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Error Expansion of Classical Trapezoidal Rule for Computing Cauchy Principal Value Integral

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Abstract: The composite classical trapezoidal rule for the computation of Cauchy principal value integral with the singular kernel $1/(x-s)$ is discussed. Based on the investigation of the superconvergence phenomenon, i.e., when the singular point coincides with some priori known point, the convergence rate of the classical trapezoidal rule is higher than the globally one which is the same as the Riemann integral for classical trapezoidal rule. The superconvergence phenomenon of the composite classical trapezoidal rule occurs at certain local coordinate of each subinterval and the corresponding superconvergence error estimate is obtained. Some numerical examples are provided to validate the theoretical analysis.

Keywords: Cauchy principal value integral, classical trapezoidal rule, error expansion, superconvergence

1 Introduction

Consider the Cauchy principal value integral

$$\fint_a^b \frac{f(x)}{x-s}dx = g(s), s \in (a,b) \tag{1}$$

where $f(x)$ is Holder continuous on interval [a,b], $\fint_a^b$ denotes a Cauchy principal value integral and s the singular point.

There are several different definitions which can be proved mathematically equal, such as the definition of subtraction of the singularity, regularity definition, direct definition and so on. In this paper we adopt the following one

$$\fint_a^b \frac{f(x)}{x-s}dx = \lim_{\varepsilon \to 0} \left\{ \int_a^{s-\varepsilon} \frac{f(x)}{x-s}dx + \int_{s+\varepsilon}^b \frac{f(x)}{x-s}dx \right\}, \tag{2}$$

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Cauchy principal value integrals have recently attracted a lot of attention. The main reason for this interest is probably due to the fact that Cauchy principal value integral equations have shown to be an adequate tool in boundary element methods for the modeling of many physical situations, such as acoustics, fluid mechanics, elasticity, fracture mechanics and electromagnetic scattering problems and so on. Numerous work has been devoted in developing efficient quadrature formulas, such as the Gaussian method[Criscuolo and Mastroianni (1989); Diethelm (1995a); Hasegawa (2004); Monegato (1984)] the Newton-Cote methods[Koler (1997); Amari (1994); Diethelm (1994); Liu Zhang and Wu (2010); Li (2011); Li (2011)], spline methods [Orsi (1990); Dagnino and Santi (1990)] and some other method [Natarajan and Mohankumar (1995); Kim and Choi (2000); Kim and Yun (2002); Behforooz (1992); Xu and Yao (1998); Junghanns and Silbermann (1998); Chen and You (1999); Chen and Hong (1999)].

The general (composite) Newton-Cotes method for the computation of Cauchy principal value integrals and Hadamard finite-part integrals with the singular kernel calculated analysis and the density function numerical approximated were studied in [Liu Zhang and Wu (2010)] and [Wu and Sun (2008)] which focus on its pointwise superconvergence phenomenon, which means that the rate of convergence of the Newton-Cotes quadrature rule is higher than what is globally possible when the singular point coincides with some a priori known point. The necessary and sufficient conditions satisfied by the superconvergence point are also given. Moreover, the superconvergence estimates are obtained and the properties of the superconvergence points are investigated.

It is the aim of this paper to investigate the superconvergence phenomenon of classical trapezoidal rule for it and, in particular, to derive error estimates. This paper focuses on the superconvergence phenomenon of classical trapezoidal rule for Cauchy principal integrals with the density function $\frac{f(x)}{x-s}$ is replaced by the approximation function $\frac{f(x_i)}{2(x_i-s)}, i = 0, 1, \cdots, n$, where x_i is the mesh point. This method can be considered as the direct method to compute the Cauchy principal integral different from the idea presented by [Linz (1985)] in the paper to calculate the hypersingular integral on interval. We prove both theoretically and numerically that the composite classical trapezoidal rule can reach the superconvergence rate $O(h^2)$ when the singular point s is far away from the end of the interval with the local coordinate of the singular point s equal to 0.

The rest of this paper is organized as follows. In Sect.2, after introducing some basic formulas of the classical trapezoidal rule, we present the main results. In Sect.3, the proof of the main results is finished. Finally, several numerical examples are provided to validate our analysis.

2 Main result

Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a uniform partition of the interval $[a,b]$ with mesh size $h = (b-a)/n$. Define by $f_L(x)$ the linear interpolant for $f(x)$

$$f_L(x) = \frac{f(x_i)}{2}, i = 0, 1, \cdots, n. \tag{3}$$

and a linear transformation

$$x = \hat{x}_i(\tau) := \frac{(\tau+1)(x_{i+1}-x_i)}{2} + x_i, \quad \tau \in [-1,1], \tag{4}$$

from the reference element $[-1,1]$ to the subinterval $[x_i, x_{i+1}]$. Replacing $f(x)$ in Eq. 2 with $f_L(x)$ gives the composite classical trapezoidal rule:

$$I_n(f;s) := \int_a^b \frac{f_L(x)}{x_i - s} dx = \sum_{i=0}^{n} \omega_i(s) f(x_i) = \int_a^b \frac{f(x)}{x-s} dx - E_n(f;s), \tag{5}$$

where $E_n(f;s)$ is the error functional and

$$\omega_i(s) = \frac{h}{2(x_i - s)} \tag{6}$$

is the Cote coefficients.

In the following analysis, C will denote a generic constant that is independent of h and s and it may have different values in different places.

