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The Sinh Transformation for Curved Elements Using the General Distance Function

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Abstract: Accurate numerical evaluation of the nearly singular boundary integrals is a major concerned issue in the implementation of boundary element method (BEM). In this paper, a general distance function independent on the nearly singular point is proposed. Combined with an iteration process, the position of the nearly singular point can be obtained more easily. Then, an extended form of the sinh transformation using the general distance function, which automatically takes into account the intrinsic coordinate of the nearly singular point and the minimum distance from source point to the element in the intrinsic parameter plane, is developed to deal with the nearly singular integrals. The iterated sinh transformation can also be achieved in a straightforward fashion. Results obtained by the one and two iterations of sinh transformation for various orders of singularities demonstrate the high efficiency and accuracy of presented method. Comparisons with other variable transformation methods are also carried out to show the superiority of the presented method.

Keywords: Boundary element method, Nearly singular integrals, Sinh transformation, Distance transformation.

1 Introduction

The implementation of the boundary element method (BEM) involves many numerical evaluations of line or surface integrals. For all the integrals, they can be categorized into three types depending on the position of the source point: non-singular integrals, singular integrals and nearly singular integrals. For non-singular integrals (the source point is away from the evaluation element), a straightforward application of Gaussian quadrature is sufficient to obtain accurate numerical values. For singular integrals (the source point is on the evaluation element),

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several transformation methods [Tanaka, Sladek, and Sladek (1994); Sladek and Sladek (1998); Guiggiani and Gigante (1990); Guiggiani, Krishnasamy, Rudolphi, and Rizzo (1992); Liu and Rudolphi (1999); Liu (2000); Gao (2010); Li, Wu, and Yu (2009); Li and Yu (2011)] have been devised to improve the accuracy of the numerical evaluation. In other boundary integral methods, such as symmetric Galerkin boundary integral equations [Han and Atluri (2007); Dong and Atluri (2012, 2013)], the nearly singular integrals should also be considered. In this paper, the nearly singular integrals are concerned, which lie between the two types of integrals defined above as the source point is close to the interval of integration but not on it. Although the nearly singular integrals are actually regular in nature, they cannot be evaluated accurately by the standard Gaussian quadrature due to that the denominator r , the distance between the source and the field point, is close to zero, which leads to a spiked integrand.

The accurate numerical evaluation of nearly singular integrals plays an important role in many engineering applications. In general, these involve with the accurate solution near the boundary in potential and elasticity problems. Particular cases include sensitivity problems [Zhang, Rizzo, and Rudolphi (1999)], unknowns around crack tips [Dirgantara and Aliabadi (2000)], contact problems [Aliabadi and Martin (2000)] and thin structures [Liu (1998)].

In the past decades, various numerical techniques have been proposed to remove the near singularities [Eberwien, Duenser, and Moser (2005); Chen, Lu, Huang, and Williams (1998); Sladek, Sladek, and Tanaka (1993); Zhou, Niu, Cheng, and Guan (2008)], among which the most popular approaches are based on the various nonlinear transformations, such as cubic polynomial transformation [Telles (1987)], coordinate optimization transformation [Sladek, Sladek, and Tanaka (2000)], rational transformation [Huang and Cruse (1993)], sigmoidal transformation [Johnston (1999, 2000)], distance transformation [Ma and Kamiya (2001, 2002a,b, 2003)], the PART method [Hayami (2005)], exponential transformation [Zhang, Gu, Chen, et al. (2009); Xie, Zhang, Qin, and Li (2011)] and the sinh transformation [Johnston and Elliott (2005); Elliott and Johnston (2008); Johnston, Johnston, and Elliott (2007)]. The key aspect of the transformation methodology is to cluster more Gaussian points towards the 'nearly singular point'(the projection point of the source point to the element).

The distance transformation method proposed by Ma employed an asymptotic distance function via Taylor expansion and has been successfully applied into the evaluation of nearly singular integrals in 2D and 3D BEM. However, the nearly singular point should be calculated before performing nonlinear transformation, which may be a time-consuming work. A general distance function has been proposed by the authors in Ref. [Lv, Miao, and Zhu (2013)]. However, we still need to know the approximate position of the projection point. In order to get fully rid of the limit of the projection point, an iteration process is introduced using the asymptotic behavior of the general distance function to get the accurate position of the nearly singular point more easily.

