no electric boundary conditions are prescribed on the crack surfaces, however boundary conditions on the outer edges are overspecified as both

potentials and surface charge densities are prescribed there.

Specification of proper boundary conditions on the crack edges in piezoelectric and piezomagnetic materials is cumbersome, because one has to model the medium inside the crack. Depending on the ratio of material properties of the medium inside the crack and cracked solid, two extreme cases can be considered, namely permeable and impermeable boundary conditions. In reality, boundary conditions on the crack faces are in between these two extreme cases, thus special techniques must be adopted to determine the actual situation in the crack. Sladek et al. (2010c) developed MLPG approach combined with iterative solution algorithm to consider energetically consistent boundary conditions on the crack faces of piezoelectric solid. Similar approach was applied also to magnetoelectroelastic solids [Sladek et al. (2012d)]. Sladek et al. (2011) applied meshless analysis to compute fracture parameters in continuously nonhomogeneous magnetoelectroelastic solids with use of interaction integral method to replace the asymptotic expansion technique. Further works in the field of magnetoelectroelastic solids include MLPG fracture analyses [Sladek et al. (2008f); Sladek and Sladek (2011)], analyses considering thermal load [Feng, Han and Li (2009); Sladek et al. (2010d)] or coupling of MLPG together with FEM for axisymmetric problems [Li, Feng and Xu (2009)].

4   Fluid flow and heat conduction

Shortcomings of the mesh-based methods mostly related to time-consuming generation of good quality mesh (mainly in 3-D problems), prevention of element distortions, cumbersome adaptive calculations or problems of moving boundaries can be observed also in the analyses of fluid flow and heat transfer problems. These difficulties can be overcome easily by meshless methods. A number of meshless methods have been applied by many researchers to numerically compute problems of heat transfer and fluid flow. RKPM method was applied to viscous compressible flow [Gunther et al. (2000)] and to 3-D heat transfer [Cheng and Liew (2012)], SPH was used for conduction modeling [Cleary and Monaghan (1999)] or solitary water wave mechanics [Lo and Shao (2002)], EFG for steady state heat conduction [Singh, Sandeep and Prakash (2002)], finite point method [Onate et al. (1996)], meshfree weak-strong (MWS) form method [Liu, Wu and Ding (2004)] for incompressible fluid flow and virtual boundary collocation boundary method (VCBM) for heat conduction in FGMs by Wang, Quin and Kang (2006) to mention just a few. Applications of the MLPG approach to heat transfer and fluid flow problems follows below.

4.1 Heat conduction

Problems associated with heat conduction arise in many kinds of engineering applications and thus have attracted much research attention. Analytical solutions are hard to obtain for complex problems that is leading to the use of numerical solutions.

First applications of MLPG approach for the heat transfer problem in anisotropic media dates back to paper by Sladek, Sladek and Atluri (2004b). Even sooner the LBIE concept was applied to thermoelasticity [Sladek, Sladek and Atluri (2001)] and to transient heat conduction in FGMs [Sladek, Sladek and Zhang (2003b); Sladek et al. (2003)]. Batra, Porfiri and Spinello (2004) proposed efficient treatment of material discontinuity in MLPG formulations of axisymmetric transient heat conduction. Solutions for heat conduction in axisymmetric FGM bodies can be found in [Sladek et al. (2007f)] and for 3-D FGM solids in [Sladek et al. (2008g)]. MLPG collocation method have been developed for 2-D heat conduction problems [Wu, Shen and Tao (2007); Wu and Tao (2008)] and compared to commercial mesh-based code FLUENT based on the finite volume method (FVM) with excellent results in favor of the MLPG.

A standard inverse heat conduction problem is characterized by aim to compute unknown temperature and heat flux at an unreachable boundary form scattered temperature measurements at reachable interior or boundary of the domain. Sladek, Sladek and Hon (2006) applied the MLPG for inverse heat conduction problems in 2-D and 3-D axisymmetric bodies. It is well known that inverse problems are in general unstable, thus singular value decomposition technique has been applied by the authors to solve the ill-conditioned linear system of algebraic equations obtained from the LIEs after application of MLS approximation. The unstabilities are often caused by a noise in temperature measurements. Ling and Atluri (2006) have analyzed the propagation of the solution-stabilities and the propagation of computed temperature errors for the inverse heat conduction problem. The MLPG solutions to inverse heat conduction problems in 3-D anisotropic FGM solids [Sladek et al. (2012e)] and inverse problems of determining the unknown heat conduction coefficients were also recently presented [Sladek et al. (2009d)].

Improvement of computational efficiency of the MLPG in heat conduction problems also attracted many researchers. Precise time step integration method has been proposed by Li, Chen and Kou (2011) for the transient heat conduction analysis. They used the three-node triangular FEM shape functions as test functions to reduce the order of integrands involved in domain integrals. Three types of LIEs for transient heat conduction in FGM and anisotropic media are presented in [Sladek et al. (2005b)]. The MLS approximation has been proposed in [Mirzaei and De-

hghan (2011)] for performing the approximations in both time and space domains, thus avoiding the use of the time difference discretization or Laplace transform method to treat the time variable. The technique was applied to continuously non-homogeneous functionally graded materials. Pini, Mazzia and Sartoretto (2008) showed that the accurate solutions of 3-D potential problems can be obtained if suitable cubature rules are identified, sparse data structures are efficiently stored and certain strategies of avoiding unnecessary integral evaluations are used. So called Direct MLPG (DMLPG) has been developed by Mazzia, Pini and Sartoretto (2012) to alleviate some "tricky" numerical integration of non-polynomial factors in weak forms. DMLPG solutions for 2-D and 3-D potential problems have been presented. A moving Kriging interpolation scheme [Lam, Wang and Hua (2004)] was employed with MLPG method for solving the partial differential equations that govern the heat flow in 2-D and 3-D spaces by Chen and Liew (2011). For the evaluation of the integrals in 3-D problems a local subdomain of polyhedral shape was considered instead of spherical one. Baradaran and Mahmoodarabadi (2010) have applied the genetic algorithm to determine the optimum parameters for radius of local subdomains and radius of support domains on the accuracy and efficiency of the MLPG solution for the 3-D heat conduction problem.
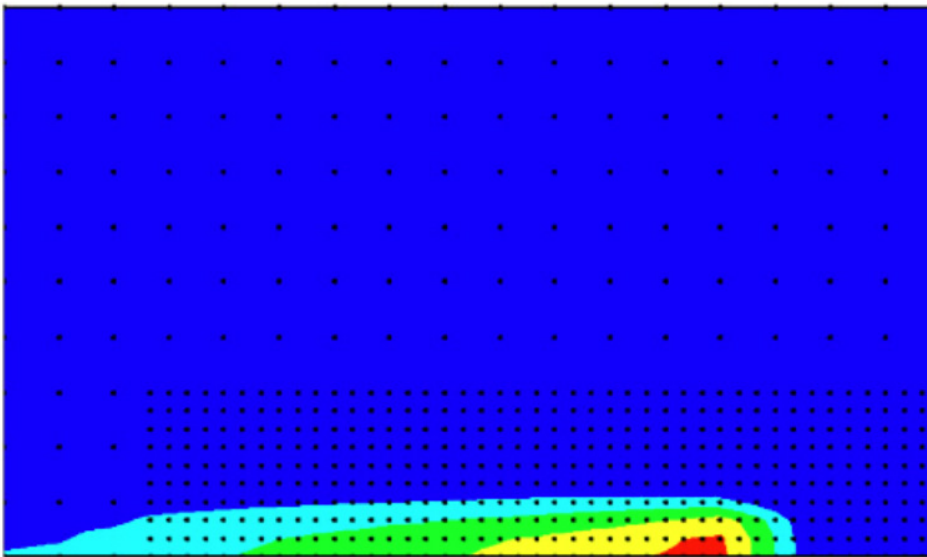

Figure 4: Transient temperature distributions 15 sec. after the welding process starts as calculated by the MLPG with adaptive nodal density. [Shibahara and Atluri (2011)].

The MLPG method was also applied to several interesting problems of heat conduction including heating of composite FGM strips by the Gaussian laser beam [Ching and Chen (2006)], radiative heat transfer [Liu (2006)] or coupled radiative and conductive heat transfer [Liu, Tan and Li (2006); Liu and Tan (2007)]. Shibahara and Atluri (2011) have recently developed the MLPG approach for the analysis of transient heat conduction due to a moving heat source that occurs in the welding process of metals. Adaptive approach has been proposed (see Fig. 4) involving the addition and elimination of nodal points which has lead to higher accuracy of the solution.

4.2 Fluid flow problems

The mesh-free or meshless methods are an extensive research area of computational fluid dynamics (CFD) problems; and steeply growing mainly in the recent few years. Two major fields of interest are analyzed, namely convection-diffusion problems and incompressible flow problems. In these problems certain numerical oscillations are present that are produced by the convection term, which produces some artificial diffusion.

First application of the MLPG method for the convection-diffusion problems has been developed by Lin and Atluri (2000) and followed by the application for the incompressible flow described by Navier-Stokes equation [Lin and Atluri (2001)]. In their pioneering works Lin and Atluri applied modified mixed formulation based on the primitive variable methodology. The small perturbation term was added to continuity equation in order to satisfy Babuška-Brezzi condition, which provides sufficient conditions for a stable mixed formulation. Two types of upwind schemes were developed to overcome existing numerical oscillations. The first upwind scheme (US1) is based on the shift of the maximum value of the test function opposite to the streamline direction, but the position of test function and the integration domain is not changed, while second upwind scheme (US2) shifts the local subdomain opposite to the streamline direction as shown in Fig. 5

Arefmanesh, Najafi and Abdi (2005) proposed so called "meshless control volume method" based on the MLPG formulation. They presented solutions for 1-D and 2-D transient heat conduction and 1-D and 2-D advection-diffusion problems. The MLPG solution of the Navier-Stokes equation for the incompressible fluid flow in terms of stream function and vorticity formulation was given in [Wu, Liu and Gu (2005)]. Arefmanesh, Najafi and Abdi (2008) considered the energy equations together with the stream-vorticity formulation to compute non-isothermal fluid flow problem and the unity was applied as weight function in their approach. Wu et al. (2010) proposed the streamline upwind Petrov-Galerkin (SUPG) scheme [Brooks and Hughes (1982)] to overcome the influence of false diffusion in the incompress-
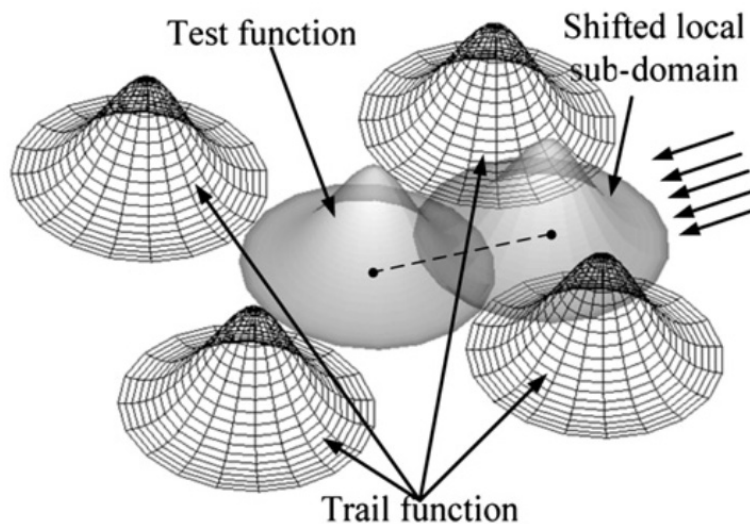
Figure 5: Shift of the local subdomain in the upwind scheme(US2) developed by Lin and Atluri (2000)

ible fluid flow analysis. The SUPG method only adds a stability term in the upwind direction, thus it is convenient to implement to the MLPG solution approach. Comparison of the convection-diffusion problem solutions stabilized by SUPG method and upwind scheme US2 developed by Lin and Atluri (2000) together with non-stabilized MLPG solution given in [Wu et al. (2012)] proved SUPG approach is robust also at high Peclet numbers, while the MLPG without any stabilization gives somewhat poor results. Mohammadi (2008) presented new type of upwind scheme to stabilize the convection operator in the streamline direction. In this up-winding technique, instead of moving subdomains, the weight function is shifted in the direction of flow. Application to incompressible fluid flow described by stream-vorticity formulation was developed using the Heaviside step function and quadratic spline as the test functions, and RBF interpolation was employed for the creation of a shape function.