Theorem 1 *Assume $f(x) \in C^\alpha[a,b], \alpha \in (0,1]$. For the classical trapezoidal rule $I_n(f,s)$ defined as Eq. 5. Assume that $s = x_{[m]} + (1+\tau)h/2$, there exist a positive constant C, independent of h and s, such that*

$$|E_n(f;s)| \le C(|\ln h| + \gamma^{-1}(\tau))h^\alpha, \tag{7}$$

where

$$\gamma(\tau) = \min_{0 \le i \le n} \frac{|s - x_i|}{h} = \frac{1-|\tau|}{2}. \tag{8}$$

Proof: Let $R(x) = f(x) - \frac{f(x_i)}{2} - \frac{f(x_{i+1})}{2}$, then we have $|R(x)| \le Ch^{\alpha}$. As

$$\begin{aligned}
E_n(f;s) &= \int_a^b \frac{f(x)}{x-s}dx - \sum_{i=0}^{n-1}\left[\frac{f(x_i)h}{2(x_i-s)} + \frac{f(x_{i+1})h}{2(x_{i+1}-s)}\right] \\
&= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{f(x)}{x-s} - \frac{f(x_i)}{2(x_i-s)} - \frac{f(x_{i+1})}{2(x_{i+1}-s)}\right]dx \\
&= \sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}\frac{f(x)-\frac{f(x_i)}{2}-\frac{f(x_{i+1})}{2}}{x-s}dx + \frac{1}{2}\int_{x_i}^{x_{i+1}}\frac{f(x_i)(x_i-x)}{(x-s)(x_i-s)}dx\right] \\
&+ \sum_{i=0}^{n-1}\frac{1}{2}\int_{x_i}^{x_{i+1}}\frac{f(x_{i+1})(x_{i+1}-x)}{(x-s)(x_{i+1}-s)}dx \\
&= \sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}}\frac{R(x)}{x-s}dx + \frac{1}{2}\int_{x_i}^{x_{i+1}}\frac{f(x_i)(x_i-x)}{(x-s)(x_i-s)}dx\right] \\
&+ \sum_{i=0}^{n-1}\frac{1}{2}\int_{x_i}^{x_{i+1}}\frac{f(x_{i+1})(x_{i+1}-x)}{(x-s)(x_{i+1}-s)}dx \\
&= \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{R(x)}{x-s}dx + \int_{x_m}^{x_{m+1}}\frac{R(x)}{x-s}dx \\
&+ \frac{1}{2}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f(x_i)(x_i-x)}{(x-s)(x_i-s)}dx + \frac{1}{2}\int_{x_m}^{x_{m+1}}\frac{f(x_m)(x_m-x)}{(x-s)(x_m-s)}dx \\
&+ \frac{1}{2}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f(x_{i+1})(x_{i+1}-x)}{(x-s)(x_{i+1}-s)}dx \\
&+ \frac{1}{2}\int_{x_m}^{x_{m+1}}\frac{f(x_{m+1})(x_{m+1}-x)}{(x-s)(x_{m+1}-s)}dx \\
&:= \mathscr{R}_1 + \mathscr{R}_2 + \mathscr{R}_3 + \mathscr{R}_4 + \mathscr{R}_5 + \mathscr{R}_6
\end{aligned} \tag{9}$$

where

$$\mathscr{R}_1 = \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{R(x)}{x-s}dx \tag{10}$$

$$\mathscr{R}_2 = \int_{x_m}^{x_{m+1}}\frac{R(x)}{x-s}dx \tag{11}$$

$$\mathscr{R}_3 = \frac{1}{2}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f(x_i)(x_i-x)}{(x-s)(x_i-s)}dx \tag{12}$$

$$\mathscr{R}_4 = \frac{1}{2} \fint_{x_m}^{x_{m+1}} \frac{f(x_m)(x_m - x)}{(x-s)(x_m - s)}\, dx \tag{13}$$

$$\mathscr{R}_5 = \frac{1}{2} \sum_{i=0, i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{f(x_{i+1})(x_{i+1} - x)}{(x-s)(x_{i+1} - s)}\, dx \tag{14}$$

$$\mathscr{R}_6 = \frac{1}{2} \fint_{x_m}^{x_{m+1}} \frac{f(x_{m+1})(x_{m+1} - x)}{(x-s)(x_{m+1} - s)}\, dx. \tag{15}$$

For Eq. 10,we have

$$\begin{aligned} \left| \sum_{i=0, i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{R(x)}{x-s} dx \right| &\leq Ch^{\alpha} \left(\int_a^{x_m} \frac{1}{s-x} dx + \int_{x_{m+1}}^{b} \frac{1}{x-s} dx \right) \\ &= Ch^{\alpha} \ln \frac{(b-s)(s-a)}{(x_{m+1}-s)(s-x_m)} \\ &\leq C(|\ln h| + |\ln \gamma(\tau)|) h^{\alpha}. \end{aligned} \tag{16}$$

As for Eq. 11,

$$\begin{aligned} \left| \fint_{x_m}^{x_{m+1}} \frac{R(x)}{x-s} dx \right| &\leq \left| \int_{x_m}^{x_{m+1}} \frac{R(x) - R(s)}{x-s} \right| + \left| R(s) \ln \frac{x_{m+1} - s}{s - x_m} \right| \\ &\leq Ch^{\alpha} |\ln \gamma(\tau)|. \end{aligned} \tag{17}$$

For Eq. 12, we have

$$\begin{aligned} &\left| \frac{1}{2} \sum_{i=0, i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{f(x_i)(x_i - x)}{(x-s)(x_i - s)}\, dx \right| \\ &\leq Ch^{1+\alpha} \sum_{i=0, i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x-s)^2}\, dx \\ &= Ch^{1+\alpha} \left[\frac{1}{s-x_m} - \frac{1}{s-a} + \frac{1}{b-s} - \frac{1}{x_{m+1}-s} \right] \\ &\leq C\gamma^{-1}(\tau) h^{\alpha}. \end{aligned} \tag{18}$$