The previous sinh transformation, which automatically takes into account the position of the nearly singular point and the distance from source point to the element, is thought to be a promising method to deal with nearly singular integrals due to its high accuracy and straightforward implementation. However, the sinh transformation is only limited to straight-line or arc elements. The combination of the general distance function and the sinh transformation makes it possible for curved boundary elements. Numerical results presented in this paper demonstrate the high efficiency and accuracy of the presented method compared with other nonlinear transformations.

This paper is organized as follows. The general forms of nearly singular integrals are described in Section 2. The conventional distance function is briefly reviewed and a general distance function combined with its iteration process is proposed in Section 3. Section 4 presents the extended form of the sinh transformation based on the distance function in detail. Numerical examples are given in Section 5 to verify the efficiency and accuracy of presented method. The paper ends with conclusions in Section 6.

2 General descriptions

Considering 2D potential problems in the domain Ω enclosed by boundary Γ, the two boundary integrals concerned in the present work are written in the usual forms in terms of the potential u and the flux q on the boundary as follows:

$$c(\mathbf{y})u(\mathbf{y}) = \int_{\Gamma} q(\mathbf{x})u^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) - \int_{\Gamma} u(\mathbf{x})q^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) \tag{1}$$

$$c(\mathbf{y})u_k(\mathbf{y}) = \int_{\Gamma} q(\mathbf{x})u_k^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) - \int_{\Gamma} u(\mathbf{x})q_k^*(\mathbf{x},\mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) \tag{2}$$

where $\mathbf{y}$ and $\mathbf{x}$ are the source and the field points, respectively. c is a coefficient depending on the smoothness of the boundary at $\mathbf{y}$. $u^*(\mathbf{x},\mathbf{y})$ represents the fundamental solution for 2D potential problems

$$u^*(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\log(\frac{1}{r}) \tag{3}$$

and $u_k^*(\mathbf{x},\mathbf{y})$, $q^*(\mathbf{x},\mathbf{y})$ and $q_k^*(\mathbf{x},\mathbf{y})$ are the derived fundamental solutions

$$u_k^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial x_k}, \quad q^*(\mathbf{x},\mathbf{y}) = \frac{\partial u^*(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}}, \quad q_k^*(\mathbf{x},\mathbf{y}) = \frac{\partial q^*(\mathbf{x},\mathbf{y})}{\partial x_k} \tag{4}$$

where r denotes the Euclidean distance between the source and the field point and $\mathbf{n}$ is the unit outward normal on the boundary. To evaluate the boundary integrals numerically, the boundary Γ is discretized into a number of linear or quadratic elements characterized by element nodes and then the boundary integrations are performed on each element. In this paper, quadratic elements are considered, because the nearly singular integrals for linear elements can be obtained easily. The distribution of element nodes and the nodal intrinsic coordinates over an element are shown in Fig. 1.

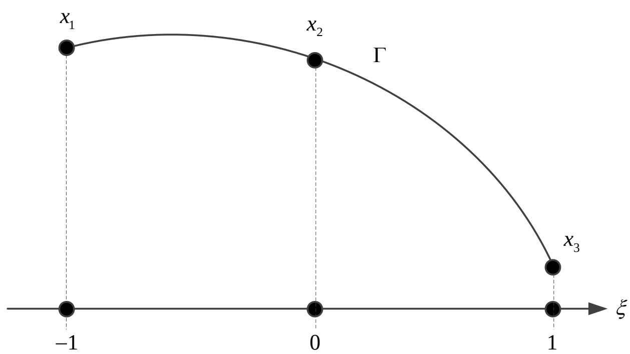

Figure 1: Quadratic element for 2D boundary

For each element, the global coordinate vector $\mathbf{x}$ is interpolated through the coordinates of the element nodes

$$\mathbf{x}(\xi) = \sum_{i=1}^{3} \phi_i(\xi)\mathbf{x}_i \tag{5}$$

where $\mathbf{x}_i$ is the coordinates at node i, ξ is the intrinsic coordinate taking values from -1.0 to +1.0 and $\phi_i(\xi)$ is the interpolation functions and commonly referred as shape functions.