In the abovementioned studies [Wu, Liu and Gu (2005); Arefmanesh, Najafi and Abdi (2008); Mohammadi (2008)] the vorticity-stream function method was applied which can satisfy the incompressible mass condition automatically. However, this method has certain limitations as it cannot be directly extended to solve 3-D and most of 2-D complex geometries. That's why primitive variable method is often considered as an alternative [Wu et al. (2010), (2012)]. Najafi, Arefmanesh and Enjilela (2012) presented MLPG solution to the incompressible fluid flow in terms of

primary variables using the characteristic-based split (CBS) scheme for discretization. The investigators were able to obtain stable results for higher Reynolds numbers fluid flow applications compared to results by Lin and Atluri (2001), however still considering the laminar flow region.

The MLPG method was applied for the laminar incompressible fluid flow problems in 2-D domains characterized by non-steady fluid motion around flexible boundaries with harmonic, undulatory or contraction-expansion movements [Avila and Atluri (2009)]. A fully implicit pressure correction approach, which requires at each time step an iterative process to solve the equations which govern the flow field, and the equations that model the corrections of pressure and velocities, has been used. Avila, Han and Atluri (2011) developed a novel MLPG-Finite-Volume mixed method for analyzing steady state Stokesian flows, that is based on the independent meshless interpolations of the deviatoric velocity strain tensor, the volumetric velocity strain tensor, the velocity vector and the pressure. Loukopoulos and Bourantas (2012) presented MLPG6 approach for the solution of the Navier-Stokes and energy equations. Note that in the MLPG6 method, the test function is chosen to be the same as the trial function, thus leading to Galerkin formulation.

Water wave problems are special class of fluid flow problems as free surface of the fluid must be considered. Mesh-based solutions require intense remeshing as elements can frequently become over-distorted during the simulation of water wave evolutions. Ma (2005) first introduced MLPG method for 2-D nonlinear water wave problems. In his formulation a time marching scheme is applied that at each time step solves the boundary value problem for the pressure by the MLPG, and the velocity and nodal positions are updated by numerical integration. Ma and Zhou (2009) have used the MLPG method based on the Rankine source solution (MLPG_R) for the analysis of 2-D breaking waves. For the indentification of the free surface particles a new technique called Mixed Particle Number Density and Auxiliary Function Method (MPAM) has been suggested. MLPG_R technique has been further developed also for the 3-D breaking waves [Zhou and Ma (2010)].

Nanofluid flow in a complex geometry cavity has been studied by Arefmanesh, Najafi and Nikfar (2010). The governing equations for the nanofluid flow have been determined in terms of the stream-vorticity formulation and a parametric study has been conducted to observe the nanofluid convective heat transfer performance. The formulation has been then extended to the analysis of the natural convection of Al_2O_3–water nanofluid in a cavity with wavy side walls given by Nikfar and Mahmoodi (2012).

The scientific study of the flow of conducting fluids in the presence of transverse magnetic fields has attracted attention owing to its applications in such diversified fields as astrophysics, geology, power generation, flow-metry or design of ther-

monuclear reactors. The unsteady magnetohydrodynamic (MHD) flow through pipe has been analyzed using the MLPG by Dehghan and Mirzaei (2009). Maxwell equations have been considered together with fluid flow equations to determine the induced magnetic field accross the various sections of the pipes with various wall conductivities

5 Advanced numerical techniques for the improvements of the MLPG

Since its introduction to scientific community, MLPG method has proved to be an efficient and accurate numerical technique for solution of broad range of problems. However, there are still great possibilities for improvements of the MLPG formulations in order to increase the robustness, accuracy and stability, improve convergence rate or decrease the computational time. Several numerical techniques related to these issues are presented next.

Kim and Atluri (2000) were among the first ones, who analysed and controlled the errors of the MLPG method together with MLS approximation. They introduced the concept of primary and secondary nodes that can be applied at any location where the solution quality needs to be improved, leading to the adaptive calculation technique.

It is a well-known fact that the accuracy of the numerical approximation decreases with increased order of derivative to be approximated. Atluri, Han and Atluri (2004) proposed MLPG "Mixed" approach for solving elasto-static problems in which both strains and displacements are interpolated separately through MLS scheme. Strain-displacement relationships are then enforced directly by collocation at nodal points. The proposed approach leads to a meshless finite-volume method and the expensive process of differentiation of the MLS interpolation for displacements is eliminated. Similar mixed scheme has been proposed for elasticity problems with independent interpolations of nodal displacements and stresses [Atluri, Liu and Han (2006a)]. Atluri, Liu and Han (2006b) developed the mixed MLPG approach combined with the finite difference method (FDM). FDM was used for the evaluation of derivatives of stresses using the scattered nodal values in the local domain of definition. Comparison of various primal and mixed MLPG approaches for the 4th order ordinary differential equation can be found in [Atluri and Shen (2005)]. The mixed MLPG methods avoid the need for a direct evaluation of high order derivatives of the primary variables in the local weak forms, and thus reduce the continuity requirement on the trial function.

Various meshless interpolation schemes can be applied for the approximation of trial functions in the MLPG method and each of them can have significant influence on the quality of obtained results. A critical assessment of the MLPG and LBIE by

Atluri, Kim and Cho (1999) investigated characteristics of MLS method, Shepard function and partition of unity, and pointed out their advantages and disadvantages. MLS method have been used probably most extensively within the MLPG. The MLS scheme is strongly influenced by the support radius of weight function applied for the creation of the MLS shape function. Nie, Atluri and Zuo (2006) have presented technique for the specification of the optimal value of the radius of the support domain. Good insight to the behavior of various meshless approximation techniques based on the RBFs can be found in the paper by Sladek, Sladek and Zhang (2006). Meshless polynomial interpolations combined with multiquadric RBFs are compared in [Sladek, Sladek and Tanaka 2005] for potential problems in non-homogeneous media. For higher accuracy of polynomial-based interpolations higher-order polynomials as interpolants are generally used. Numerical interpolation by high-order polynomials and treatment of ill-posed problems that may arise has been presented by Liu and Atluri (2009). Numerical stability, accuracy and cost effectivity of MLS – central approximation node (MLS-CAN) approximation has been investigated by Sladek, Sladek and Zhang (2008d) for the problems involving non-homogeneous elastic solids.

Sellountos, Vavourakis and Polyzos (2005) have proposed meshless method based on the LBIE approach called singular/hypersingular MLPG (LBIE) to avoid the derivatives of the MLS shape functions and thus treat displacement and tractions as independent variables in elastostatics. The representation of the displacement field at the internal points is accomplished with the aid of the displacement local boundary integral equation, while for the boundary nodes both the displacement and the corresponding traction local boundary integral equations are employed. Numerical integration of singularities appearing in LBIEs has been investigated in [Sladek et al. (2000)]. The authors recomend to recast the singular integrands into smooth functions, which can be integrated by standard quadratures of the numerical integration with sufficient accuracy. Vavourakis, Sellountos and Polyzos (2006) have provided detailed comparison study of five different MLPG (LBIE) formulations and concluded that derivatives of shape functions decrease solution accuracy and uniform distribution of nodes provides best results. The effect of nodal distribution on the accuracy of MLPG formulations was also presented in [Augarde and Deeks (2005)]. Han and Atluri (2007) have used the regularization technique to avoid the hypersingularities in the BIEs by the systematic decomposition of the kernel functions of BIEs.

Prolonged numerical evaluation of integrals in LIEs is often considered as a disadvantage of meshless methods. Use of analytical integration instead of Gauss-Legendre or other numerical quadrature schemes allows for significant reduction of CPU times needed for the creation of the system matrix. Sladek and Sladek (2010)

have reduced the emount of evaluations of the shape functions and their derivatives to nodal points instead of the integration points by introducing the analytical integrations in LIEs implemented by meshless approximations for field variables. Furthermore, they have developed a modified differentiation scheme for approximation of higher order derivatives of displacements appearing in the discretized formulations. Applications of this approach for heat conduction problem have been also proposed [Sladek, Sladek and Zhang (2010), (2011)]. Soares, Sladek and Sladek (2012) have extended the idea of analytical integrations to the solution of elastodynamic problems by the MLPG. So called „modified methodology" has been proposed involving analytical evaluation of integrals and Taylor series expansion of unknown field variables. Such MLPG formulation has lead to better computational efficiency, especially for large scale problems. Very recently, Wen and Aliabadi (2013) have derived exact forms of integrals in the meshless LBIE method for elastostatic problems. A completed set of closed forms of the local boundary integrals with RBFs has been obtained. Significant reduction of computational time has been reported.

Certain computational time savings can be obtained if proper solver is applied for the solution of the system of LIEs. Yuan, Chen and Liu (2007) have proposed new system solver for the direct solution method of quasi-unsymmetric sparse matrix (QUSM) which is arising in the MLPG. They utilized the fact that QUSM is unsymmetric in its numerical values, but nearly symmetric in its nonzero distribution of upper and lower triangular portions. In order to efficiently treat a large system of non-linear algebraic equations (NAE), Liu and Atluri (2008) have developed a new iterative time integration method. Fictituous time integration has been considered in order to derive a natural system of explicit ordinary differential equations from the given system of NAEs.

A detailed convergence study of the MLPG1 method (MLS as a weight function) was performed for the diffusion equation by Sterk and Trobec (2008) to optimize the number of support nodes, quadrature domain size and other parameters. Abbasbandy and Shirzadi (2011) have presented a new treatment of non-classical boundary conditions for the MLPG solution of diffusion equation based on the finite differences and the MLS approximations. Ferronato, Mazzia and Pini (2010) have suggested a new technique for the improvement of solution accuracy of finite element mesh. In their approach a limited number of moving MLPG nodes is added over a coarse mesh in order to increase the accuracy in specific regions of the solution domain without remeshing or mesh refinement.

With massive use of multiprocessor computers also the parallel computation techniques become more available for the researchers. Trobec, Sterk and Robic (2009) have presented approach for parallelization of the MLPG1 and analyzed computa-

tional complexity of the MLPG1 with respect to FEM and finite difference method (FDM). They have showed that MLPG1 remains competitive for larger scale problems and significant speed up can be obtained by parallelisation as shown in Fig. 6.
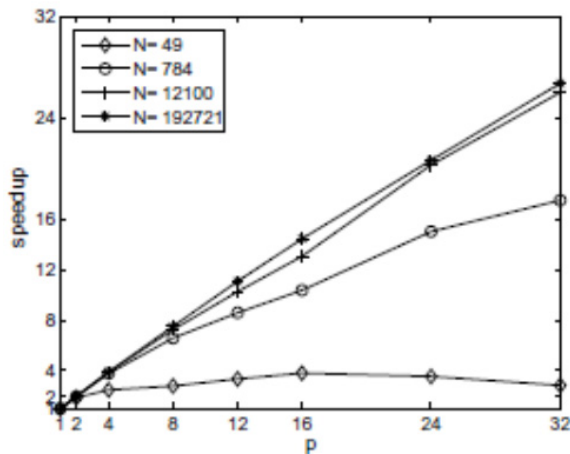

Figure 6: Measured speed up of the parallel MLPG1 with optimal parameters as a function of the number of processors *p*. [Trobec et al., 2009].