As for Eq. 13,

$$
\begin{aligned}
&\left|\frac{1}{2}\int_{x_m}^{x_{m+1}}\frac{f(x_m)(x_m-x)}{(x-s)(x_m-s)}\,dx\right| \\
&=\left|\frac{1}{2}\frac{f(x_m)}{(x_m-s)}\int_{x_m}^{x_{m+1}}\frac{(x_m-x)}{(x-s)}\,dx\right| \\
&=\left|\frac{1}{2}\frac{f(x_m)}{(x_m-s)}\int_{x_m}^{x_{m+1}}\frac{(x_m-s)-(x-s)}{(x-s)}\,dx\right| \\
&=\left|\frac{1}{2}\frac{f(x_m)}{(x_m-s)}\left[\int_{x_m}^{x_{m+1}}\frac{x_m-s}{(x-s)}\,dx+\int_{x_m}^{x_{m+1}}dx\right]\right| \\
&=\left|-\frac{1}{2}\frac{f(x_m)h}{(x_m-s)}+\frac{f(x_m)}{2}\ln\frac{x_{m+1}-s}{s-x_m}\right| \\
&\leq C\gamma^{-1}(\tau)h^{\alpha}+Ch^{\alpha}\left|\ln\frac{x_{m+1}-s}{s-x_m}\right| \\
&\leq C\gamma^{-1}(\tau)h^{\alpha}.
\end{aligned}
\tag{19}
$$

Similarly we have

$$
\left|\frac{1}{2}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f(x_{i+1})(x_{i+1}-x)}{(x-s)(x_{i+1}-s)}\,dx\right|\leq C\gamma^{-1}(\tau)h^{\alpha} \tag{20}
$$

and

$$
|\mathscr{R}_6|=\left|\frac{1}{2}\int_{x_m}^{x_{m+1}}\frac{f(x_{m+1})(x_{m+1}-x)}{(x-s)(x_{m+1}-s)}\,dx\right|\leq C\gamma^{-1}(\tau)h^{\alpha}. \tag{21}
$$

Combining from Eq. 16 to Eq. 21 together, the proof is completed.

Now we present the error estimate for the classical trapezoidal rule in the following theorem. We firstly define

$$
\phi_0(x)=\begin{cases}-\dfrac{1}{2}\displaystyle\int_{-1}^{1}\frac{1}{\tau-x}+\frac{x}{x^2-1}d\tau, & |x|<1,\\[2ex] -\dfrac{1}{2}\displaystyle\int_{-1}^{1}\frac{1}{\tau-x}+\frac{x}{x^2-1}d\tau, & |x|>1\end{cases} \tag{22}
$$

and

$$
S_0(\tau):=\phi_0(\tau)+\sum_{i=1}^{\infty}\phi_0(2i+\tau)+\phi_0(-2i+\tau),\tau\in(-1,1). \tag{23}
$$

Theorem 2 *Assume $f(x) \in C^2[a,b]$. For the classical trapezoidal rule $I_n(f,s)$ defined as Eq. 5. Assume that $s = x_{[m]} + (1+\tau)h/2$, there exist a positive constant C, independent of h and s, such that*

$$E_n(f;s) = -f(s)S_0(\tau) + \mathscr{R}_n(s), \tag{24}$$

where

$$|\mathscr{R}_n(s)| \le C(|\ln h| + |\ln \gamma(\tau)| + \eta^2(s))h^2 \tag{25}$$

$\gamma(\tau)$ is defined as Eq. 8 and

$$\eta(s) = \max\left\{\frac{1}{s-a}, \frac{1}{b-s}\right\}. \tag{26}$$

It is known that the global convergence rate of the composite classical trapezoidal rule is lower than Riemann integral. For the influence of the $\eta(s)$, when the singular point is near the end of the interval, there are no convergence rate; while the singular point is far away from the end of the interval, the convergence rate can reach $O(h^2)$ which is the same as the convergence rate of Riemann integral.

3 Proof of theorem 2

In this section, we study the superconvergence phenomenon of the composite classical trapezoidal rule for Cauchy principal integrals.

3.1 Preliminaries

Lemma 1 *Under the same assumptions of theorem 2, it holds that*

$$\begin{aligned} &2(x_i - s)(x_{i+1} - s)f(x) - (x-s)(x_{i+1} - s)f(x_i) \\ &-(x-s)(x_i - s)f(x_{i+1}) = [(x_{i+1} - s)(x_i - x) + (x_i - s)(x_{i+1} - x)]f(s) \\ &+R_f^1(x) + R_f^2(x) + R_f^3(x) + R_f^4(x) \end{aligned} \tag{27}$$

where

$$R_f^1(x) = \frac{1}{2}[(x_{i+1} - s)(x_i - x) + (x_i - s)(x_{i+1} - x)](x-s)^2 f''(\beta_{i1}) \tag{28}$$

$$R_f^2(x) = -[(x_{i+1} - s)(x_i - x) + (x_i - s)(x_{i+1} - x)](x-s)^2 f''(\beta_{i2}) \tag{29}$$

$$R_f^3(x) = -\frac{1}{2}f''(\alpha_{i1})(x-s)(x_{i+1} - s)(x_i - x)^2 \tag{30}$$

and

$$R_f^4(x) = -\frac{1}{2}f''(\alpha_{i2})(x-s)(x_i-s)(x_{i+1}-x)^2 \tag{31}$$

here $\beta_{i1},\beta_{i2},\alpha_{i1},\alpha_{i2} \in (x_i,x_{i+1})$.