When the source point is very close to but not on the integration element, nearly singular integrals arise with different orders. In this paper, we deal with these boundary integrals of the following forms:

$$\mathrm{I} = \int_{-1}^{1} f(\xi)\ln(r(\xi))\mathrm{d}\xi \tag{6}$$

$$\mathrm{II} = \int_{-1}^{1} f(\xi) \frac{1}{r^m(\xi)} \mathrm{d}\xi \tag{7}$$

where $f(\xi)$ is a smooth function consists of shape function, Jacobian and coefficients from the derivation of the kernels. Eq. (6) denotes the nearly weak singular integrals and Eq. (7) represents the strongly integrals or the hyper-singular integrals as $m = 1$ or $m = 2$, respectively.

3 Construction of general distance function

3.1 Conventional distance function

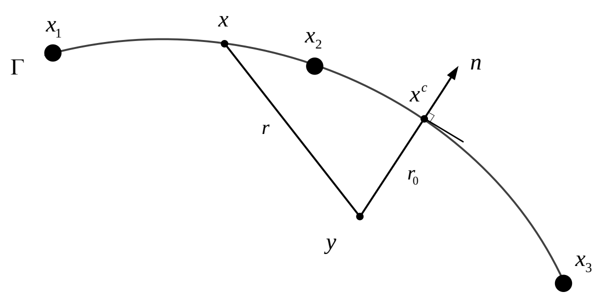

Figure 2: Conventional distance function

In this section, we will review the definition of the conventional distance function. As shown in Fig. 2, the minimum distance r_0 from the source point to the element is defined perpendicular to the tangential line, through the nearly singular point $\mathbf{x}^c$ and the source point $\mathbf{y}$. By employing the first-order Taylor expansion in the neighborhood of the projection point $\mathbf{x}^c$, we have

$$x_k - y_k = x_k - x_k^c + x_k^c - y_k = \frac{\partial x_k}{\partial \xi}|_{\xi=c}(\xi - c) + r_0 n_k(c) + O(|\xi - c|^2) \tag{8}$$

where c is the intrinsic coordinate of the nearly singular point $\mathbf{x}^c$. Then the real distance can be expanded to the following form:

$$\begin{aligned} r^2(\xi) &= (x_k - y_k)(x_k - y_k) \\ &= r_0^2 + \frac{\partial x_k}{\partial \xi}\frac{\partial x_k}{\partial \xi}|_{\xi=c}(\xi - c)^2 + 2r_0 \frac{\partial x_k}{\partial \xi}|_{\xi=c} n_k(c)(\xi - c) + O(|\xi - c|^3) \\ &= r_0^2 + G_c^2(\xi - c)^2 + O(|\xi - c|^3) \\ &= G_c^2(\alpha^2 + (\xi - c)^2) + O(|\xi - c|^3) \end{aligned} \tag{9}$$

where G_c stands for the Jacobian at nearly singular point $\mathbf{x}^c$, and $\alpha = r_0/G_c$, corresponding the minimum distance in the intrinsic parametric plane. The above deductions result in an asymptotic expression for the distance via Taylor expansion, and have been used to deal with nearly singular integrals successfully.

3.2 General distance function

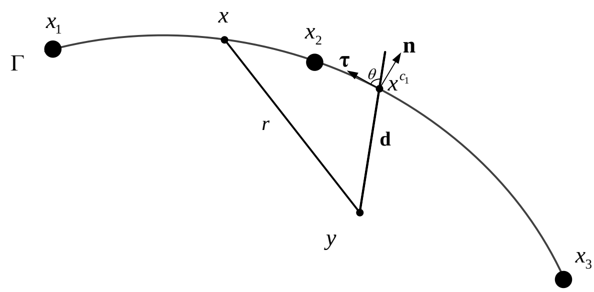

Figure 3: General distance function

For the conventional distance function, the nearly singular point should be evaluated for each source point before the nonlinear variable transformation, which will lower the efficiency of the computation process. In this section, a general distance function independent on the nearly singular point is constructed as follows.