6   Special applications & further research perspectives

Except of the traditional problems of solid and fluid mechanics, the MLPG method is capable to efficiently analyze also some special problems that appeared mostly in the recent past. These new problems can also point out to several new directions in the development of MLPG family of methods.

Elimination of the domain mesh in meshless method enables efficient simulation of nonlinear and multiscale problems. Shen and Atluri (2004a) applied the MLPG method for the multiscale simulations of the interactions between atomistic and continuum regions. They introduced several alternate time-dependent interfacial conditions by decomposing the displacement of atoms into long and short-wave components. Tangent stiffness formulation for the MLPG multiscale analysis has been given in [Shen and Atluri (2005)]. More information on the multiscale simulation, nanotechnology and micro-mechanics of materials can be found in review paper by Shen and Atluri (2004b) and monograph by Atluri (2004). The MLPG approach for the higher order gradient theories has been introduced in [Tang, Shen and Atluri (2003b)]. The two-dimensional Toupin-Mindlin strain gradient theory

applied by the researchers is from a mathematical point of view a generalization of the Poisson-Kirchhoff plate theories, involving, in addition to the fourth-order derivatives of the displacements, also a second-order derivative. Treatment of discontinuity between two and different material regions in meshless methods becomes also important field of study [Matsuda and Noguchi (2006); Wang, Sun and Li (2009)]. Nematic liquid crystals are the key component of the modern liquid crystal displays (LCD). The behavior of nematic liquid crystals involving topological defects can be described by the Q-tensor based model. The MLPG solution of Q-tensor equations by Pecher, Elston and Raynes (2006) have shown a high degree of continuity and high accuracy of the meshfree approach used. The MLPG solution of magnetic diffusion in non-magnetic conductors has been developed by Johnson and Owen (2007). Another possible new research direction of the MLPG method has been pointed out by Dehghan and Mirzaei (2008) who have performed the MLPG solution of the generalized 2-D non-linear Schrodinger equation.

In the theory of self-organization of biological systems one can observe the coupled pair of nonlinear reaction-diffusion equations responsible for appearance of diffusion driven instabilities. Such cases may also appear in the mixing of two chemical agents under specific conditions. Abbasbandy et al. (2011) applied MLPG method for solution of these equations and Shirzadi, Sladek and Sladek (2013) proposed LBIE solution with the test function in the form of modified fundamental solution of the Laplace operator.

Interesting application of the MLPG method in cloth simulation has been presented by Yuan et al. (2008). The micro-mechanical material model of woven fabric composite material has been proposed in [Wen and Alliabadi (2010)]. The material models considered in the paper are based on a repeated unit cell approach and two smooth fibre modes. Elastic moduli of such composite have been determined numerically.

Meshless methods are well suited also for the shape optimization, since nodal points can be added or eliminated easily. Structural topology-optimization based on the MLPG mixed collocation method has been applied in the paper by Li and Atluri (2008a,b). Problem of compliance minimization of elastic structures has been pursued. Design sensitivity analysis (DSA) and topology optimization by meshless natural neighbour Petrov-Galerkin (NNPG) method has been successfully used by Wang et al. (2008). The NNPG is considered as a special case of generalized MLPG method.

Regarding the analyzed problems presented in this section and previous sections as well, the possible future research directions of the MLPG method can be outlined as:

- improvment of the unified and somewhat universal theory of the approximation properties and stability of the MLPG for various meshless interpolation schemes to reduce CPU

- crack propagation and evaluation of time dependent fracture parameters with use of adaptive calculations

- large –scale stabilized fluid flow problems in 3-D domains

- analysis of new types of materials such as carbon nanotubes or quasicrystals [Shechtman et al., 1984]

- biomechanical analysis – investigation of mechanical behavior of living tissue

- analysis of problems outside of the field of classical mechanics, such as quantum mechanics, chemistry or biology

- coupling of the MLPG with other numerical methods such as FEM, BEM in order to exploit the benefits of each formulation

- increase of the computational effectiveness through development of new system solvers or parallelization of computation for multiprocessor and cloud systems

- possible development of the MLPG code suitable for commercial use

The research in the field of meshless methods still advances and new computational methods are being developed. Several truly meshless methods have recently received increased attention such as point interpolation method (PIM) [Liu and Gu (2001)], finite cloud method [Aluru and Li (2001)], method of fundamental solutions (MFS) [Marin (2008); Tsai (2011)] or meshless Trefftz based methods [Liu (2008a),(2008b); Liu, Yeih and Atluri (2009); Dong, Atluri (2012abd,2013a); Bishay, Atluri(2012,2013); Bishay, Sladek, Sladek, Atluri(2012)]. However, MLPG still remains competitive and has a good perspective in the future.

7   Conclusion

In this paper we have tried to give an overview of the principles and application of the meshless local Petrov-Galerkin (MLPG) method. Properties and advantages of the MLPG are discussed and compared to standard mesh-based methods. MLPG is a truly meshless method thus it involves not only a meshless interpolation of

trial functions but also a meshless integration of the local weak form and as a consequence no background mesh or elements are required. The MLPG approach provides the flexibility in choosing the trial and test functions as well as the size and shape of local subdomains. Therefore, MLPG is characterized as more flexible and capable to handle in easier way the problems from which the conventional mesh-based methods suffer.

Extensive literature review related to broad range of topics is presented. Significant research papers from the very recent past and others published since 1998 when the MLPG has been first introduced [Atluri and Zhu (1998a)] are included. Broad range of research areas in the field of computational mechanics are included and advantages of MLPG in each field are emphasized. Absence of the finite element mesh has prooved to be convenient for the analysis of continuously non-homogeneous solids. Benefits of higher order continuity are conveniently utilized when solving fracture mechanics problems, since a smoother stress distribution around the crack tip can be obtained. Flexibility and simple implementation are suitable for various coupled problems including piezoelectricity or magnetoelectroelasticity. Elimination of the shear locking in the analysis of thin waled structures, plates and shells gives good promise for the future applications. Handling of large deformations is simplified since the distortion of nodal points positions have a smaller influence on accuracy compared to finite elements. Adaptive calculation and shape optimization is also carried out more naturally. After elimination of artificial numerical oscillations the advantages of meshless concept can be fully exploited also in case of fluid flow problems. Application of sophisticated techniques for analytical integration of integrals in the weak form offers fast computation of large-scale problems.

Increased number of journals, researchers and scholars give good promise for the future that the MLPG will receive adequate attention also among the community of university students and design engineers such as strongly established finite element method.

Acknowledgement

References

Abbasbandy, S.; Shirzadi, A. (2011): MLPG method for two-dimensional diffusion equation with Neumann's and non-classical boundary conditions. *Applied*

Numerical Mathematics, vol. 61, No. 2, pp. 170-180.

Abbasbandy, S.; Sladek, V.; Shirzadi, A.; Sladek, J. (2011): Numerical simulations for coupled pair of diffusion equations by MLPG method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 71, No. 1, pp. 15-37.

Akbari, A.; Bagri, A.; Bordas, S. P. A.; Rabczuk, T. (2010): Analysis of Thermoelastic Waves in a Two-Dimensional Functionally Graded Materials Domain by the Meshless Local Petrov-Galerkin (MLPG) Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 65, No. 1, pp. 27-74.

Aluru, N. R.; Li, G. (2001): Finite cloud method: a true meshless technique based on reproducing kernel approximation. *Int. J. Numer. Methods Eng.*, vol. 50, pp. 2373-2410.

Anderson T. L. (2005): *Fracture Mechanics – Fundamentals and Applications*, 3rd ed.,CRC Press, Boca Raton.

Andreaus, U.; Batra, R. C.; Porfiri, M. (2005): Vibrations of Cracked Euler-Bernoulli Beams using Meshless Local Petrov-Galerkin (MLPG) Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 9, No. 2, pp. 111-131.

Arefmanesh, A.; Najafi, M.; Abdi, H. (2005): A meshless local Petrov-Galerkin method for fluid dynamics and heat transfer applications. *Journal of Fluids Engineering*, vol. 127, pp. 647-655.

Arefmanesh, A.; Najafi, M.; Abdi, H. (2008): Meshless local Petrov-Galerkin method with unity test function for non-isothermal fluid flow. *CMES: Computer Modeling in Engineering & Sciences*, vol. 25, No. 1, pp. 9-22.

Arefmanesh, A.; Najafi, M.; Nikfar, M. (2010): MLPG Application of Nanofluid Flow Mixed Convection Heat Transfer in a Wavy Wall Cavity. *CMES: Computer Modeling in Engineering & Sciences*, vol. 69, No. 2, pp. 91-118.

Atluri, S. N. (2004): *The Meshless Method (MLPG) for Domain and BIE Discretizations*, Tech Science Press, Forsyth.

Atluri, S. N.; Cho, J. Y.; Kim, H. G. (1999): Analysis of the beams, using the meshless local Petrov-Galerkin method, with generalized moving least squares interpolations, *Comput. Mech.,* vol. 24, pp. 334-347.

Atluri, S. N.; Han, Z. D.; Shen, S. (2003): Meshless Local Patrov-Galerkin (MLPG) approaches for weakly-singular traction & displacement boundary integral equations, CMES: *Computer Modeling in Engineering & Sciences*, vol. 4, no. 5, pp. 507-517.

Atluri, S. N.; Han, Z. D.; Rajendran, A. M. (2004): A New Implementation of the Meshless Volume Method, Through the MLPG "Mixed" Approach. *CMES: Computer Modeling in Engineering & Sciences*, vol. 6, No. 6, pp. 491-513.

Atluri, S. N.; Kim, H. G.; Cho, J. Y. (1999): A critical assessment of the truly meshless local Petrov-Galerkin (MLPG), and local boundary integral equation (LBIE) methods. *Comput. Mech.,* vol. 24, pp. 348-372.

Atluri, S. N.; Liu, H. T.; Han, Z. D. (2006a): Meshless local Petrov-Galerkin (MLPG) mixed collocation method for elasticity problems, CMES: *Computer Modeling in Engineering & Sciences*, vol. 14, no. 3, pp. 141-152.

Atluri, S. N.; Liu, H. T.; Han, Z. D. (2006b): Meshless Local Petrov-Galerkin (MLPG) Mixed Finite Difference Method for Solid Mechanics. *CMES: Computer Modeling in Engineering & Sciences*, vol. 15, No. 1, pp. 1-16.

Atluri, S. N.; Shen, S. (2002a): *The Meshless Local Petrov-Galerkin (MLPG) Method*, Tech Science Press, Forsyth, GA.

Atluri, S. N.; Shen, S. (2002b): The Meshless Local Petrov-Galerkin (MLPG) Method: A Simple & Less-costly Alternative to the Finite Element and Boundary Element Methods. *CMES: Computer Modeling in Engineering & Sciences*, vol. 3, No. 1, pp. 11-51.