Proof: Performing Taylor expansion of $f_L(x)$ at the point x, we have

$$f(x_i) = f(x)+f'(x)(x_i-x)+\frac{f''(\alpha_{i1})}{2}(x_i-x)^2 \tag{32}$$

$$f(x_{i+1}) = f(x)+f'(x)(x_{i+1}-x)+\frac{f''(\alpha_{i2})}{2}(x_{i+1}-x)^2. \tag{33}$$

Similarly, we have

$$f(x) = f(s)+f'(s)(x-s)+\frac{1}{2}f''(\beta_{i1})(x-s)^2 \tag{34}$$

and

$$f'(x) = f'(s)+f''(\beta_{i2})(x-s). \tag{35}$$

Combining Eq. 32 to Eq. 35 together we get the results.

Lemma 2 *Assume* $s \in (x_m,x_{m+1})$ *for some* m *and let* $c_i = 2(s-x_i)/h-1, 0 \le i \le n-1$. *Then, we have*

$$\phi_0(c_i) = \begin{cases} -2\,⨍_{x_m}^{x_{m+1}}\left[\frac{2}{x-s}-\frac{1}{x_m-s}-\frac{1}{x_{m+1}-s}\right]dx, & i=m, \\ -2\int_{x_i}^{x_{i+1}}\left[\frac{2}{x-s}-\frac{1}{x_i-s}-\frac{1}{x_{i+1}-s}\right]dx, & i\neq m. \end{cases} \tag{36}$$

Proof: Following the definition of Eq. 2 and the linear transformation Eq. 4, we have

$$\begin{aligned} &⨍_{x_m}^{x_{m+1}}\left[\frac{2}{x-s}-\frac{1}{x_m-s}-\frac{1}{x_{m+1}-s}\right]dx \\ &= ⨍_{x_m}^{x_{m+1}}\frac{2(x_m-s)(x_{m+1}-s)-(x-s)(x_{m+1}-s)-(x-s)(x_m-s)}{(x-s)(x_m-s)(x_{m+1}-s)}dx \\ &= \lim_{\varepsilon\to 0}\left\{\left(\int_{x_m}^{s-\varepsilon}+\int_{s+\varepsilon}^{x_{m+1}}\right)\frac{(x_m-x)(x_{m+1}-s)+(x_{m+1}-x)(x_m-s)}{(x-s)(x_m-s)(x_{m+1}-s)}\right\}dx \\ &= -\frac{1}{2}⨍_{-1}^{1}\frac{\tau c_m-c_m-1}{(\tau-c_m)(c_m^2-1)}d\tau \\ &= -\frac{1}{2}\phi_0(c_m). \end{aligned} \tag{37}$$

The case $j \neq m$ can be proved by applying the same approach to the correspondent Riemann integral.
Setting

$$E_m(x) = f(x) - \frac{(x-s)}{2}\left[\frac{f(x_m)}{x_m - s} - \frac{f(x_{m+1})}{x_{m+1} - s}\right] - \frac{f(s)}{2}\left[2 - \frac{x-s}{x_m - s} - \frac{x-s}{x_{m+1} - s}\right]. \tag{38}$$

Lemma 3 *Let $f(x) \in C^2[a,b]$, denote $E_m(x)$ to be the error functional for the composite classical trapezoidal rule as Eq. 38. Assume $s \neq x_i$ for any $i = 0, 1, \cdots, n$, then there holds*

$$\left| \int_{x_m}^{x_{m+1}} \frac{E_m(x)}{x-s} dx \right| \leq Ch^2 |\ln \gamma(\tau)|. \tag{39}$$

Proof: As $f(x) \in C^2[a,b]$, we get $E_m(x) \in C^2[a,b]$. By the definition of

$$\int_a^b \frac{f(x)}{x-s} dx = \int_a^b \frac{f(x) - f(s)}{x-s} dx + f(s) \ln \left| \frac{b-s}{s-a} \right|, \tag{40}$$

then we have

$$\int_{x_m}^{x_{m+1}} \frac{E_m(x)}{x-s} dx = \int_{x_m}^{x_{m+1}} \frac{E_m(x) - E_m(s)}{x-s} dx + E_m(s) \ln \frac{x_{m+1} - s}{s - x_m}, \tag{41}$$

then we get

$$\begin{aligned} \left| \int_{x_m}^{x_{m+1}} \frac{E_m(x)}{x-s} dx \right| &\leq \left| \int_{x_m}^{x_{m+1}} \frac{E_m(x) - E_m(s)}{x-s} dx \right| + \left| E_m(s) \ln \frac{x_{m+1} - s}{s - x_m} \right| \\ &\leq Ch^2 |\ln \gamma(\tau)| \end{aligned} \tag{42}$$

and the proof is completed.

Lemma 4 *For $\tau \in (-1,1)$, and $m \geq 1$, we have*

$$\left| \sum_{i=m}^{\infty} \phi_0(2i + \tau) + \sum_{i=n-m+1}^{\infty} \phi_0(-2i + \tau) \right| \leq Ch^2 \eta^2(s). \tag{43}$$

Proof: By the definition of $\phi_0(\tau)$,we have

$$\phi_0(\tau) = -\log \left| \frac{1-\tau}{1+\tau} \right| - \frac{2\tau}{\tau^2 - 1} = -2Q_1'(\tau), \tag{44}$$

which means

$$\phi_0(x) = -\frac{1}{2}\int_{-1}^{1} \frac{1-\tau^2}{(x-\tau)^3} d\tau = -2Q_1'(x). \tag{45}$$