Firstly,a general point $\mathbf{x}^{c_1}$ is defined arbitrarily, which can be located inside the integration interval or on one node of the element as shown in Fig. 3. τ and $\mathbf{n}$ are the unit tangential and outward normal vector at $\mathbf{x}^{c_1}$, respectively. A new distance vector $\mathbf{d}$ from the source point $\mathbf{y}$ to the general point $\mathbf{x}^{c_1}$ is defined additionally, which is not required to be perpendicular to the tangential line through $\mathbf{x}^{c_1}$ as the conventional distance function. By applying the first-order Taylor expansion in the neighborhood of point $\mathbf{x}^{c_1}$, we have

$$x_k - y_k = x_k - x_k^{c_1} + x_k^{c_1} - y_k = \frac{\partial x_k}{\partial \xi}|_{\xi=c_1}(\xi - c_1) + d_k + O(|\xi - c_1|^2) \tag{10}$$

where c_1 is the intrinsic coordinate of the general point $\mathbf{x}^{c_1}$, and d_k represents the components of $\mathbf{d}$. The real distance can be expanded to the following form:

$$\begin{aligned} d^2(\xi) &= (x_k - y_k)(x_k - y_k) \\ &= d^2 + \frac{\partial x_k}{\partial \xi}\frac{\partial x_k}{\partial \xi}|_{\xi=c_1}(\xi - c_1)^2 + 2d_k\frac{\partial x_k}{\partial \xi}|_{\xi=c_1}(\xi - c_1) + O(|\xi - c_1|^3) \end{aligned} \tag{11}$$

Note that

$$d_k\frac{\partial x_k}{\partial \xi}|_{\xi=c_1} = G_{c_1}(\mathbf{d}\cdot\tau) = G_{c_1} d\cos\theta_1 \tag{12}$$

where G_{c_1} stands for the Jacobian at general point $\mathbf{x}^{c_1}$ and θ_1 is the angle between $\mathbf{d}$ and τ, which is only related to $\mathbf{x}^{c_1}$ and $\mathbf{y}$.The real distance can be rewritten as

$$\begin{aligned} d^2(\xi) &= (x_k - y_k)(x_k - y_k) \\ &= d^2 + G_{c_1}^2(\xi - c_1)^2 + 2G_{c_1} d\cos\theta_1(\xi - c_1) + O(|\xi - c_1|^3) \\ &= G_{c_1}^2[\lambda_1^2 + (\xi - c_1 + \alpha_1\cos\theta_1)^2] + O(|\xi - c_1|^3) \end{aligned} \tag{13}$$

where $\alpha_1 = d/G_{c_1}$, and $\lambda_1 = \sqrt{\alpha_1^2 - \alpha_1^2\cos^2\theta_1}$. By taking the derivative of Eq. (13), the intrinsic coordinate of the approximate nearly singular point is located at $c_1 - \alpha_1\cos\theta_1$. This point may be inaccurate enough for the evaluation of the nearly singular integrals. Then, an iteration process is introduced to obtain the accurate position of the nearly singular point. We place the general point at the approximate nearly singular point which has been obtained in the last iteration and perform the above process again until get the accurate nearly singular point.

$$c_n = c_1 - \sum_{i=1}^{n} \alpha_i\cos\theta_i \tag{14}$$

where n is the number of iterations. And the minimum distance can be rewritten as

$$r^2(\xi) = d_{min}^2 = G_{c_n}^2[\lambda_n^2 + (\xi - c_n)^2] + O(|\xi - c_n|^3) \tag{15}$$

where c_n is the intrinsic coordinate of nearly singular point obtained by n iterations, and λ_n is the corresponding coefficient.

4 The sinh transformation

Using the distance function mentioned above, the nearly singular integrals can be rewritten as

$$\mathrm{I} = \int_{-1}^{1} f(\xi)\ln(G_c)\mathrm{d}\xi + \frac{1}{2}\int_{-1}^{1} f(\xi)\ln[\lambda^2 + (\xi - c)^2]\mathrm{d}\xi \tag{16}$$

$$\mathrm{II} = \frac{1}{G_c^m}\int_{-1}^{1} \frac{f(\xi)}{[\lambda^2 + (\xi - c)^2]^{m/2}}\mathrm{d}\xi \tag{17}$$