Atluri, S. N.; Shen, S. (2005): Simulation of a 4th Order ODE: Illustrations of Various Primal & Mixed MLPG Methods. *CMES: Computer Modeling in Engineering & Sciences*, vol. 7, No. 3, pp. 241-268.

Atluri, S. N.; Sladek, J. (Eds.) (2009): *Advances in the MLPG Meshless Methods*. Tech. Sci. Press, Duluth.

Atluri, S. N.; Sladek, J.; Sladek, V.; Zhu, T. (2000): The local boundary integral equation (LBIE) and its meshless implementation for linear elasticity. *Comput. Mech.*, vol. 25, pp. 180-198.

Atluri, S. N.; Zhu, T. (1998a): "A New Meshless Local Petrov-Galerkin (MLPG) Approach in Computational Mechanics*", Computational Mechanics*, vol. 22, pp. 117-127.

Atluri, S. N.; Zhu, T. (1998b): A new meshless local Petrov-Galerkin (MLPG) approach to nonlinear problems in computer modeling and simulation. *Comput. Model. Simul. Eng.,* vol. 3, pp. 187-196.

Atluri, S. N.; Zhu, T. (2000): The meshless local Petrov-Galerkin (MLPG) approach for solving problems in elasto-statics. *Computational Mechanics* , vol. 25, pp. 169-179.

Augarde, C. E.; Deeks, A. J. (2005): On the effects of nodal distributions for imposition of essential boundary conditions in the MLPG meshfree method. *Communications in Numerical Methods in Engineering*, vol. 21, pp. 389–395.

Avila, R.; Atluri, S. N. (2009): Numerical Solution of Non-steady Flows, Around Surfaces in Spatially and Temporally Arbitrary Motions, by using the MLPG method.

CMES: Computer Modeling in Engineering & Sciences, vol. 54, No. 1, pp. 15-64.

Avila, R.; Han, Z.; Atluri, S. N. (2011): A novel MLPG-Finite-Volume Mixed Method for Analyzing Stokesian Flows & Study of a New Vortex Mixing Flow. *CMES: Computer Modeling in Engineering & Sciences*, vol. 71, No. 4, pp. 363-396.

Babuska, I.; Melenk, J. M. (1997): The partition of unity method. *Int. J. Num. Meth. Engrg,.* vol. 40, pp. 727-758.

Balas, J.; Sladek, J.; Sladek, V. (1989): *Stress analysis by boundary element method.* Elsevier, Amsterdam.

Baltacioglu, A. K.; Civalek, O. (2010): Geometrically Nonlinear Analysis of Anisotropic Composite Plate Resting on Nonlinear Elastic Foundations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 68, No. 1, pp. 1-23.

Banerjee, P. K. (1994): *The Boundary Element Methods in Engineering.* Mc Graw-Hill, London.

Baradaran, G. H.; Mahmoodarabadi, M. J. (2010): Parametric study of the MLPG method for the analysis of three-dimensional steady state heat conduction problems. *Journal of Mechanical Engineering*, vol. 61, pp. 31-61.

Batra, R, C.; Ching, H. K. (2002): Analysis of Elastodynamic Deformations near a Crack/Notch Tip by the Meshless Local Petrov-Galerkin (MLPG) Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 3, No. 6, pp. 717-730.

Batra, R. C.; Porfiri, M.; Spinello, D. (2004): Treatment of material discontinuity in two mehless local Petrov-Galerkin (MLPG) formulations of axisymmetric transient heat conduction. *Int J Numer Methods Eng*, vol. 61, pp. 2461-2479.

Belytschko, T.; Krogauz, Y.; Organ, D.; Fleming, M.; Krysl, P. (1996): Meshless methods; an overview and recent developments. Comp. Meth. Appl. Mech. Engn., vol. 139, pp. 3-47.

Belytschko, T.; Lu, Y. Y.; Gu, L. (1994): Element-free Galerkin methods. *Int. J. Numer. Methods. Eng.,* vol. 37, pp. 229-256.

Benjeddou, A. (2000): Advances in piezoelectric finite element modeling of adaptive structural elements: a survey. *Computers and Structures*, vol. 76, pp. 347-363.

Bergamaschi, L.; Martinez, A.; Pini, G. (2009): An Efficient Parallel MLPG Method for Poroelastic Models. *CMES: Computer Modeling in Engineering & Sciences*, vol. 49, No. 3, pp. 191-215.

Bishay P. L.; Atluri, S. N. (2012): High-Performance 3D Hybrid/Mixed, and Simple 3D Voronoi Cell Finite Elements, for Macro-& Micro-mechanical Modeling of Solids, Without Using Multi-field Variational Principles, *CMES: Computer Modeling in Engineering and Sciences*, Volume: 84, Issue: 1, pp. 41-97.

Bishay P. L.; Atluri, S. N. (2013): 2D and 3D Multiphysics Voronoi Cells, Based on Radial Basis Functions, for Direct Mesoscale Numerical Simulation (DMNS) of the Switching Phenomena in Ferroelectric Polycrystalline Materials, *CMC: Computers Materials and Continua*, Volume: 33, Issue: 1, pp. 19-62.

Bishay P. L.; Sladek, J.; Sladek, V.; Atluri, S. N. (2012): Analysis of Functionally Graded Magneto-Electro-Elastic Composites Using Hybrid/Mixed Finite Elements and Node-Wise Material Properties, *CMC: Computers, Materials & Continua*, Volume: 29, Issue: 3, pp. 213-262.

Brebbia, C. A.; Telles, J. C. F.; Wrobel, L. C. (1984): *Boundary Element Techniques*. Springer, Berlin.

Brooks, A. N.; Hughes, T. J. R. (1982): Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Eng.*, vol. 32, pp. 199-259.

Chen, L.; Liew, K. M. (2011): A local Petrov-Galerkin approach with moving Kriging interpolation for solving transient heat conduction problems. *Computational Mechanics*, vol. 47, No. 4, pp. 455-467.

Chen, W. (2000): New RBF collocation methods and kernel RBF with applications *Meshfree methods for partial differential equations*. Springer Verlag, pp. 75-86.

Cheng, R. J.; Liew, K. M. (2012): A meshless analysis of three-dimensional transient heat conduction problems. *Eng. Anal. Bound. Elem.*, vol. 36, pp. 203-210.

Ching H. K.; Batra R. C. (2001): Determination of Crack Tip Fields in Linear Elastostatics by the Meshless Local Petrov-Galerkin (MLPG) Method. *Computer Modeling in Engineering & Sciences*, Vol. 2(2), pp. 273-289.

Ching, H. K.; Chen, J. K. (2006): Thermomechanical Analysis of Functionally Graded Composites under Laser Heating by the MLPG Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, No. 3, pp. 199-217.

Cleary, P. W.; Monaghan, J. J. (1999): Conduction modeling using smoothed particle hydrodynamics. *J. Comput. Phys.*, vol. 148, pp. 227-264.

Dang, T. B.; Sankar, B. V. (2008): Meshless Local Petrov-Galerkin Micromechanical Analysis of Periodic Composites Including Shear Loading. *CMES: Computer Modeling in Engineering & Sciences*, vol. 26, No. 3, pp. 169-187.

Dehghan, M.; Mirzaei, D. (2008): The meshless local Petrov-Galerkin (MLPG) method for the generalized two-dimensional non-linear Schrodinger equation. *Eng. Anal. Bound. Elem.,* vol. 32, pp. 747-756.

Dehghan, M.; Mirzaei, D. (2009): Meshless Local Petrov-Galerkin (MLPG) method for the unsteady magnetohydrodynamic (MHD) flow through pipe with arbitrary

wall conductivity. *Applied Numerical Mathematics,* vol. 59, pp. 1043-1058.

Dominguez, J. (1993): *Boundary Elements in Dynamics*. CMP, Southampton, Elsevier Applied Science, London.

Dong, L.; Atluri, S. N. (2012a): T-Trefftz Voronoi Cell Finite Elements with eastic/rigid inclusions or voids for micromechanical analysis of composite and porous materials, *CMES: Computer Modeling in Engineering and Sciences*, Volume: 83, Issue: 2, pp. 183-220.

Dong, L.; Atluri, S. N. (2012b): A Simple Multi-Source-Point Trefftz Method for Solving Direct/Inverse SHM Problems of Plane Elasticity in Arbitrary Multiply-Connected Domains, *CMES: Computer Modeling in Engineering and Sciences*, Volume: 85, Issue: 1, pp. 1-43.

Dong, L.; Atluri, S. N. (2012c): SGBEM (Using Non-hyper-singular Traction BIE), and Super Elements, for Non-Collinear Fatigue-growth Analyses of Cracks in Stiffened Panels with Composite-Patch Repairs. CMES: *Computer Modeling in Engineering & Sciences*, vol.89, no. 5, pp. 417-458.

Dong, L.; Atluri, S. N. (2012d): Development of 3D T-Trefftz Voronoi Cell Finite Elements with/without Spherical Voids &/or Elastic/Rigid Inclusions for Micromechanical Modeling of Heterogeneous Materials, *CMC: Computers Materials and Continua*, Volume: 29, Issue: 2, pp. 169-211.

Dong, L.; Atluri, S. N. (2012e): Development of 3D Trefftz Voronoi Cells with Ellipsoidal Voids and/or Elastic/Rigid Inclusions for Micromechanical Modeling of Heterogeneous Materials, *CMC: Computers Materials and Continua*, Volume: 30, Issue: 1, pp. 39-81.

Dong, L.; Atluri, S. N. (2013a): SGBEM Voronoi Cells (SVCs), with Embedded Arbitrary-Shaped Inclusions, Voids, and/or Cracks, for Micromechanical Modeling of Heterogeneous Materials. CMC: *Computers, Materials & Continua*, Volume: 33, Issue: 2, pp. 111-154.

Dong, L.; Atluri, S. N. (2013b): Fracture & Fatigue Analyses: SGBEM-FEM or XFEM? Part 1: 2D Structures, *CMES: Computer Modeling in Engineering & Sciences*, Volume: 90, Issue: 2, pp. 91-146.

Dong, L.; Atluri, S. N. (2013c): Fracture & Fatigue Analyses: SGBEM-FEM or XFEM? Part 2: 3D Solids, *CMES: Computer Modeling in Engineering & Sciences*, Volume: 90, Issue: 5, pp. 379-413.

Feng, W. J.; Han, X.; Li, Y. S. (2009): Fracture Analysis for Two-dimensional Plane Problems of Nonhomogeneous Magneto-electro-thermo-elastic Plates Subjected to Thermal Shock by Using the Meshless Local Petrov-Galerkin Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 48, No. 1, pp. 1-26.

Ferreira, A. J. M.; Roque, C. M. C.; Jorge, R. M. N. (2006): Modelling cross-ply laminated elastic shells by a higher-order theory and multiquadrics. *Computers and Structures,* vol. 84, pp. 1288-1299.

Ferronato, M.; Mazzia, A.; Pini, G. (2010): A Finite Element enrichment technique by the Meshless Local Petrov-Galerkin method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 62, No. 2, pp. 205-224.

Gao, L.; Liu, K.; Liu, Y. (2006): Applications of MLPG Method in Dynamic Fracture Problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 12, No. 3, pp. 181-195.

Garcia-Sanchez, F.; Rojas-Diaz, R.; Saez, A.; Zhang, Ch. (2007): Fracture of magnetoelectroelastic composite materials using boundary element method (BEM). *Theoretical and Applied Fracture Mechanics,* vol. 47, pp. 192-204.