Noting that $s = x_m + \frac{\tau+1}{2}h = a + (m + \frac{\tau+1}{2})h$, we have $2(s-a)/h = \tau + 2m + 1$ and

$$\begin{aligned} \left|\sum_{i=m}^{\infty} \phi_0(2i+\tau)\right| &\leq C\sum_{i=m}^{\infty}\int_{-1}^{1} \frac{dt}{|2i+\tau-t|^3} \\ &= C\int_{\tau+2m+1}^{\infty} \frac{dx}{x^3} = \frac{C}{(\tau+2m+1)^2} = \frac{Ch^2}{(s-a)^2}. \end{aligned} \tag{46}$$

On the other hand, since $b = a + nh$, we have $2(b-s)/h = 2(n-m) - 1 - \tau$ and

$$\begin{aligned} \left|\sum_{i=n-m+1}^{\infty} \phi_0(\tau-2i)\right| &\leq C\sum_{i=n-m+1}^{\infty}\int_{-1}^{1} \frac{dt}{|2i-\tau+t|^3} \\ &= C\int_{2(n-m)-1-\tau}^{\infty} \frac{dx}{x^3} = \frac{C}{[2(n-m)-1-\tau]^2} = \frac{Ch^2}{(b-s)^2}. \end{aligned} \tag{47}$$

Combining Eq. 46 and Eq. 47 together and the proof of this Lemma is completed.

Proof of Theorem 2: By Lemma 1, we have

$$\begin{aligned} &\left(\int_a^{x_m} + \int_{x_{m+1}}^{b}\right)\frac{f(x)}{x-s}dx - \sum_{i=0,i\neq m}^{n-1}\left[\frac{f(x_i)h}{2(x_i-s)} + \frac{f(x_{i+1})h}{2(x_{i+1}-s)}\right] \\ &= \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{f(x)}{x-s} - \frac{f(x_i)}{2(x_i-s)} - \frac{f(x_{i+1})}{2(x_{i+1}-s)}\right]dx \\ &= \frac{f(s)}{2}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{2}{x-s} - \frac{1}{x_i-s} - \frac{1}{x_{i+1}-s}\right]dx \\ &+ \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})(x-s)^2}{4}\left[\frac{2}{x-s} - \frac{1}{x_i-s} - \frac{1}{x_{i+1}-s}\right]dx \\ &- \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i2})(x-s)^2}{2}\left[\frac{2}{x-s} - \frac{1}{x_i-s} - \frac{1}{x_{i+1}-s}\right]dx \\ &- \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\alpha_{i1})(x_i-x)^2}{4(x_i-s)}dx \\ &- \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\alpha_{i2})(x_{i+1}-x)^2}{4(x_{i+1}-s)}dx. \end{aligned} \tag{48}$$

By the definition of $E_m(x)$, we have

$$\begin{aligned}
&\int_{x_m}^{x_{m+1}}\left[\frac{f(x)}{x-s}-\frac{f(x_m)}{2(x_m-s)}-\frac{f(x_{m+1})}{2(x_{m+1}-s)}\right]dx \\
&=\int_{x_m}^{x_{m+1}}\frac{E_m(x)}{x-s}\,dx+f(s)\int_{x_m}^{x_{m+1}}\left[\frac{2}{x-s}-\frac{1}{x_m-s}-\frac{1}{x_{m+1}-s}\right]dx.
\end{aligned} \tag{49}$$

Putting Eq. 48, Eq. 49 together yields

$$\begin{aligned}
&\int_a^b\frac{f(x)}{x-s}dx-\sum_{i=0}^{n-1}\left[\frac{f(x_i)h}{2(x_i-s)}+\frac{f(x_{i+1})h}{2(x_{i+1}-s)}\right] \\
&=\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{f(x)}{x-s}-\frac{f(x_i)}{2(x_i-s)}-\frac{f(x_{i+1})}{2(x_{i+1}-s)}\right]dx \\
&=\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{f(x)}{x-s}-\frac{f(x_i)}{2(x_i-s)}-\frac{f(x_{i+1})}{2(x_{i+1}-s)}\right]dx \\
&+\int_{x_m}^{x_{m+1}}\left[\frac{f(x)}{x-s}-\frac{f(x_m)}{2(x_m-s)}-\frac{f(x_{m+1})}{2(x_{m+1}-s)}\right]dx \\
&=-f(s)S_0(\tau)+\mathscr{R}_n(s)
\end{aligned} \tag{50}$$

where

$$\mathscr{R}_n(s)=R_1+R_2+R_3+R_4$$

and

$$\begin{aligned}
R_1 &= \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})(x-s)^2}{4}\left[\frac{2}{x-s}-\frac{1}{x_i-s}-\frac{1}{x_{i+1}-s}\right]dx \\
&-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i2})(x-s)^2}{2}\left[\frac{2}{x-s}-\frac{1}{x_i-s}-\frac{1}{x_{i+1}-s}\right]dx
\end{aligned} \tag{51}$$

$$\begin{aligned}
R_2 &= -\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\alpha_{i1})(x_i-x)^2}{4(x_i-s)}dx \\
&-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\alpha_{i2})(x_{i+1}-x)^2}{4(x_{i+1}-s)}dx
\end{aligned} \tag{52}$$

$$R_3=\int_{x_m}^{x_{m+1}}\frac{E_m(x)}{x-s}dx, \tag{53}$$

$$R_4 = f(s)\left[\sum_{i=m}^{\infty}\phi_0(2i+\tau)+\sum_{i=n-m+1}^{\infty}\phi_0(-2i+\tau)\right]. \tag{54}$$