Note that the integrals I and II all contain an argument of the form $\lambda^2 + (\xi - c)^2$. Therefore, the sinh transformation can be applied in a straightforward fashion as follows:

$$\xi = a_1 + b_1 \sinh(\mu_1\xi_1 - \eta_1) \tag{18}$$

where a_1 is the intrinsic coordinate of the nearly singular point and b_1 represents the minimum distance in the intrinsic parameter plane, i.e. $a_1 = c$ and $b_1 = \lambda$. μ_1 and η_1 are chosen such that the transformation maps [-1, 1] onto [-1, 1], so that the Gaussian quadrature can be applied directly. Evaluating μ_1 and η_1 yields

$$\mu_1 = \frac{1}{2}\{\operatorname{arc\,sinh}(\frac{1+a_1}{b_1}) + \operatorname{arc\,sinh}(\frac{1-a_1}{b_1})\} \tag{19}$$

$$\eta_1 = \frac{1}{2}\{\operatorname{arc\,sinh}(\frac{1+a_1}{b_1}) - \operatorname{arc\,sinh}(\frac{1-a_1}{b_1})\} \tag{20}$$

Substituting the transformation of Eq. (18) into Eqs. (16) and (17) yields

$$\begin{aligned}\mathrm{I} = &\int_{-1}^{1} f(\xi)\ln(G_c)\mathrm{d}\xi \\ &+ \frac{1}{2}b_1\mu_1\int_{-1}^{1} f(\xi_1)\ln[b_1^2 + b_1^2\sinh^2(\mu_1\xi_1 - \eta_1)]\cosh(\mu_1\xi_1 - \eta_1)\mathrm{d}\xi_1\end{aligned} \tag{21}$$

$$\mathrm{II} = \frac{1}{G_c^m} b_1^{1-m}\mu_1 \int_{-1}^{1} \frac{f(\xi_1)}{[1 + \sinh^2(\mu_1\xi_1 - \eta_1)]^{m/2}}\cosh(\mu_1\xi_1 - \eta_1)\mathrm{d}\xi_1 \tag{22}$$

In Eqs. (21) and (22), the kernel function has been transformed into a smoother one as $1 + \sinh^2(\mu_1\xi_1 - \eta_1)$ is always greater than 1. But the integrand may have singularities

$$\cosh(\mu_1\xi_1 - \eta_1) = 0 \tag{23}$$

Then a similar transformation can be applied again to further remove the singularities by letting

$$\xi_1 = a_2 + b_2 \sinh(\mu_2 \xi_2 - \eta_2) \tag{24}$$

where

$$\mu_2 = \frac{1}{2}\{\operatorname{arc\,sinh}(\frac{1+a_2}{b_2}) + \operatorname{arc\,sinh}(\frac{1-a_2}{b_2})\} \tag{25}$$

$$\eta_2 = \frac{1}{2}\{\operatorname{arc\,sinh}(\frac{1+a_2}{b_2}) - \operatorname{arc\,sinh}(\frac{1-a_2}{b_2})\} \tag{26}$$

with $a_2 = \frac{\eta_1}{\mu_1}$ and $b_2 = \frac{\pi}{2\mu_1}$.

Similar iterated sinh transformation can be achieved for the general distance transformation with n iterations using Eq. (15) by setting $a_1 = c_n$ and $b_1 = \lambda_n$. For brevity, the detailed deductions are omitted.

5 Numerical examples

In this section, a number of numerical examples for quadratic element are presented to verify the accuracy and efficiency of presented method. The relative distance is given in terms of r_0/l to describe the influence of the nearly singular integrals over each element, where r_0 is the minimum distance as shown in Fig. 2 and l stands for the length of the element. For the purpose of error estimation, the relative error is defined as follows:

$$\text{error} = \frac{\mathrm{I}_{num} - \mathrm{I}_{ref}}{\mathrm{I}_{ref}} \tag{27}$$

where the subscripts *num* and *ref* refer to the numerical and reference solutions, respectively. The reference solutions are obtained by subdivision method with enough subelements and denoted as 'reference' in the table. Results obtained by one and two iterations of sinh transformation using the general distance function are denoted by 'ge_sinh1' and 'ge_sinh2', respectively.