Gingold, R. A.; Monaghan, J. J. (1977): Smoothed particle hydrodynamics: theory and application to non-spherical stars. *Mon. Not. Roy. Astron. Soc.,* vol. 181, pp. 375-389.

Gu Y. T.; Liu G. R. (2001a): A Meshless Local Petrov-Galerkin (MLPG) Method for Free and Forced Vibration Analyses for Solids. *Computational Mechanics*, vol. 27, pp. 188-198.

Gu, Y. T.; Liu, G. R. (2001b): A meshless local Petrov-Galerkin (MLPG) formulation for static and free vibration analyses of thin plates. *Computer Model. Engng. Sci.,* vol. 2, pp. 463-476.

Gunther, F.; Liu, W. K.; Diachin, D.; Christon, M. A. (2000): Multi-scale meshfree parallel computations for viscous, compressible flows. *Comput. Methods Appl. Mech. Eng.* , vol. 190, No. (3–4), pp.279–303.

Hagihara, S.; Tsunori, M.; Ikeda, T.; Miyazaki, N. (2007): Application of Meshfree Method to Elastic-Plastic Fracture Mechanics Parameter Analysis. *CMES: Computer Modeling in Engineering & Sciences*, vol. 17, No. 2, pp. 63-72.

Han, Z. D.; Atluri, S. N. (2002): SGBEM (for Cracked Local Subdomain) – FEM (for uncracked global Structure) Alternating Method for Analyzing 3D Surface Cracks and Their Fatigue-Growth. CMES: *Computer Modeling in Engineering & Sciences*, vol.3, no.6, pp.699-716.

Han, Z. D.; Atluri, S. N. (2003a): On Simple Formulations of Weakly-Singular Traction & Displacement BIE, and Their Solutions through Petrov-Galerkin Approaches. CMES: *Computer Modeling in Engineering & Sciences*, vol. 4, no. 1, pp.5-20.

Han, Z. D.; Atluri, S. N. (2003b): Truly Meshless Local Petrov-Galerkin (MLPG) Solutions of Traction & Displacement BIEs. *CMES: Computer Modeling in Engi-*

neering & Sciences, vol. 4, No. 6, pp. 665-678.

Han, Z. D.; Atluri, S. N. (2004a): A Meshless Local Petrov-Galerkin (MLPG) approaches for solving 3-D Problems in elasto-statics. *CMES: Computer Modeling in Engineering & Sciences*, vol. 6, No. 2, pp. 169-188.

Han, Z. D.; Atluri, S. N. (2004b): A Meshless Local Petrov-Galerkin (MLPG) Approach for 3-Dimensional Elasto-dynamics. *CMC: Computers, Materials & Continua*, vol. 1, No. 2, pp. 129-140.

Han, Z. D.; Atluri, S. N. (2007): A Systematic Approach for the Development of Weakly-Singular BIEs. *CMES: Computer Modeling in Engineering & Sciences*, vol. 21, No. 1, pp. 41-52.

Han Z. D.; Liu H. T.; Rajendran, A. M; Atluri, S. N. (2006): The Applications of Meshless Local Petrov Galerkin (MLPG) Approaches in High-Speed Impact, Penetration and Perforation Problems, *CMES: Computer Modeling in Engineering & Sciences*, vol. 14, no. 2, pp. 119-128.

Han, Z. D.; Rajendran, A. M.; Atluri, S. N. (2005): Meshless Local Petrov-Galerkin (MLPG) Approaches for Solving Nonlinear Problems with Large Deformations and Rotations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 10, No. 1, pp. 1-12.

Heaney, C.; Augarde, Ch.; Deeks, A. (2010): Modelling Elasto-Plasticity Using the Hybrid MLPG Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 56, No. 2, pp. 153-177.

Hosseini, S. M.; Shahabian, F.; Sladek, J.; Sladek, V. (2011): Stochastic Meshless Local Petrov-Galerkin (MLPG) Method for Thermo-Elastic Wave Propagation Analysis in Functionally Graded Thick Hollow Cylinders. *CMES: Computer Modeling in Engineering & Sciences*, vol. 71, No. 1, pp. 39-66.

Hosseini, S. M.; Sladek, J.; Sladek, V. (2011): Meshless local Petrov-Galerkin method for coupled thermoelasticity analysis of a functionally graded thick hollow cylinder. *Eng. Anal. Bound. Elem.*, vol. 35, pp. 827-835.

Jarak, T.; Soric, J. (2008): Analysis of rectangular squatre plates by the mixed Meshless Local Petrov-Galerkin (MLPG) approach. *CMES: Computer Modeling in Engineering & Sciences*, vol. 38, No. 3, pp. 231-261.

Jarak, T.; Soric, J.; Hoster, J. (2007): Analysis of Shell Deformation Responses by the Meshless Local Petrov-Galerkin (MLPG) Approach. *CMES: Computer Modeling in Engineering & Sciences*, vol. 18, No. 3, pp. 235-246.

Johnson, J. N.; Owen, J. M. (2007): A Meshless Local Petrov-Galerkin Method for Magnetic Diffusion in Non-magnetic Conductors. *CMES: Computer Modeling in Engineering & Sciences*, vol. 22, No. 3, pp. 165-188.

Kansa, E. J. (1990): Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics-II. Solutions to parabolic, hyperbolic and elliptic partial differential equations. *Computers and Mathematics with Applications,* vol. 19, pp. 147-161.

Kim, H. G.; Atluri, S. N. (2000): Arbitrary placement of secondary nodes, and error control, in the meshless local Petrov-Galerkin (MLPG) method. *CMES: Computer Modeling in Engineering & Sciences*, vol.1, No. 3, pp. 11-32.

Lancaster, P.; Salkauskas, K. (1981): Surfaces generated by moving least square method. *Mathematics of Computations,* vol. 37, pp. 141-158.

Lam, K. Y.; Wang, Q. X.; Hua, L. (2004): A novel meshless approach—local Kriging (LoKriging) method with two-dimensional structural analysis. *Comput. Mech.*, vol. 33, No. 3, pp. 235–244.

Li, Q.; Shen, S.; Han, Z. D.; Atluri, S. N. (2003): Application of Meshless Local Petrov-Galerkin (MLPG) to Problems with Singularities, and Material Discontinuities, in 3-D Elasticity. *CMES: Computer Modeling in Engineering & Sciences*, vol. 4, No. 5, pp. 571-585.

Li, Q. H.; Chen, S. S.; Kou, G. X. (2011): Transient heat conduction analysis using the MLPG method and modified precise time step integration method. *Journal of Computational Physics*, vol. 230, No. 7, pp. 2736-2750.

Li, Q.; Soric, J.; Jarak, T.; Atluri, S. N. (2005): A locking-free meshless local Petrov-Galerkin formulation for thick and thin plates. *Journal of Computational Physics*, vol. 208, No. 1, pp. 116-133.

Li, S.; Atluri, S. N. (2008a): Topology-optimization of Structures Based on the MLPG Mixed Collocation Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 26, No. 1, pp. 61-74.

Li, S.; Atluri, S. N. (2008b): The MLPG mixed collocation method for material orientation and topology optimization of anisotropic solids and structures. *CMES: Computer Modeling in Engineering & Sciences*, vol. 30, No. 1, pp. 37-56.

Li, S.; Liu, W. K. (2002): Meshfree and particle methods and their application. *Applied Mechanics Reviews,* vol. 55, pp. 1-34.

Li, S.; Liu, W. K. (2004): *Meshfree Particle Methods*. Springer-Verlag, Berlin.

Li, Y. S.; Feng, W. J.; Xu, Z. H. (2009): Fracture analysis of cracked 2D planar and axisymmetric problems of magneto-electro-elastic materials by the MLPG coupled with FEM. *Computer Methods in Applied Mechanics and Engineering.* vol. 198, No. 30-32, pp. 2347-2359.

Liew, K. M.; Zhao, X.; Ferreira, A. J. M. (2011): A review of meshless methods for laminated and functionally graded plates and shells. *Composite Structures,* vol.

93, pp. 2031-2041.

Lin, H.; Atluri, S. N. (2000): Meshless Local Petrov-Galerkin (MLPG) Method for Convection-Diffusion Problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 1, No. 2, pp. 45-60.

Lin, H.; Atluri, S. N. (2001): The Meshless Local Petrov-Galerkin (MLPG) Method for Solving Incompressible Navier-Stokes Equation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 2, pp. 117-142.

Ling, X.; Atluri, S. N. (2006): Stability analysis of inverse heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, No. 3, pp. 219-228.

Liu, C. S. (2008a): A Highly Accurate MCMT for Direct and Inverse Problems of Biharmonic Equation in Arbitrary Plane Domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 30, No. 2, pp. 65-75.

Liu, C. S. (2008b): A Highly Accurate MCMT for Inverse Cauchy Problems of Laplace Equation in Arbitrary Plane Domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 35, No. 2, pp. 91-111.

Liu, C. S.; Atluri, S. N. (2008): A Novel Time Integration Method for Solving a Large System of Non-Linear Algebraic Equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 20, No. 3, pp. 147-156.

Liu, C. S.; Atluri S. N. (2009): A Highly Accurate Technique for Interpolations Using Very High-Order Polynomials, and Its Applications to Some Ill-Posed Linear Problems, *CMES: Computer Modeling in Engineering & Sciences*, vol. 43, No. 3, pp. 253-276.

Liu, C. S.; Yeih, W.; Atluri, S. N. (2009): On Solving the Ill-Conditioned System Ax=b: General-Purpose Conditioners Obtained From the Boundary-Collocation Solution of the Laplace Equation, Using Trefftz Expansions With Multiple Length Scales. *CMES: Computer Modeling in Engineering & Sciences*, vol. 44, No. 3, pp. 281-311.

Liu, G. R.; Gu, Y. T. (2001): A point interpolation method for two-dimensional solids. *Int. J. Numer. Methods Eng.*, vol. 50, pp. 937-951.

Liu, G. R.; Wu, Y. L.; Ding, H. (2004): Meshfree weak-strong (MWS) form method and its application to incompressible flow problems. *Int. J. Numer. Meth. Fluids*, vol. 46, pp. 1025-1047.

Liu, H. T.; Han, Z. D.; Rajendran, A. M.; Atluri, S. N. (2006): Computational Modeling of Impact Response with the RG Damage Model and the Meshless Local Petrov-Galerkin (MLPG) Approaches. *CMC: Computers, Materials & Continua*, vol. 4, No. 1, pp. 43-53.

Liu, L. H. (2006): Meshless local Petrov–Galerkin method for solving radiative transfer equation. *Journal of Thermophysics and Heat Transfer.*, vol. 20, pp. 150–154.

Liu, L. H.; Tan, J. Y.; Li, B. X. (2006): Meshless approach for coupled radiative and conductive heat transfer in one-dimensional graded index medium. *Journal of Quantitative Spectroscopy & Radiative Transfer,* vol. 101, pp. 237-248.

Liu, L. H.; Tan, J. Y. (2007): Meshless local Petrov–Galerkin approach for coupled radiative and conductive heat transfer. *Int. J. Therm. Sci.*, vol. 46, pp. 672–681.

Liu, K.; Long, S.; Li, G. (2006): A simple and less-costly meshless local Petrov-Galerkin (MLPG) method for the dynamic fracture problem. *Eng. Anal. Bound. Elem.,* vol. 30, pp. 72-76.