Now we estimate $\mathscr{R}_n(s)$ term by term. For the first part of R_1,we have

$$\begin{aligned}
&\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})(x-s)^2}{4}\left[\frac{2}{x-s}-\frac{1}{x_i-s}-\frac{1}{x_{i+1}-s}\right]dx\\
&=\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}\left[\frac{(x_i-x)(x-s)}{x_i-s}+\frac{(x_{i+1}-x)(x-s)}{x_{i+1}-s}\right]dx\\
&=\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}\left[\frac{(x_i-x)(x-x_i+x_i-s)}{x_i-s}+\frac{(x_{i+1}-x)(x-x_{i+1}+x_{i+1}-s)}{x_{i+1}-s}\right]dx\\
&=\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}(x_i+x_{i+1}-2x)\,dx\\
&-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}\frac{(x_i-x)^2}{x_i-s}\,dx-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}\frac{(x_{i+1}-x)^2}{x_{i+1}-s}\,dx
\end{aligned} \tag{55}$$

For the first part of Eq. 55, there are no singularity, then we get

$$\begin{aligned}
&\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}(x_i+x_{i+1}-2x)\,dx\right|\\
&=\left|\frac{1}{2}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}f''(\beta_{i1})(x_i+\frac{h}{2}-x)dx\right|\\
&=\left|\frac{1}{2}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}f''(\beta_{i1})(x_i-x)dx+\frac{h}{4}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}f''(\beta_{i1})dx\right|\\
&=\left|\frac{1}{2}\sum_{i=0,i\neq m}^{n-1}f''(\xi_{1i})\int_{x_i}^{x_{i+1}}(x_i-x)dx+\frac{h}{4}\sum_{i=0,i\neq m}^{n-1}f''(\xi_{2i})\int_{x_i}^{x_{i+1}}dx\right|\\
&=\left|\frac{1}{2}\sum_{i=0,i\neq m}^{n-1}f''(\xi_{1i})\int_{x_i}^{x_{i+1}}(x_i+\frac{h}{2}-x)dx+\frac{h^2}{4}\sum_{i=0,i\neq m}^{n-1}(f''(\xi_{2i})-f''(\xi_{1i})\right|\\
&=\left|\frac{h^2}{4}\sum_{i=0,i\neq m}^{n-1}(f''(\xi_{2i})-f''(\xi_{1i})\right|\\
&\leq Ch^2
\end{aligned} \tag{56}$$

where $\xi_{1i},\xi_{2i},\beta_{i1}\in(x_i,x_{i+1})$.

For the second part and third part of Eq. 55, there are no singularity, then we have

$$\begin{aligned}
&\left|-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}\frac{(x_i-x)^2}{x_i-s}\,dx\right| \\
&\leq Ch^2\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{1}{|x_i-s|}\,dx\right| \\
&\leq Ch^2\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{1}{|x-s|}\,dx \\
&\leq Ch^\alpha\left(\int_a^{x_m}\frac{1}{s-x}dx+\int_{x_{m+1}}^b\frac{1}{x-s}dx\right) \\
&= Ch^\alpha\ln\frac{(b-s)(s-a)}{(x_{m+1}-s)(s-x_m)} \\
&\leq C(|\ln h|+|\ln\gamma(\tau)|)h^\alpha
\end{aligned} \tag{57}$$

and

$$\left|-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i1})}{4}\frac{(x_{i+1}-x)^2}{x_{i+1}-s}\,dx\right|\leq C\left(|\ln h|+|\ln\gamma(\tau)|\right)h^2. \tag{58}$$

Similarly, for the second part of R_1, we have

$$\begin{aligned}
&\left|-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\beta_{i2})(x-s)^2}{2}\left[\frac{2}{x-s}-\frac{1}{x_i-s}-\frac{1}{x_{i+1}-s}\right]dx\right| \\
&\leq C\left(|\ln h|+|\ln\gamma(\tau)|\right)h^2
\end{aligned} \tag{59}$$

For R_2, we have

$$\left|-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\alpha_{i1})(x_i-x)^2}{4(x_i-s)}\,dx\right|\leq C\left(|\ln h|+|\ln\gamma(\tau)|\right)h^2 \tag{60}$$

and

$$\left|-\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{f''(\alpha_{i2})(x_{i+1}-x)^2}{4(x_{i+1}-s)}\,dx\right|\leq C\left(|\ln h|+|\ln\gamma(\tau)|\right)h^2. \tag{61}$$

As for R_3 and R_4, by Lemma 3 and lemma 4, the we get

$$|\mathscr{R}_n(s)|\leq|R_1|+|R_2|+|R_3|+|R_4|\leq C(|\ln h|+|\ln\gamma(\tau)|+\eta^2(s))h^2 \tag{62}$$

and the proof is completed.

3.2 The calculation of $S_0(\tau)$

Let $Q_n(x)$ be the function of the second kind associated with the Legendre polynomial $P_n(x)$, defined by [Andrews (2002)]

$$Q_0(x) = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|, \quad Q_1(x) = xQ_0(x) - 1. \tag{63}$$

We also define

$$W(f,\tau) := f(\tau) + \sum_{i=0}^{\infty}[f(2i+\tau) + f(-2i+\tau)], \tau \in (-1,1). \tag{64}$$