The numerical example in Ref. [Ma and Kamiya (2002a)] is chosen as a representative quadratic element. The example is computed over a curved boundary element with the node coordinates of (2.0, 0.0), (1.0, 1.0), and (0.0, 0.5). The intrinsic coordinate of the nearly singular point is set at $c = -0.5$. For other cases, similar results can be obtained. Twenty Gaussian points are always used in the element interval for the convenience of comparisons.

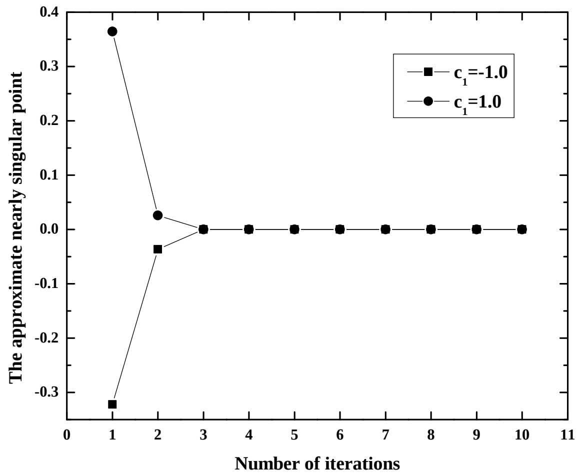

Figure 4: Convergence of iteration process

5.1 Convergence of iteration process

The accurate calculation of the position of nearly singular point is critical as the Gaussian points will be clustered around it by nonlinear transformation. In order to determine the proper number of iterations, the nearly point is set at $c = 0.0$ with $r_0/l = 10^{-3}$ and the initial general point is placed at the two ends of the element, i.e. $c_1 = -1.0$ or $c_1 = 1.0$, respectively. Fig. 4 gives a plot of the approximate nearly singular point corresponding to number of iterations. It can be seen that after 5 iterations, an accurate intrinsic coordinate which is enough for the evaluation of the nearly singular integrals can be generally obtained. Thus, the number of iterations is taken as 5 for all the computations presented later.

5.2 Nearly weak singular integral

The first case is concerned with the evaluation of the integral I in Eq. (6), which is representative of the nearly weakly singular integral with a kernel of $\log(1/r)$. The kernel multiplied by different shape functions ϕ_i is evaluated as the relative distance r_0/l varies from 10^{-1} to 10^{-6}, which is thought to be enough for general computational applications. The relative errors for the one and two iterations of sinh transformation using the general distance function are listed in Table 1. It can be seen that both one and two iterations of the sinh transformation can get acceptable results as well as the general trend of increasing relative error with decreasing relative distance, due to the fact that the integrand becomes more peaked. Besides, one iteration of sinh transformation leads to more accurate results than that of the

Table 1: The relative errors of nearly singular integrals with different orders using conventional and general distance transformation

r_0/l	Methods	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
	reference	0.07515265	0.11666812	0.12144209	0.12192656	0.12197508	0.12197993
$\phi_1 u^*$	ge_sinh1	-5.50238E-09	7.00528E-08	-4.91681E-07	5.16788E-06	-3.50909E-06	7.02376E-05
	ge_sinh2	-3.21283E-09	7.48015E-06	6.24399E-04	3.80387E-03	6.70809E-03	1.04164E-02
	reference	0.11491031	0.20922006	0.22016906	0.22128126	0.22139266	0.22140380
$\phi_2 u^*$	ge_sinh1	-5.31991E-08	-1.74173E-07	2.48061E-06	-1.27493E-05	-1.18736E-05	-3.65333E-05
	ge_sinh2	-4.71654E-08	-1.34746E-05	-7.90056E-04	-6.91772E-03	-1.56041E-02	-2.05425E-02
	reference	-0.04759340	-0.04873418	-0.04921692	-0.04927009	-0.04927546	-0.04927599
$\phi_3 u^*$	ge_sinh1	8.87921E-08	-1.18303E-07	4.65970E-06	-1.14517E-06	-6.36069E-05	-2.27678E-04
	ge_sinh2	2.30801E-08	-1.63890E-05	-2.18980E-03	-2.25169E-02	-7.23486E-02	-1.39055E-01

second iteration, especially at small relative distance values. A possible explanation is that the second sinh transformation unnecessarily clustered too many integration points around the nearly singular point, resulting in insufficient evaluation of the integrand away from this point. Thus, one iteration of sinh transformation is sufficient in this case.

