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The Second Kind Chebyshev Wavelet Method for Fractional Differential Equations with Variable Coefficients

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Abstract: In this article, the second kind Chebyshev wavelet method is presented for solving a class of multi-order fractional differential equations (FDEs) with variable coefficients. We first construct the second kind Chebyshev wavelet, prove its convergence and then derive the operational matrix of fractional integration of the second kind Chebyshev wavelet. The operational matrix of fractional integration is utilized to reduce the fractional differential equations to a system of algebraic equations. In addition, illustrative examples are presented to demonstrate the efficiency and accuracy of the proposed method.

Keywords: Fractional calculus, the second kind Chebyshev wavelet, operational matrix, fractional differential equations, Block Pulse Function.

1 Introduction

During the past decades, the field of fractional differential equations has attracted the interest of researchers in several areas including physics, chemistry, engineering and even finance and social sciences [Garrappa and Popolizio (2011); Podlubny(1999)] and there has been significant in developing numerical schemes for their solution. These methods include Laplace transforms[Podlubny(1999)], Fourier transforms[Gaul, Klein and Kemple(1991)], eigenvector expansion [Suarez and Shokooh(1997)], Adomian decomposition method [Momani(2007); Jafari and Seifi (2009)], Variational Iteration Method [Sweilam, Khader and Al-Bar(2007); Das (2009)], Fractional Differential Transform Method [Arikoglu and Ozkol (2009); Erturk, Momani and Odibat (2008)], Fractional Difference Method [Meerschaert and Tadjeran (2006)]and Power Series Method[Odibat and Shawagfeh (2007)]. But, few papers reported application of wavelet to solve the fractional order differential

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equations[Wu (2009); Lepik (2009); Wei, Chen, Li, and Yi (2012); Zhou, Wang, Wang and Liu (2011)].

In view of successful application of wavelet operational matrix in system analysis[Chen and Hsiao(1997); Bujurke, Salimath and Shiralashetti(2008)], system identification[Karimi, Lohmann, Maralani and Moshiri (2004); Pawlak and Hasiewicz (1998)], optimal control[Hsiao and Wang (1999); Karimi, Moshiri, Lohmann and Maralani (2005); Sadek, Abualrub and Abukhaled (2007)] and numerical solution of integral and differential equations[Bujurke, Shiralashetti and Salimath (2009); Babolian and Masouri (2009); Kajani and Vencheh(2008); Reihani and Abadi (2007); Khellat and Yousefi (2006); Razzaghi and Yousefi (2001)], together with the characteristic of wavelet functions, we hold that they should be applicable to solve the fractional order systems. Although traditional computational methods such as finite element and boundary element[Dong and Atluri (2011); Yao (2009)] are widely applied to numerical solutions of differential equations, they are not suitable for fractional differential equations with variable coefficients.

My purpose is to introduce the second kind Chebyshev wavelet method to solve a class of multi-order arbitrary differential equations with variable coefficients. First, we construct the second kind Chebyshev wavelet and derive the operational matrix of fractional integration; Then the underlying fractional differential equation is converted into a fractional integral equation via fractional integration; subsequently, the various signals involved in the fractional integral equation are approximated by representing them as linear combinations of the wavelet functions and truncating them at optimal levels; Fnally, the integral equation is converted to an algebraic equation by introducing the wavelet operational matrix of the fractional integration. In this paper, by using the second kind Chebyshev wavelet, we solve numerically the following multi-order fractional differential equations (FDEs) with variable coefficients(called the problem 1)

$$D_*^{v}u(t)+\sum_{i=1}^{r-1}\gamma_i(t)D_*^{\beta_i}u(t)+\gamma_r(t)u(t)=g(t),t\in[0,1) \tag{1}$$

$$u^{(i)}(0)=d_i,\quad i=0,1,\cdots,n-1 \tag{2}$$

where $0<\beta_1<\beta_2<\cdots<\beta_{r-1}<v$, $n-1<v\leq n$ are constants. Moreover, D_*^{v} denotes the Caputo fractional derivative of order v and the values of $d_i(i=0,1,\cdots,n-1)$ describe the initial state of $u(t)$ and $g(t)$ is a given source function. The existence and uniqueness of solutions of FDEs have been studied by [Deng and Ma (2010)].