Then, by the definition of W,

$$\begin{aligned} W(Q_0)(\tau) &= \frac{1}{2}\ln\frac{1+\tau}{1-\tau} + \frac{1}{2}\sum_{i=1}^{\infty}\left(\ln\frac{2i+1+\tau}{2i-1+\tau} + \ln\frac{2i-1-\tau}{2i+1-\tau}\right) \\ &= \frac{1}{2}\lim_{i\to\infty}\ln\frac{2i+1+\tau}{2i+1-\tau} = 0, \end{aligned}$$

$$\begin{aligned} W(xQ_0')(\tau) &= \frac{\tau}{1-\tau^2} - \sum_{i=1}^{\infty}\left(\frac{2i+\tau}{(2i+\tau)^2-1} + \frac{-2i+\tau}{(-2i+\tau)^2-1}\right) \\ &= \frac{1}{2}\lim_{n\to\infty}\sum_{k=-n}^{k=n}\frac{1}{k+\frac{1}{2}-\frac{\tau}{2}} = \frac{\pi}{2}\tan\frac{\pi\tau}{2}. \end{aligned}$$

it follows that

$$S_0(Q_1',\tau) = W(Q_0 + xQ_0',\tau) = \pi\tan\frac{\pi\tau}{2} \tag{65}$$

For $S_0(\tau) = 0$, we have

Corollary 1 *Under the same assumption of theorem 2,we have*

$$|E_n(f;s)| \le C(|\ln h| + \eta^2(s))h^2 \tag{66}$$

and $\eta(s)$*is defined as Eq. 26.*

Based on the theorem 2, we present the modify classical trapezoidal rule

$$\tilde{I}_n(f;s) = I_n(f;s) - f(s)S_0(\tau), \tag{67}$$

and

$$\tilde{E}_n(f;s) = \int_a^b \frac{f(x)}{x-s}dx - \tilde{I}_n(f;s) \tag{68}$$

then we have

Corollary 2 *Under the same assumption of theorem 2,we have*

$$\tilde{E}_n(f;s) \leq C(|\ln h| + |\ln \gamma(\tau)| + \eta^2(s))h^2 \tag{69}$$

where $\gamma(\tau)$ and $\eta(s)$ is defined as Eq. 8 and Eq. 26,respectively.

4 Numerical Examples

In this section, some computational results are given to illustrated our theoretical analysis.

Example 1 Consider the Cauchy principal value integral with $f(x) = x^6, a = -1, b = 1$ and the exact value is

$$2s^5 + \frac{2s^3}{3} + \frac{2s}{5} + s^6 \log \frac{1-s}{1+s}.$$

We adopt the uniform meshes to examine the convergence rate of the trapezoidal rule $I_n(x^6, s)$ and the modified classical trapezoidal rule $\tilde{I}_n(x^6, s)$ with the dynamic point with $s = x_{[n/4]} + (1+\tau)h/2$ and $s = a + (\tau+1)h/2$.

The left of table 1 show that when the local coordinate of singular point $\tau = 0$, the quadrature reach the convergence rate of $O(h^2)$ as for the non-supersingular point the there are no convergence rate which agree with our theorematically analysis. From the right of the table 1 shows the modify classical trapezoidal rule have the convergence rate of $O(h^2)$ at both the superconvergence point and non-superconvergence point which coincide with our Corollary 2. For the case of $s = a + (\tau+1)h/2$, table 2 show that there are no superconvergence phenomenon for the classical trapezoidal rue and the modify classical trapezoidal rule which coincide with our theoretically analysis.

Table 1: Errors of the (modified) classical trapezoidal rule $I_n(x^6, s)$ and $\tilde{I}_n(x^6, s)$ with $s = x_{[n/4]} + (1+\tau)h/2$

	$I_n(x^6,s)$			$\tilde{I}_n(x^6,s)$	
n	$\tau = 0$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$
32	1.3437e-003	-2.5900e-002	-4.2668e-002	1.2926e-003	1.2756e-003
64	3.4884e-004	-3.6459e-002	-6.1341e-002	3.4242e-004	3.4029e-004
128	8.8818e-005	-4.2489e-002	-7.2485e-002	8.8016e-005	8.7749e-005
256	2.2405e-005	-4.5713e-002	-7.8570e-002	2.2305e-005	2.2272e-005
512	5.6264e-006	-4.7381e-002	-8.1749e-002	5.6139e-006	5.6097e-006
1024	1.4097e-006	-4.8229e-002	-8.3373e-002	1.4082e-006	1.4076e-006
h^α	1.9793	-	-	1.9685	1.9647

Table 2: Errors of the (modified) classical trapezoidal rule $I_n(x^6,s)$ and $\tilde{I}_n(x^6,s)$ with $s = a + (\tau+1)h/2$

	$I_n(x^6,s)$			$\tilde{I}_n(x^6,s)$	
n	$\tau = 0$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$
32	-2.1969e-001	-2.4502e+000	-4.0226e+000	-9.4925e-002	-7.5008e-002
64	-2.4502e-001	-2.8380e+000	-4.7362e+000	-1.1311e-001	-9.1582e-002
128	-2.5769e-001	-3.0494e+000	-5.1299e+000	-1.2229e-001	-9.9966e-002
256	-2.6403e-001	-3.1596e+000	-5.3365e+000	-1.2689e-001	-1.0418e-001
512	-2.6719e-001	-3.2160e+000	-5.4423e+000	-1.2920e-001	-1.0629e-001
1024	-2.6878e-001	-3.2444e+000	-5.4958e+000	-1.3036e-001	-1.0735e-001
h^α	-	-	-	-	-

Table 3: Errors of the (modified) classical trapezoidal rule $I_n(x^2-1,s)$ and $\tilde{I}_n(x^2-1,s)$ with $s = x_{[n/4]} + (1+\tau)h/2$

	$I_n(x^2-1,s)$			$\tilde{I}_n(x^2-1,s)$	
n	$\tau = 0$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$
32	7.8101e-004	2.4973e+000	4.3504e+000	7.4135e-004	7.2854e-004
64	2.0592e-004	2.4283e+000	4.2193e+000	2.0064e-004	1.9890e-004
128	5.2854e-005	2.3926e+000	4.1510e+000	5.2171e-005	5.1945e-005
256	1.3388e-005	2.3745e+000	4.1163e+000	1.3301e-005	1.3272e-005
512	3.3688e-006	2.3654e+000	4.0987e+000	3.3579e-006	3.3542e-006
1024	8.4495e-007	2.3608e+000	4.0899e+000	8.4358e-007	8.4312e-007
h^α	1.9705	-	-	1.9559	1.9510