5.3 Nearly strong singular integral

The derived fundamental solutions of u_k^* and q^* are characterized with nearly strong singular kernel of r^{-1} , as described in the integral II in Eq. (7) with m being 1. u_k^* and q^* multiplied with shape function ϕ_2 are taken as the integrand. Results of the relative errors for the one and two iterations of sinh transformation using the general distance function are given in Table 2. It can be concluded that both one and two iterations of sinh transformation can get acceptable results. For most cases, the results are of the same accuracy. This fact indicates that one iteration would be sufficient for the evaluation of nearly strong singular integrals.

5.4 Nearly hyper-singular integral

Now consider the evaluating of the hyper-singular integral II in Eq. (7) as $m = 2$. Note that the spike of the integrand is considerably narrower and higher than in the previous case, suggesting that more integration points should be clustered around the nearly singular point. The basis function $f(\xi)$ is taken as 1 and ϕ_i, respectively. Table 3 shows the relative errors for the one and two iterations of sinh transformation using the general distance function. It can be observed that while one iteration sinh transformation remarkably improves the numerical evaluation of the integral, a second iteration improves the numerical evaluation again by several orders of magnitude. The improvement is often more dramatic at smaller values of relative distance. Therefore, the second iteration sinh transformation is a better choice for evaluation of nearly hyper-singular integrals.

5.5 Comparison of various variable transformation methods

Results obtained by the sinh transformation method are compared with the transformation methods of Ma [Ma and Kamiya (2002a)] and Zhang [Zhang, Gu, Chen, et al. (2009)]. Comparisons are made for $\log(1/r)$, r^{-1} and r^{-2} type kernels multiplied by ϕ_1. It should be noted that the transformations of Ma and Zhang split the interval of the integration into two subintervals at the nearly singular point. Hence in order to achieve a fair comparison in term of computational load, only half of the indicated number of Gaussian nodes is used in each subinterval, giving the same total number of function evaluations for all methods. The relative errors of results obtained by different transformation methods are given in Table 4. For

Table 2: The relative errors of nearly singular integrals with different orders using conventional and general distance transformation

r_0/l	Methods	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
	reference	-0.34076743	-0.48003434	-0.49842677	-0.50032007	-0.50050995	-0.50048047
$\phi_2 u_1^*$	ge_sinh1	-2.14616E-08	1.51910E-06	2.00999E-04	2.08438E-04	-4.27329E-03	-1.86032E-02
	ge_sinh2	-1.20730E-08	6.95445E-06	-3.25716E-04	-6.31734E-04	3.43986E-03	1.18196E-02
	reference	-0.15040401	-0.16390054	-0.16414734	-0.16415235	-0.16415264	-0.16409556
$\phi_2 u_2^*$	ge_sinh1	-2.10561E-08	-1.05629E-05	5.07570E-04	8.10884E-03	3.02879E-02	6.26996E-02
	ge_sinh2	-2.33897E-08	-1.35518E-05	-1.11944E-05	5.18541E-03	1.17203E-02	1.37284E-02
	reference	0.19811938	0.25327993	0.26009706	0.26079347	0.26086326	0.26079558
$\phi_2 q^*$	ge_sinh1	-4.83766E-08	-2.86112E-06	4.99852E-04	3.86525E-03	7.66829E-03	2.92658E-03
	ge_sinh2	-5.82992E-08	-8.05325E-06	2.89079E-04	1.27422E-03	-8.15975E-04	-9.72356E-03