The paper is organized as follows. In Section2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus. In Section3, after describing the basic formulation of wavelets and the second kind Chebyshev wavelet, we derive convergence of the second kind Chebyshev wavelet and the second kind Chebyshev wavelet operational matrix of the fractional differential equation. In Section4, the proposed method is applied to some examples. Also a conclusion is given in Section5.

2 Definitions and notations

We give some necessary definitions and mathematical preliminaries of the fractional calculus theory which are used further in this paper.

Definition1.The Riemann–Liouville fractional integral operator $I^{\alpha}$of ordera $\alpha > 0$ on usual Lebesgue space $L_1[a,b]$ is given by

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, t > a, \tag{3}$$

$$I^0 f(t) = f(t) \tag{4}$$

and its fractional derivative of order$\alpha > 0$ is normally used:

$$D^{\alpha}f(t) = \frac{d^n}{dt^n}(I^{n-\alpha}f(t)), \quad n-1 < \alpha \leq n, \tag{5}$$

Where $n$is an integer. For Riemann–Liouvilles definition, one has

$$I^{\alpha}t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}t^{\alpha+\nu} \tag{6}$$

The Riemann–Liouville derivation have certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differential operator D_*^{α} proposed by Caputo.

Definition2. The Caputo definition of fractional differential operator is given by

$$D_*^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, n-1 < \alpha \leq n, \tag{7}$$

Where n is an integer.

It has the following two basic properties for $n-1 < \alpha \leq n$ and $f \in L_1[a,b]$

$$D_*^{\alpha}I^{\alpha}f(t) = f(t) \tag{8}$$

and

$$I^{\alpha}D_*^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{t^k}{k!}, t > a \tag{9}$$

3 Convergence of the second kind Chebyshev wavelet and the second kind Chebyshev wavelet operational matrix of the fractional integration

In this section, we use the second kind Chebyshev polynomial to construct the second kind Chebyshev wavelet and give some properties of this wavelet.

3.1 The second kind Chebyshev wavelet

Wavelets are a family of functions constructed from dilation and translation of a single function $\psi(t)$ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [Kajani and Vencheh (2008)]

$$\psi_{ab}(t) = |a|^{-\frac{1}{2}}\,\psi(\frac{t-b}{a}),\ a,b \in R, a \neq 0 \tag{10}$$

If we restrict the parametersaandbto discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 0$, $b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{kn}(t) = |a_0|^{\frac{k}{2}}\,\psi(a_0^k t - nb_0), \tag{11}$$

Where ψ_{kn} from a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ *and* $b_0 = 1$, ψ_{kn} forms an orthogonal basis.

The second Chebyshev wavelet $\psi_{nm}(t) = \psi(k,n,m,t)$ involve four arguments, $n = 1,2,\cdots,2^{k-1}$, k is assumed any positive integer, m is the degree of the second kind Chebyshev polynomials and $t$is the normalized time. They are defined on the interval $[0,1)$

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{U}_m(2^{\frac{k}{2}}t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0, & otherwise, \end{cases} \tag{12}$$

Where

$$\tilde{U}_m(t) = \sqrt{\frac{2}{\pi}}U_m(t) \tag{13}$$

and $m = 0,1,\cdots,M-1$. Here $U_m(t)$ are the second kind Chebyshev polynomials of degree m which respect to the weight function $\omega(t) = \sqrt{1-t^2}$ on interval $[-1,1]$ and satisfy the following recursive formula

$$U_0(t) = 1, U_1(t) = 2t, U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), m = 1,2,\cdots. \tag{14}$$