Liu, K. Y.; Long, S. Y.; Li, G. Y. (2008): A Meshless Local Petrov-Galerkin Method for the Analysis of Crack in the Isotropic Functionally Graded Material. *CMC: Computers, Materials & Continua*, vol. 7, No. 1, pp. 43-57.

Liu, W. K.; Chen, Y.; Jun, S.; Chen, J. S.; Belytschko, T.; Pan, C.; et al. (1996): Overview and applications of reproducing kernel particle methods. *Arch. Comput. Methods Eng.,* vol. 3, pp. 3-80.

Liu, W. K.; Jun, S.; Zhang, Y. F. (1995): Reproducing kernel particle methods. *Int. J. Numer. Meth. Fluids*. vol. 20, pp. 1081-1106.

Lo, E. Y. M.; Shao, S. (2002): Simulation of near-shore solitary wave mechanics by an incompressible SPH method, *Appl. Ocean Res.*, vol. 24, No. 5, pp. 275-286.

Long, S.; Atluri, S. N. (2002): A Meshless Local Petrov-Galerkin Method for Solving the Bending Problem of a Thin Plate. *CMES: Computer Modeling in Engineering & Sciences*, vol. 3, No. 1, pp. 53-63.

Long, S. Y.; Liu, K. Y.; Li, G. I. (2008): An Analysis for the Elasto-Plastic Fracture Problem by the Meshless Local Petrov-Galerkin Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 28, No. 3, pp. 203-216.

Loukopoulos, V. C.; Bourantas, G. C. (2012): MLPG6 for the Solution of Incompressible Flow Equations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 88, No. 6, pp. 531-558.

Ma, Q. W. (2005): Meshless local Petrov-Galerkin method for two-dimensional nonlinear water wave problems. *J. Comput. Phys.*, vol. 205, pp. 611-625.

Ma, Q. W.; Zhou, J. T. (2009): MLPG_R Method for Numerical Simulation of 2D Breaking Waves. *CMES: Computer Modeling in Engineering & Sciences*, vol. 43, No. 3, pp. 277-303.

Manolis, G. D.; Beskos, D. E. (1988): *Boundary Element Methods in Elastody-*

namics. Unwin-Hyman, London.

Marin, L. (2008): Stable MFS Solution to Singular Direct and Inverse Problems Associated with the Laplace Equation Subjected to Noisy Data. *CMES: Computer Modeling in Engineering & Sciences*, vol. 37, No. 3, pp. 203-242.

Masuda, S.; Noguchi, H. (2006): Analysis of Structure with Material Interface by Meshfree Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 11, No. 3, pp. 131-143.

Mazzia, A.; Pini, G.; Sartoretto, F. (2012): Numerical Investigation on Direct MLPG for 2D and 3D Potential Problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 88, No. 3, pp. 183-210.

Mindlin, R. D. (1951): Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates. *Journal of Applied Mechanics ASME,* vol. 18, pp. 31-38.

Mirzaei, D.; Dehghan, M. (2011): MLPG Method for Transient Heat Conduction Problem with MLS as Trial Approximation in Both Time and Space Domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 72, No. 3, pp. 185-210.

Mohammadi, M. H. (2008): Stabilized meshless local Petrov-Galerkin (MLPG) method for incompressible viscous fluid flows. *CMES: Computer Modeling in Engineering & Sciences*, vol. 29, No. 2, pp. 75-94.

Monaghan, J. J. (1988): An introduction to SPH. *Comput. Phys. Commun.*, vol. 48, pp. 89-96.

Najafi, M.; Arefmanesh, A.; Enjilela, V. (2012): Meshless local Petrov-Galerkin method – higher Reynolds numbers fluid flow applications. *Eng. Anal. Bound. Elem.*, vol. 36, pp. 1671-1685.

Nayroles, B.; Touzot, G.; Villon, P. (1992): Generalizing the finite element method: diffuse approximation and diffuse elements. *Comput. Mech.,* vol. 10, pp. 307-318.

Nguyen, V. P.; Rabczuk, T.; Bordas, S.; Duflot, M. (2008): Meshless methods: A review and computer implementation aspects. *Mathematics and Computers in Simulation.* vol. 79, pp. 763-813.

Nie, Y. F.; Atluri, S. N.; Zuo, C. W. (2006): The Optimal Radius of the Support of Radial Weights Used in Moving Least Squares Approximation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 12, No. 2, pp. 137-142.

Nikfar, M.; Mahmoodi, M. (2012): Meshless local Petrov-Galerkin analysis of free convection of nanofluid in a cavity with a wavy side walls. *Eng. Anal. Bound. Elem.*, vol. 36, No. 3, pp. 433-445.

Nikishkov, G. P.; Park, J. H.; Atluri, S. N. (2001): SGBEM-FEM alternating method for analyzing 3D non-planar cracks and their growth in structural compo-

nents. CMES: *Computer Modeling in Engineering & Sciences*, vol.2, no.3, pp.401-422.

Ohs, R. R.; Aluru, N. R. (2001): Meshless analysis of piezoelectric devices. *Computational Mechanics,* vol. 27, pp. 23-36.

Okada, H.; Rajiyah, H.; Atluri, S. N. (1988): A Novel Displacement Gradient Boundary Element Method for Elastic Stress Analysis with High Accuracy, *Journal of Applied Mechanics*, vol. 55, pp. 786-794.

Okada, H.; Rajiyah, H.; Atluri, S. N. (1989): Non-hyper-singular integral representations for velocity (displacement) gradients in elastic/plastic solids (small or finite deformations), *Computational Mechanics.*, vol. 4, no. 3, pp. 165-175.

Onate, E.; Idelsohn, S.; Zienkiewicz, O. C.; Taylor, R. L. (1996): A finite point method in computational mechanics: applications to convective transport and fluid flow. *Int. J. Numer. Methods Eng.*, vol. 39, pp. 3839-3866.

Pini, G.; Mazzia, A.; Sartoretto, F. (2008): Accurate MLPG Solution of 3D Potential Problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 36, No. 1, pp. 43-64.

Pecher, R.; Elston, S.; Raynes, P. (2006): Meshfree Solution of Q-tensor Equations of Nematostatics Using the MLPG Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, No. 2, pp. 91-102.

Qian, L. F.; Batra, R.C. (2004): Transient thermoelastic deformations of thick functionally graded plate. *Jour. Thermal Stresses*, vol. 27, pp. 705-740

Qian, L. F.; Batra, R. C.; Chen, L. M. (2004a): Analysis of cylindrical bending thermoelastic deformation of functionally graded plates by a meshless local Petrov-Galerkin method. *Computational Mechanics,* vol. 33, pp. 263-273.

Qian, L. F.; Batra, R. C.; Chen, L. M. (2004b): Static and dynamic deformations of thick functionally graded elastic plate by using higher-order shear and normal deformable plate theory and meshless local Petrov-Galerkin method. *Composites, Part B Engineering.* vol. 35, No. 6-8, pp. 685-697.

Raju, I. S.; Phillips, D. R. (2003): Further developments in the MLPG method for beam problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 4, No. 1, pp. 141-160.

Reddy, J. N. (1997): *Mechanics of Laminated Composite Plates: Theory and Analysis.* CRC Press, Boca Raton.

Rezaei Mojdehi, A.; Darvizeh, A.; Basti, A. (2012): Nonlinear Dynamic Analysis of Three-Dimensional Elasto-Plastic Solids by the Meshless Local Petrov-Galerkin (MLPG) Method. *CMC: Computers, Materials & Continua*, vol. 29, No. 1, pp. 15-39.

Sellountos, E. J.; Polyzos, D. (2003): A MLPG (LBIE) method for solving frequency domain elastic problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 4, No. 6, pp. 619-636.

Sellountos, E. J.; Polyzos, D.; Atluri, S. N. (2012): A New and Simple Meshless LBIE-RBF Numerical Scheme in Linear Elasticity. *CMES: Computer Modeling in Engineering & Sciences*, vol. 89, No. 6, pp. 513-551.

Sellountos, E. J.; Sequeira, A.; Polyzos, D. (2009): Elastic transient analysis with MLPG(LBIE) method and local RBFs. *CMES: Computer Modeling in Engineering & Sciences*, vol. 41, No. 3, pp. 215-241.

Sellountos, E. J.; Vavourakis, V.; Polyzos, D. (2005): A new singular/hypersingular MLPG (LBIE) method for 2D elastostatics. *CMES: Computer Modeling in Engineering & Sciences*, vol. 7, pp. 35-48.

Shen, S.; Atluri, S. N. (2004a): Multiscale Simulation Based on The Meshless Local Petrov-Galerkin (MLPG) Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 5, No. 3, pp. 235-256.

Shen, S.; Atluri, S. N. (2004b): Computational nanomechanics and multiscale simulation. *CMC: Computers, Materials,& Continuua*, vol. 1, No. 1, pp. 59-90.

Shen, S. P.; Atluri, S. N. (2005): A tangent stiffness MLPG method for atom/continuum multiscale simulation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 7, No. 1, pp. 49-67.

Shibahara, M.; Atluri, S. N. (2011): The meshless local Petrov-Galerkin method for the analysis of heat conduction due to a moving heat source, in welding. *International Journal of Thermal Sciences,* vol. 50, pp. 984-992.

Shirzadi, A.; Sladek, V.; Sladek, J. (2013): A local integral equation formulation to solve coupled nonlinear reaction-diffusion equations by using moving least square approximation. *Eng. Anal. Bound. Elem.,* vol. 37, pp. 8-14.

Singh, I. V.; Sandeep, K.; Prakash, R. (2002): The element free Galerkin method in three- dimensional steady state heat conduction. *Int. J. Comput. Eng. Sci.*, vol. 3, pp. 291–303.

Sladek, J.; Sladek, V. (2011): The MLPG analysis of crack problems in magneto-electroelastic solids. *Computer Methods in Material Science,* vol. 11, pp. 81-87.

Sladek, J.; Sladek, V.; Atluri, S. N. (2000): Local boundary integral equation (LBIE) method for solving problems of elasticity with nonhomogeneous material properties, *Computational Mechanics* , vol. 24, pp. 456-462.

Sladek, J.; Sladek, V.; Atluri, S. N. (2001): A pure contour formulation for the meshless local boundary integral equation method in thermoelasticity. *CMES: Computer Modeling in Engineering & Sciences*, vol. 2, No. 4, pp. 423-434.

Sladek, J.; Sladek, V.; Atluri, S. N. (2004a): Meshless Local Petrov-Galerkin Method in Anisotropic Elasticity. *CMES: Computer Modeling in Engineering & Sciences*, vol. 6, No. 5, pp. 477-489.

Sladek, J.; Sladek, V.; Atluri, S. N. (2004b): Meshless Local Petrov-Galerkin Method for Heat Conduction Problem in an Anisotropic Medium. *CMES: Computer Modeling in Engineering & Sciences*, vol. 6, No. 3, pp. 309-318.

Sladek, J.; Sladek, V.; Hellmich, Ch.; Eberhardsteiner, J. (2007a): Analysis of thick functionally graded plates by local integral equation method. *Communications in Numerical Method in Engineering*, vol. 23, pp. 733-754.

Sladek, J.; Sladek, V.; Hellmich, Ch.; Eberhardsteiner, J. (2007f): Heat conduction analysis of 3-D axisymmetric and anisotropic FGM bodies by meshless local Petrov-Galerkin method. *Computational Mechanics*, vol. 39, No. 3, pp. 323-333.

Sladek, J.; Sladek, V.; Hon, Y. C. (2006): Inverse heat conduction problems by meshless local Petrov-Galerkin method. *Eng. Anal. Bound. Elem.,* vol. 30, pp. 650-661.