Table 3: The relative errors of nearly singular integrals with different orders using conventional and general distance transformation

r_0/l	Methods	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$1/r^2$	ge_sinh1	-2.28485E-08	-2.06753E-06	3.47015E-04	2.69613E-03	5.35234E-03	2.01682E-03
	ge_sinh2	-2.10589E-08	-3.86034E-08	-1.68154E-07	-1.28294E-07	-1.52890E-08	1.99119E-04
ϕ_1/r^2	ge_sinh1	-2.23142E-08	-2.31350E-06	3.47023E-04	2.69742E-03	5.35304E-03	2.01700E-03
	ge_sinh2	-2.19713E-08	-1.49003E-07	4.03191E-07	1.33724E-07	-6.67664E-08	1.99095E-04
ϕ_2/r^2	ge_sinh1	-2.94859E-08	-1.96577E-06	3.47361E-04	2.69576E-03	5.35205E-03	2.01673E-03
	ge_sinh2	-2.56522E-08	3.44666E-08	-3.44192E-07	-4.62417E-07	-5.05842E-08	1.99126E-04
ϕ_3/r^2	ge_sinh1	-9.24039E-08	-2.20371E-06	3.49128E-04	2.69776E-03	5.35266E-03	2.01683E-03
	ge_sinh2	-7.63100E-08	7.36528E-08	4.93560E-07	-1.34760E-06	-3.81512E-07	1.99086E-04

Table 4: Comparison of various variable transformation methods with different orders of kernels

r_0/l	Methods	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$\phi_1 u^*$	Ma's	-1.54295E-07	5.97917E-06	1.54269E-04	1.28800E-03	3.37731E-03	5.19860E-03
	Zhang's	5.29227E-08	3.65865E-06	5.21903E-05	8.07333E-04	2.71862E-03	4.73709E-03
	ge_sinh1	-5.50238E-09	7.00528E-08	-4.91681E-07	5.16788E-06	-3.50909E-06	7.02376E-05
	ge_sinh2	-3.21283E-09	7.48015E-06	6.24399E-04	3.80387E-03	6.70809E-03	1.04164E-02
$\phi_1 q^*$	Ma's	-1.61302E-08	4.27025E-06	-5.93319E-05	-2.86087E-04	-3.77568E-04	2.02984E-04
	Zhang's	-2.27208E-07	-9.62095E-06	-1.47263E-04	1.91577E-03	1.71991E-03	-4.79344E-03
	ge_sinh1	-1.89265E-08	-2.48429E-06	3.72181E-04	2.89635E-03	5.74252E-03	2.12330E-03
	ge_sinh2	-1.58334E-08	2.67390E-06	-1.59669E-04	-3.14368E-04	1.65654E-03	7.25416E-03
ϕ_1/r^2	Ma's	-2.19956E-08	-2.65716E-07	-2.31089E-06	1.77922E-05	9.65704E-05	2.98464E-04
	Zhang's	-1.95143E-07	-1.13830E-05	-1.19232E-04	1.99110E-03	2.03762E-03	-4.18565E-03
	ge_sinh1	-2.23142E-08	-2.31350E-06	3.47023E-04	2.69742E-03	5.35304E-03	2.01700E-03
	ge_sinh2	-2.19713E-08	-1.49003E-07	4.03191E-07	1.33724E-07	-6.67664E-08	1.99095E-04

the nearly weak and hyper- singular integrals, the sinh transformation is superior to the methods of Ma and Zhang by several orders of magnitude. The relative error for sinh transformation decreases less rapidly than that of other methods as the relative distance decreases. For the nearly strong singular integrals, both one and two iterations of sinh transformation can get acceptable result, but not as accurate as that of Ma's.

6 Conclusions

In this paper, a general distance function independent on the nearly singular point is proposed. The position of the nearly singular point can be obtained more easily combined with an iteration process. Then, an extended form of the sinh transformation using the general distance function is developed to deal with various orders of singularities. The iterated sinh transformation is also implemented in a straightforward fashion. A representative curved boundary element is chosen to validate the efficiency and accuracy of presented method.

Some significant conclusions have been obtained from the numerical results. For nearly weak singular integrals, one iteration of sinh transformation leads to more accurate results than that of the second iteration, especially at small relative distance values. Acceptable results of the same accuracy can be obtained using both one and two iterations of sinh transformation. For nearly hyper-singular integrals, while one iteration sinh transformation remarkably improves the numerical evaluation of the integral, a second iteration improves the numerical evaluation again by several orders of magnitude. Comparisons with the transformations of Ma and Zhang prove that the extended sinh transformation is a superior choice to deal with the nearly singular integrals.

The proposed method also offers great advantages in the numerical evaluation of the nearly singular integrals in 3D BEM. Subsequent work is already underway and will be reported in the future papers.

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