A function $f(t)$ defined on the interval $[0,1)$ may be expanded as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(t), \tag{15}$$

Where

$$c_{nm}=(f(t),\psi_{nm}(t))_{\omega_n}=\int_0^1\omega_n(t)\psi_{nm}(t)f(t)dt$$

If the infinite series in Eq.(15) is truncated, then Eq.(15) can be written as

$$f(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=C^T\Psi(t) \tag{16}$$

where T indicates transposition, C and $\Psi(t)$ are $2^{k-1}M\times 1$ matrices given by

$$C=[c_{10},c_{11},\cdots,c_{1(M-1)},c_{20},\cdots,c_{2(M-1)},\cdots,c_{2^{k-1}0},\cdots,c_{2^{k-1}(M-1)}]^T$$

$$\Psi(t)=[\psi_{10},\psi_{11},\cdots,\psi_{1(M-1)},\psi_{20},\cdots,\psi_{2(M-1)},\cdots,\psi_{2^{k-1}0},\cdots,\psi_{2^{k-1}(M-1)}]^T \tag{17}$$

Taking the collocation points as following:

$$t_i=\frac{2i-1}{2^kM},i=1,2,\cdots,2^{k-1}M, \tag{18}$$

We define the second Chebyshev wavelet matrix $\Phi_{m'\times m'}$ as

$$\Phi_{m'\times m'}=[\Psi(t_1),\Psi(t_2),\cdots,\Psi(t_{m'})] \tag{19}$$

where $m'=2^{k-1}M$.
For example, when $M=3$ and $k=2$ the second Chebyshev wavelet matrix is expressed as

$$\Phi_{6\times 6}=\begin{bmatrix}1.5958 & 1.5958 & 1.5958 & 0 & 0 & 0\\ -2.1277 & 0 & 2.1277 & 0 & 0 & 0\\ 1.2412 & -1.5958 & 1.2412 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.5958 & 1.5958 & 1.5958\\ 0 & 0 & 0 & -2.1277 & 0 & 2.1277\\ 0 & 0 & 0 & 1.2412 & -1.5958 & 1.2412\end{bmatrix}$$

3.2 Convergence of the second kind Chebyshev wavelet bases

Theorem 3.1 Let A function $f(t)$ defined on the interval $[0,1)$ may be expanded as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(t),\quad \tilde{f}(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t),$$

Then $\sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=0}^{M-1}c_{nm}^2\le\int_0^1 f^2(t)dt$

Where

$$c_{nm}=(f(t),\psi_{nm}(t))_{\omega_n}=\int_0^1\omega_n(t)\psi_{nm}(t)f(t)dt.$$

Proof. Since

$$0\le\int_0^1(f(t)-\tilde{f}(t))^2dt=\int_0^1 f^2(t)dt+\int_0^1\tilde{f}^2(t)dt-2\int_0^1 f(t)\tilde{f}(t)dt$$

$$=\int_0^1 f^2(t)dt-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}^2$$

Hence, $\sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=0}^{M-1}c_{nm}^2\le\int_0^1 f^2(t)dt$

Theorem 3.2 Let A function $f(t)$ be $L^2[0,1]$, and

$$R_{K,M}=f(t)-\tilde{f}(t)$$

Then $\lim\limits_{K,M\to\infty}\|R_{K,M}\|=0.$

Where $f(t),\tilde{f}(t)$ be defined as above and $K=2^{k-1}$.

Proof. $\|R_{K,M}\|^2=\int_0^1(f(t)-\tilde{f}(t))^2dt=\sum\limits_{n=2^{k-1}+1}^{\infty}\sum\limits_{m=M}^{\infty}c_{nm}^2$

by **Theorem 5.1,** $\sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=0}^{M-1}c_{nm}^2\le\int_0^1 f^2(t)dt$ Is true for arbitrary natural numbers K, M, and $\int_0^1 f^2(t)dt$ is a limited value, so $\sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{\infty}c_{nm}^2$ is limited, that is, $\sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{\infty}c_{nm}^2$ is convergent.