Sladek, J.; Sladek, V.; Krahulec, S.; Pan, E. (2012c): Enhancement of the magnetoelectric coefficient in functionally graded multiferroic composites. *Journal of Intelligent Material Systems and Structures,* vol. 23, pp. 1649-1658.

Sladek, J.; Sladek, V.; Krahulec, S.; Wunsche, M.; Zhang, Ch. (2012b): MLPG Analysis of Layered Composites with Piezoelectric and Piezomagnetic Phases. *CMC: Computers, Materials & Continua*, vol. 29, No. 1, pp. 75-101.

Sladek, J.; Sladek, V.; Krivacek, J.; Aliabadi, M. H. (2007c): Local boundary integral equations for orthotropic shallow shells, *Int. J. Solids and Structures*, vol. 44, pp. 2285-2303.

Sladek, J.; Sladek, V.; Krivacek, J.; Wen, P. H.; Zhang, Ch. (2007b): Meshless Local Petrov-Galerkin (MLPG) method for Reissner-Mindlin plates under dynamic load. *Comput. Meth. Appl. Mech. Engng.,* vol. 196, pp. 2681-2691.

Sladek, J.; Sladek, V.; Krivacek, J.; Zhang, Ch. (2003): Local BIEM for transient heat conduction analysis in 3-D axisymmetric functionally graded solids. *Comput. Mech.*, vol. 32, pp. 169-176.

Sladek, J.; Sladek, V.; Krivacek, J.; Zhang, Ch. (2005a): Meshless Local Petrov-Galerkin Method for Stress and Crack Analysis in 3-D Axisymmetric FGM Bodies. *CMES: Computer Modeling in Engineering & Sciences*, vol. 8, No. 3, pp. 259-270.

Sladek, J.; Sladek, V.; Mang, H. A. (2002): Meshless formulations for simply supported and clamped plate problems. *Int. J. Num. Meth. Engn.*, vol. 55, pp. 359-375.

Sladek, J.; Sladek, V.; Solek, P. (2009): Elastic analysis in 3D anisotropic functionally graded solids by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 43, No. 3, pp. 223-251.

Sladek, J.; Sladek, V.; Solek, P.; Atluri, S. N. (2008e): Modeling of Intelligent Material Systems by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 34, No. 4, pp. 273-300.

Sladek, J.; Sladek, V.; Solek, P.; Pan, E. (2008f): Fracture analysis of cracks in magneto-electro-elastic solids by the MLPG. *Computational Mechanics,* vol. 42, pp. 697-714.

Sladek, J.; Sladek, V.; Solek, P.; Saez, A. (2008d): Dynamic 3D axisymmetric problems in continuously non-homogeneous piezoelectric solids. *Int. J. Sol. Struct.*, vol. 45, No. 16, pp. 4523-4542.

Sladek, J.; Sladek, V.; Solek, P.; Tan, C. L.; Zhang, Ch. (2009a): Two- and Three-Dimensional Transient Thermoelastic Analysis by the MLPG Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 47, No. 1, pp. 61-96.

Sladek, J.; Sladek, V.; Solek, P.; Wen, P. H. (2008a): Thermal Bending of Reissner-Mindlin Plates by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 28, No. 1, pp. 57-76.

Sladek, J.; Sladek, V.; Solek, P.; Wen, P. H.; Atluri, S. N. (2008c): Thermal Analysis of Reissner-Mindlin Shallow Shells with FGM Properties by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 30, No. 2, pp. 77-97.

Sladek, J.; Sladek, V.; Solek, P.; Zhang, Ch. (2010d): Fracture analysis in continuously nonhomogeneous magneto-electro-elastic solids under a thermal load by the MLPG. *International Journal of Solids and Structures*, vol. 47, No. 10, pp. 1381-1391.

Sladek, J.; Sladek, V.; Stanak, P.; Pan, E. (2010b): The MLPG for Bending of Electroelastic Plates. *CMES: Computer Modeling in Engineering & Sciences*, vol. 64, No. 3, pp. 267297

Sladek, J.; Sladek, V.; Stanak, P.; Wen, P. H.; Atluri, S. N. (2012a): Laminated Elastic Plates with Piezoelectric Sensors and Actuators. *CMES: Computer Modeling in Engineering & Sciences*, vol. 85, No. 6, pp. 543-572.

Sladek, J.; Sladek, V.; Stanak, P.; Zhang, Ch. (2010a): Meshless Local Petrov-Galerkin (MLPG) Method for Laminate Plates under Dynamic Loading. *CMC: Computers, Materials & Continua*, vol. 15, No. 1, pp. 1-25.

Sladek, J.; Sladek, V.; Stanak, P.; Zhang, Ch.; Wunsche, M. (2011): An Interaction Integral Method for Computing Fracture Parameters in Functionally Graded Magnetoelectroelastic Composites. *CMC: Computers, Materials & Continua*, vol.

32, No. 1, pp. 35-68.

Sladek, J.; Sladek, V.; Stanak, P.; Zhang, Ch.; Wunsche, M. (2013): Analysis of the bending of circular piezoelectric plates with functionally graded material properties by a MLPG method. *Engineering Structures,* vol. 47, pp. 81-89.

Sladek, J.; Sladek, V.; Tan, C. L.; Atluri, S. N. (2008g): Analysis of Heat Conduction in 3D Anisotropic Functionally Graded Solids, by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 32, No. 3, pp. 161-174.

Sladek, J.; Sladek, V.; Van Keer, R. (2003): Meshless local boundary integral equation method for 2D elastodynamic problems. *Int. J. Num. Meth. Engn.*, vol. 57, pp. 235-249.

Sladek, J.; Sladek, V.; Wen, P. H.; Aliabadi, M. H. (2006c): Meshless Local Petrov-Galerkin (MLPG) Method for Shear Deformable Shells Analysis. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, No. 2, pp. 103-117.

Sladek, J.; Sladek, V.; Wen, P.; Hon, B. (2012e): Inverse heat conduction problems in three-dimensional anisotropic functionally graded solids. *Journal of Engineering Mathematics*, vol. 75, No. 1, pp. 157-171.

Sladek, J.; Sladek, V.; Wen, P. H.; Hon, Y. C. (2009c): Inverse fracture problems in piezoelectric solids by local integral equation method. *Eng. Anal. Bound. Elem.,* vol. 33, pp. 1089-1099.

Sladek, J.; Sladek, V.; Wen, P. H.; Hon, Y. C. (2009d): The Inverse Problem of Determining Heat Transfer Coefficients by the Meshless Local Petrov-Galerkin Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 48, pp. 191-218.

Sladek, J.; Sladek, V.; Wunsche, M.; Zhang, Ch. (2009b): Interface crack problems in anisotropic solids analyzed by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 54, No. 2, pp. 223-253.

Sladek, J.; Sladek, V.; Wunsche, M.; Zhang, Ch. (2012d): Analysis of an interface crack between two dissimilar piezoelectric solids. *Engineering Fracture Mechanics,* vol. 89, pp. 114-127.

Sladek, J.; Sladek, V.; Zhang, Ch. (2003a): Application of Meshless Local Petrov-Galerkin (MLPG) Method to Elastodynamic Problems in Continuously Non-homogeneous Solids. *CMES: Computer Modeling in Engineering & Sciences*, vol. 4, No. 6, pp. 637-647.

Sladek, J.; Sladek, V.; Zhang, Ch. (2003b): Transient heat conduction analysis in functionally graded materials by the meshless local boundary integral equation method. *Comput. Material Science*, vol. 28, pp. 494-504.

Sladek, J.; Sladek, V.; Zhang, Ch. (2008b): Local integral equation method for

viscoelastic Reissner-Mindlin plates. *Comput. Mech.,* vol. 41, pp 759-768.

Sladek, J.; Sladek, V.; Zhang, Ch. (2008c): Evaluation of the stress intensity factors for cracks in continuously nonhomogeneous solids, part II: Meshless method. *Mechanics of Advanced Materials and Structures*, vol. 15, No. 6-7, pp. 444-452.

Sladek, J.; Sladek, V.; Zhang, Ch.; Garcia-Sanchez, F.; Wunsche, M. (2006d): Meshless Local Petrov-Galerkin Method for Plane Piezoelectricity. *CMC: Computers, Materials & Continua*, vol. 4, No. 2, pp. 109-117.

Sladek, J.; Sladek, V.; Zhang, Ch.; Schanz, M. (2006a): Meshless local Petrov-Galerkin method for continuously nonhomogeneous linear viscoelastic solids. *Comput. Mech.,* vol. 37, pp. 279-289.

Sladek, J.; Sladek, V.; Zhang, Ch.; Solek, P. (2007d): Application of the MLPG to Thermo-Piezoelectricity. *CMES: Computer Modeling in Engineering & Sciences*, vol. 22, No. 3, pp. 217-233.

Sladek, J.; Sladek, V.; Zhang, Ch.; Solek, P. (2008b): Static and dynamic analysis of shallow shells with functionally graded and orthotropic material properties. *Mechanics of Advanced Materials and Structures*, vol. 15, No. 2, pp. 142-156.

Sladek, J.; Sladek, V.; Zhang, Ch.; Solek, P.; Pan, E. (2007e): Evaluation of fracture parameters in continuously nonhomogeneous piezoelectric solids. *International Journal of Fracture* , vol. 145, pp. 313–326.

Sladek, J.; Sladek, V.; Zhang, Ch.; Tan, C. L. (2006b): Meshless Local Petrov-Galerkin Method for Linear Coupled Thermoelastic Analysis. *CMES: Computer Modeling in Engineering & Sciences*, vol. 16, No. 1, pp. 57-68.

Sladek, J.; Sladek, V.; Zhang, Ch.; Wunsche, M. (2010c): Crack analysis in piezoelectric solids with energetically consistent boundary conditions by the MLPG. *CMES: Computer Modeling in Engineering & Sciences*, vol. 68, No. 2, pp. 185-219.

Sladek, J.; Sladek, V.; Zhang, Ch.; Wunsche, M. (2012d): Semi-permeable crack analysis in magnetoelectroelastic solids. *Smart Materials & Structures*, vol. 21, No. 025003.

Sladek, V.; Sladek, J. (2010): Local integral equations implemented by MLS-approximation and analytical integrations. *Eng. Anal. Bound. Elem.*, vol. 34, No. 11, pp. 904-913.

Sladek, V.; Sladek, J.; Atluri, S. N.; Van Keer, R. (2000): Numerical integration of singularities in meshless implementation of local boundary integral equations. *Comput. Mech.*, vol. 25, No. 4, pp. 394-403.

Sladek, V.; Sladek, J.; Sator, L. (2013): Physical decomposition of thin plate bending problems and their solution by mesh-free methods. *Eng. Anal. Bound.*

Elem., vol. 37, pp. 348-365.

Sladek, V.; Sladek, J.; Tanaka, M. (2005): Local Integral Equations and two Meshless Polynomial Interpolations with Application to Potential Problems in Non-homogeneous Media. *CMES: Computer Modeling in Engineering & Sciences*, vol. 7, No. 1, pp. 69-83.

Sladek, V.; Sladek, J.; Tanaka, M.; Zhang, Ch. (2005b): Transient heat conduction in anisotropic and functionally graded media by local integral equations. *Eng. Anal. Bound. Elem.*, vol. 29, pp. 1047-1065.

Sladek, V.; Sladek, J.; Zhang, C. (2006): A Comparative Study of Meshless Approximations in Local Integral Equation Method. *CMC: Computers, Materials & Continua*, vol. 4, No. 3, pp. 177-188.