Table 4: Errors of the (modified) classical trapezoidal rule $I_n(x^2-1,s)$ and $\tilde{I}_n(x^2-1,s)$ with $s = a + (\tau+1)h/2$

	$I_n(x^2-1,s)$			$\tilde{I}_n(x^2-1,s)$	
n	$\tau = 0$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$	$\tau = \frac{1}{2}$	$\tau = \frac{2}{3}$
32	1.6628e-002	2.9965e-001	5.6304e-001	1.2032e-002	1.0990e-002
64	8.3822e-003	1.5163e-001	2.8529e-001	6.0914e-003	5.5717e-003
128	4.2078e-003	7.6264e-002	1.4359e-001	3.0641e-003	2.8046e-003
256	2.1081e-003	3.8244e-002	7.2028e-002	1.5366e-003	1.4069e-003
512	1.0551e-003	1.9150e-002	3.6073e-002	7.6945e-004	7.0463e-004
1024	5.2779e-004	9.5822e-003	1.8051e-002	3.8501e-004	3.5260e-004
h^α	0.9955	0.9934	0.9926	0.9932	0.9924

Example 2 Consider the Cauchy principal value integral with $f(x) = x^2 - 1, a = -1, b = 1$ and the exact value is

$$2s + (1 - s^2)\log\left|\frac{1+s}{1-s}\right|.$$

We adopt the uniform meshes to examine the convergence rate of the trapezoidal rule $I_n(x^2-1,s)$ and the modified classical trapezoidal rule $\tilde{I}_n(x^2-1,s)$ with the dynamic point with $s = x_{[n/4]} + (1+\tau)h/2$ and $s = a + (\tau+1)h/2$.

The left of table 3 show that when the local coordinate of singular point $\tau = 0$, the quadrature reach the convergence rate of $O(h^2)$ as for the non-supersingular point there are divergence. From the right of the table 3 shows the modify classical trapezoidal rule have the convergence rate of $O(h^2)$ at both the superconvergence point and non-superconvergence point which coincide with our Corollary 2. For the case of $s = a + (\tau+1)h/2$, table 4 show that the convergence for the classical trapezoidal rue and the modify classical trapezoidal rule are both $O(h)$ for both the superconvergence point and the non-superconvergce point because of the end of the boundary condition $f(a) = f(b) = 0$.

Example 3 Now we consider an example of less regularity. Let $a = -b = -1$, and $s = 0$ and

$$f(x) = \mathscr{F}_i(x) := x^2 + (2 + \mathrm{sign}(x))|x|^{2-i+0.5}, i = -1, 0, 1.$$

Obviously, $\mathscr{F}_i(x) \in C^{1-i+0.5}[-1,1](i = -1, 0.8, 1)$. The exact value of the integral is

$$\mathscr{I}_2(\mathscr{F}_i(x), 0) = \frac{10-4i}{3-2i}.$$

The numerical results are presented in Table 5 and Table 6. When the density function $f(x)$ is smooth enough $i = -1$, the error bound is $O(h^2)$, and if the density function has less regularity $i = 0.8, 1$, the convergence rate is $O(h^{0.7})$ and $O(h^{0.5})$ respectively, there is no superconvergence phenomenon, which means the regularity of density function can not be reduced.

Table 5: Errors of the mod-classical trapezoidal rule

n	$i=-1$	h^α	$i=0.8$	h^α	$i=1$	h^α
32	1.0270e-002		7.1836e-002		3.1441e-001	
64	2.6245e-003	1.9684	3.9924e-002	8.4743e-001	2.1687e-001	5.3587e-001
128	6.6343e-004	1.9840	2.3495e-002	7.6492e-001	1.5198e-001	5.1295e-001
256	1.6680e-004	1.9918	1.4192e-002	7.2732e-001	1.0712e-001	5.0462e-001
512	4.1821e-005	1.9958	8.6680e-003	7.1126e-001	7.5659e-002	5.0164e-001
1024	1.0471e-005	1.9978	5.3188e-003	7.0460e-001	5.3478e-002	5.0058e-001

Table 6: Errors of the mod-classical trapezoidal rule

n	$i=-1$	h^{α}	$i=0.8$	h^{α}	$i=1$	h^{α}
32						
64	7.6457e-003		7.0460e-001		9.7550e-002	
128	1.9610e-003	1.9630	1.6429e-002	9.5779e-001	6.4889e-002	5.8816e-001
256	4.9663e-004	1.9814	1.6429e-002	8.2045e-001	4.4856e-002	5.3267e-001
512	1.2498e-004	1.9905	5.5236e-003	7.5216e-001	3.1461e-002	5.1176e-001
1024	3.1350e-005	1.9951	3.3492e-003	7.2178e-001	2.2182e-002	5.0418e-001

5 Conclusion

In this paper, we study the composite classical trapezoidal rule for numerical evaluation integrals defined on interval with Cauchy singular kernel. Based on the error expansion in each subinterval, the superconvergence phenomenon is obtained. This kind of Cauchy principal value integral equation is widely used in many engineering area [Yu (2002);Yu (1993);Han and Atluri (2007)]. Certainly, in using this method to evaluate the Cauchy principal value integral, the case where the singular point happens to be a superconvergence point is rare. However, by using certain mesh techniques or by extrapolation [Li Wu and Yu (2009)], the most potentially useful and important aspect of the superconvergence result is the solution of the singular integral equation. The results in this paper show a possible way to improve the accuracy of the collocation method for singular integral equations by choosing the superconvergence points to be the collocation points.

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