Then, it exists N,J, when $K>N,M>J$, $\sum\limits_{n=2^{N-1}+1}^{\infty}\sum\limits_{m=J}^{\infty}c_{nm}^2$ may be arbitrary small.

So $\lim\limits_{K,M\to\infty}\|R_{K,M}\|=0$

3.3 Operational matrix of the fractional integration

The integration of the vector $\Psi(t)$ defined in Eq.(17) can be obtained as

$$\int_0^1 \Psi(t)dt \approx \mathrm{P}\Psi(t) \tag{20}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration [Kajani and Vencheh (2008)].

Our purpose is to derive the second kind Chebyshev wavelet operational matrix of the fractional integration. For this purpose, we rewrite Riemann–Liouville fractional integration, as following

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \quad t > 0, \tag{21}$$

Now, if $f(t)$ is expanded in the second Chebyshev wavelets, as showed in Eq.(15), the Riemann–Liouville fractional integration becomes

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t) \approx C^T \frac{1}{\Gamma(\alpha)} \left\{ t^{\alpha-1} * \Psi(t) \right\} \tag{22}$$

Thus if $t^{\alpha-1} * f(t)$ can be integrated, then expanded in the second Chebyshev wavelets, the Riemann–Liouville fractional integration is solved via the Chebyshev wavelets.

Also, we define a m'-set of Block Pulse function(BPF) as

$$b_i(t) = \begin{cases} 1, & i/m' \le t < (i+1)/m', \\ 0, & otherwisw, \end{cases} \tag{23}$$

where $i = 0, 1, 2, \cdots, m'-1$.

The functions $b_i(t)$ are disjoint and orthogonal. That is

$$b_i(t)b_l(t) = \begin{cases} b_i(t), & i = l, \\ 0, & i \ne l. \end{cases} \tag{24}$$

$$\int_0^1 b_i(t)b_l(t) = \begin{cases} 1/m', & i = l, \\ 0, & i \ne l. \end{cases} \tag{25}$$

From the orthogonal property of BPF, it is possible to expand functions into their Block Pulse series, this means that for every $f(t) \in [0,1)$ we can write

$$f(t) \approx \sum_{i=0}^{m-1} f_i b_i(t) = f^T B_{m'}(t) \tag{26}$$

where

$$f^T = [f_0, f_1, \cdots, f_{m'-1}], B_m^T(t) = [b_0(t), b_1(t), \ldots, b_{m'-1}(t)],$$

such that f_i for $i = 0, 1, 2, \cdots, m'-1$ are obtained by $f_i = m' \int_0^1 f(t) b_i(t) dt$.
Similarly, the second Chebyshev wavelet may be expanded into an m'-term block pulse functions (BPF) as

$$\Psi_{m'}(t) = \Phi_{m' \times m'} B_{m'}(t), \tag{27}$$

We derive the Block Pulse operational matrix of the fractional integration F^α as following

$$I^\alpha B_{m'}(t) = F^\alpha B_{m'}(t), \tag{28}$$

where

$$F^\alpha = \frac{1}{2^\alpha m'^\alpha \Gamma(\alpha+1)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \xi_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

with $\xi_k = (2k+1)^\alpha - (2k-1)^\alpha$, $k = 1, 2, \cdots, m'-1$.
Next, we derive the Chebyshev wavelet operational matrix of the fractional integration. Let

$$I^\alpha \Psi_{m'}(t) \approx \mathrm{P}^\alpha_{m' \times m'} \Psi_{m'}(t) \tag{30}$$

where matrix $\mathrm{P}^\alpha_{m' \times m'}$ is called the second Chebyshev wavelet operational matrix of the fractional integration.
Using Eqs. (27) and (28), we have

$$I^\alpha \Psi_{m'}(t) \approx I^\alpha \Phi_{m' \times m'} B_{m'}(t) = \Phi_{m' \times m'} I^\alpha B_{m'}(t) \approx \Phi_{m' \times m'} F^\alpha B_{m'}(t) \tag{31}$$