Sladek, V.; Sladek, J.; Zhang, Ch. (2008d): Computation of stresses in non-homogeneous elastic solids by local integral equation method: a comparative study. *Computational Mechanics*, vol. 41, pp. 827-845.

Sladek, V.; Sladek, J.; Zhang, Ch. (2008a): Local integral equation formulation for axially symmetric problems involving elastic FGM. *Eng. Anal. Bound. Elem.,* vol. 32, pp. 1012-1024.

Sladek, V.; Sladek, J.; Zhang, Ch. (201): On Increasing Computational Efficiency of Local Integral Equation Method Combined with Meshless Implementations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 63, No. 3, pp. 243-263.

Sladek, V.; Sladek, J.; Zhang, Ch. (2011): Mesh-free methods and time integrations for transient heat conduction. *Computed Assisted Mechanics and Engineering Sciences*, vol. 18, pp. 115-128.

Soares, D. Jr. (2009a): Numerical Modeling of Electromagnetic Wave Propagation by Meshless Local Petrov-Galerkin Formulations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 50, No. 2, pp. 97-114.

Soares, D. Jr. (2009b): An Iterative Time-Domain Algorithm for Acoustic-Elastodynamic Coupled Analysis Considering Meshless Local Petrov-Galerkin Formulations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 54, No. 2, pp. 201-222.

Soares, D. Jr. (2010): A time-domain meshless local Petrov-Galerkin formulation for the dynamic analysis of nonlinear porous media. *CMES: Computer Modeling in Engineering & Sciences*, vol. 66, No. x, pp. 227-248.

Soares, D. Jr. (2011): Iterative analysis of pore-dynamic models discretized by meshless local Petrov-Galerkin formulations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 76, No. 1, pp. 61-82.

Soares, D. Jr.; Sladek, J.; Sladek, V. (2009): Dynamic analysis by meshless local

Petrov-Galerkin formulations considering a time-marching scheme based on implicit Green's functions. *CMES: Computer Modeling in Engineering & Sciences*, vol. 50, No. 2, pp. 115-140.

Soares, D. Jr.; Sladek, J.; Sladek, V. (2010): Non-linear dynamic analyses by meshless local Petrov-Galerkin formulations. *Int. J. Num. Meth. Eng.,* vol. 81, No. 13, pp. 1687-1699.

Soares, D. Jr.; Sladek, V.; Sladek, J. (2012): Modified meshless local Petrov-Galerkin formulations for elastodynamics. *International Journal for Numerical Methods in Engineering,* vol. 90, No. 12, pp. 1508-1528.

Soares, D. Jr.; Sladek, V.; Sladek, J.; Zmindak, M.; Medvecky, S. (2012): Porous media analysis by modified MLPG formulations. *CMC: Computers, Materials & Continua*, vol. 27, No. 2, pp. 101-127.

Soric, J.; Jarak, T. (2010): Mixed meshless formulation for analysis of shell-like structures. *Comput. Methods Appl. Mech. Engrg.,* vol. 199, pp. 1153-1164.

Soric, J.; Li, Q.; Jarak, T.; Atluri, S. N. (2004): Meshless Local Petrov-Galerkin (MLPG) Formulation for Analysis of Thick Plates. *CMES: Computer Modeling in Engineering & Sciences*, vol. 6, No. 4, pp. 349-357.

Shechtman, D.; Blech, I.; Gratias, D.; Cahn, J. W. (1984): Metallic Phase with Long-Range Orientational Order and No Translational Symmetry. *Phys. Rev. Lett.*, vol. 53, pp. 1951-1953.

Sterk, M.; Trobec, R. (2008): Meshless solution of diffusion equation with parameter optimization and error analysis. *Eng. Anal. Bound. Elem.,* vol. 32, pp. 567-577.

Suresh, S.; Mortensen A. (1998): *Fundamentals of Functionally Graded Materials.* Institute of Materials, London.

Tang, Z.; Shen, S.; Atluri, S. N. (2003a): Analysis of materials with strain-gradient effects: A meshless local Petrov-Galerkin (MLPG) approach, with nodal displacements only. *CMES: Computer Modeling in Engineering & Sciences*, vol. 4, No. 1, pp. 177-196.

Tang, Z.; Shen, S.; Atluri, S. N. (2003b): Analysis of Materials with Strain-Gradient Effects: A Meshless Local Petrov-Galerkin (MLPG) Approach, with Nodal Displacements only. *CMES: Computer Modeling in Engineering & Sciences*, vol. 4, No. 1, pp. 177-196.

Trobec, R.; Sterk, M.; Robic, B. (2009): Computational complexity and parallelization of the meshless local Petrov-Galerkin method. *Compututers and Structures,* vol. 87, pp. 81-90.

Tsai, C. C. (2011): A Meshless Numerical Method for Kirchhoff Plates under

Arbitrary Loadings. *CMC: Computers, Materials & Continua*, vol. 22, No. 3, pp. 197-218.

Vavourakis, V.; Polyzos, D. (2007): A MLPG4 (LBIE) Formulation in Elastostatics. *CMC: Computers, Materials & Continua*, vol. 5, No. 3, pp. 185-196.

Vavourakis, V.; Polyzos, D. (2008): A MLPG (LBIE) numerical methos for solving 2D incompressible and nearly incompressible elastostatic problems. *Commun. Numer. Meth. Engrg.,* vol. 24, pp. 281-296.

Vavourakis, V.; Sellountos, E. J.; Polyzos, D. (2006): A comparison study on different MLPG (LBIE) formulations. *CMES: Computer Modeling in Engineering & Sciences*, vol. 13, No. 3, pp. 171-183.

Wang, D.; Sun, Y.; Li, L. (2009): A Discontinuous Galerkin Meshfree Modeling of Material Interface. *CMES: Computer Modeling in Engineering & Sciences*, vol. 45, No. 1, pp. 57-82.

Wang, H.; Qin, Q.H.; Kang, Y.L. (2006): A meshless model for transient heat conduction in functionally graded materials. *Computational Mechanics*, vol. 38, pp. 51-60.

Wang, K.; Zhou, S.; Nie, Z.; Kong, S. (2008): Natural neighbour Petrov-Galerkin Method for Shape Design Sensitivity Analysis. *CMES: Computer Modeling in Engineering & Sciences*, vol. 26, No. 2, pp. 107-121.

Wang, L.; Brust, F. W.; Atluri, S. N. (1997a): The Elastic-Plastic Finite Element Alternating Method (EPFEAM) and the Prediction of Fracture Under WFD Conditions in Aircraft Structures Part I: EPFEAM Theory, *Computational Mechanics*, Volume: 19, Issue: 5, pp. 356-369.

Wang, L.; Brust, F. W.; Atluri, S. N. (1997b): The Elastic-Plastic Finite Element Alternating Method (EPFEAM) and the Prediction of Fracture Under WFD Conditions in Aircraft Structures, Part II: Fracture and the T* Parameter, Computational Mechanics, Volume: 19, Issue: 5, Pages: 370-379.

Wang, L.; Brust, F. W.; Atluri, S. N. (1997c):The Elastic-Plastic Finite Element Alternating Method (EPFEAM) and the Prediction of Fracture Under WFD Conditions in Aircraft Structures, Part III: Computational Predictions of the NIST Multiple Site Damage Experimental Results, Computational Mechanics, Volume: 20, Issue: 3, Pages: 199-212.

Wedland, H. (1999): Meshless Galerkin methods using radial basis function. *Math. Comput.,* vol. 68, pp. 1521-1531.

Wen, P. H.; Aliabadi, M. H. (2010): Elastic Moduli of Woven Fabric Composite by Meshless Local Petrov-Galerkin (MLPG) Method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 61, No. 2, pp. 133-154.

Wen, P. H.; Aliabadi, M. H. (2012): Analysis of functionally graded plates by meshless method: A purely analytical formulation. *Eng. Anal. Bound. Elem.,* vol. 36, pp. 639-650.

Wen, P. H.; Aliabadi, M. H. (2013): Analytical formulation of meshless local integral equation method. *Applied Mathematical Modelling,* vol. 37, pp. 2115-2126.

Wrobel, L. C.; Aliabadi, M. H. (2002): *The Boundary Element Method.* New Jersey: Wiley.

Wu, C. P.; Chiu, K. H.; Wang, Y. M. (2008a): A Differential Reproducing Kernel Particle Method for the Analysis of Multilayered Elastic and Piezoelectric Plates. *CMES: Computer Modeling in Engineering & Sciences*, vol. 27, No. 3, pp. 163-186.

Wu, C. P.; Chiu, K. H.; Wang, Y. M. (2008b): A Review on the Three-Dimensional Analytical Approaches of Multilayered and Functionally Graded Piezoelectric Plates and Shells. *CMC: Computers, Materials & Continua*, vol. 8, No. 2, pp. 93-132.

Wu, X. H.; Chang, Z. J.; Lu, Y. L.; Tao, W. Q.; Shen, S. P. (2012): An analysis of the convection-diffusion problems using meshless and meshbased methods. *Eng. Anal. Bound. Elem.,* vol. 36, pp. 1040-1048.

Wu, X. H.; Shen, S. P.; Tao, W. Q. (2007): Meshless Local Petrov-Galerkin Collocation Method for Two-dimensional Heat Conduction Problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 22, No. 1, pp. 65-76.

Wu, X. H.; Tao, W. Q. (2008): Meshless method based on the local weak-forms for steady state heat conduction problems, *Int. J. Heat Mass Tran.*, vol. 51, pp. 3103–3112.

Wu, X. H.; Tao, W. Q.; Shen, S. P.; Zhu, X. W. (2010): A stabilized MLPG method for steady state incompressible fluid flow simulation. *Journal of Computational Physics*, vol. 229, No. 22, pp. 8564-8577.

Wu, Y. L.; Liu, G. R.; Gu, Y. T. (2005): Application of meshless local Petrov-Galerkin (MLPG) approach to simulation of incompressible flow. *Numer. Heat Tran. B*, vol.48, pp. 459-475.

Yazid, A.; Abdelkader, N.; Abdelmadjid, H. (2009): A state-of-the-art review of the X-FEM for computational fracture mechanics. *Appl. Math. Model.,* vol. 33, pp. 4269-4282.

Yuan, W.; Chen, P.; Liu, K. (2007) A New Quasi-Unsymmetric Sparse Linear System Solver for Meshless Local Petrov-Galerkin Method (MLPG). *CMES: Computer Modeling in Engineering & Sciences*, vol. 17, No. 2, pp. 115-134.

Yuan, W. R.; Chen, Y. J.; Gagalowicz, A.; Liu, K. X. (2008): Application of

Meshless Local Petrov-Galerkin (MLPG) Method in Cloth Simulation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 35, No. 2, pp. 133-155.

Zhou, J. T.; Ma, Q. W. (2010): MLPG Method Based on Rankine Source Solution for Modelling 3D Breaking Waves. *CMES: Computer Modeling in Engineering & Sciences*, vol. 56, No. 2, pp. 179-210.

Zhu, T.; Zhang, J. D.; Atluri, S. N. (1998): A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach. *Comput. Mech.* vol. 21, pp. 223-235.

Zhu, T.; Zhang, J.; Atluri, S. N. (1999): A Meshless Numerical Method Based on The Local Boundary Integral Equation (LBIE) to Solve Linear and Nonlinear Boundary Value Problems. *Eng. Anal. Bound. Elem.*, vol. 23, pp. 375-389.