From Eqs. (30) and (31) we get

$$\mathrm{P}^\alpha_{m' \times m'} \Psi_{m'}(t) = P^\alpha_{m' \times m'} \Phi_{m' \times m'} B_{m'}(t) = \Phi_{m' \times m'} F^\alpha B_{m'}(t) \tag{32}$$

Then, the second Chebyshev wavelet operational matrix of the fractional integration $\mathrm{P}^\alpha_{m' \times m'}$ is given by

$$P^\alpha_{m' \times m'} = \Phi_{m' \times m'} F^\alpha \Phi^{-1}_{m' \times m'} \tag{33}$$

In particular, for $M = 3$, $k = 2$ and $\alpha = 0.5$ the second Chebyshev wavelet operational matrix of the fractional integration $\mathrm{P}^{\alpha}_{m'\times m'}$ is given by

$$\mathrm{P}^{0.5}_{6\times 6} = \begin{bmatrix} 0.1513 & -0.2077 & -0.1558 & -3.7364 & -1.5403 & -0.0746 \\ 0.2077 & 0.5841 & 0.2077 & 1.8244 & 0.1826 & 0.0033 \\ -0.1212 & -0.1615 & 0.1860 & -0.7450 & -0.2871 & -0.0096 \\ 0 & 0 & 0 & 0.1513 & -0.2077 & -0.1558 \\ 0 & 0 & 0 & 0.2077 & 0.5841 & 0.2077 \\ 0 & 0 & 0 & -0.1212 & -0.1615 & 0.1860 \end{bmatrix}$$

It should be noted that the operational matrix $\mathrm{P}^{\alpha}_{m'\times m'}$ contains many zero entries. This phenomena makes calulations fast. The calculation for the matrix $\mathrm{P}^{\alpha}_{m'\times m'}$ is carried out once and is used to solve fractional order as well as integer order differential equations.

4 Solution of the problem 1

By approximating the function $D^{\nu}_{*}u(t)$, we have

$$D^{\nu}_{*}u(t) \cong C^T\Psi(t), \tag{34}$$

together with the initial states, we get

$$D^{\beta_i}_{*}u(t) \cong C^T P^{\nu-\beta_i}_{m'\times m'}\Psi(t), i = 1,2,\cdots,\nu-1 \tag{35}$$

$$u(t) \cong C^T P^{\nu}_{m'\times m'}\Psi(t) + \sum_{k=0}^{n-1} u^{(k)}(0)\frac{t^k}{k!}. \tag{36}$$

Substituting Eq.(34) ,Eq.(35) and Eq.(36)into Eq.(1),we have

$$C^T\Psi(t) + \sum_{i=1}^{r-1}\gamma_i(t)C^T P^{\nu-\beta_i}_{m'\times m'}\Psi(t) + \gamma_r(t)(C^T P^{\nu}_{m'\times m'}\Psi(t) + \sum_{k=0}^{n-1} u^{(k)}(0)\frac{t^k}{k!}) = g(t) \tag{37}$$

Coefficients $\gamma_i(t)(i = 1,2,\cdots,r)$ can be dispersed into $\gamma_i(t_l)$ and $g(t)$ may be dispersed into $g(t_l)(l = 1,2,\cdots,m')$.

Let

$$R_i = \begin{bmatrix} \gamma_i(t_1) & 0 & \cdots & 0 \\ 0 & \gamma_i(t_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_i(t_{m'}) \end{bmatrix}, (i = 1,2,\cdots,r), \tag{38}$$

$$g = [g(t_1) - \gamma_r(t_1)\sum_{k=0}^{n-1} u^{(k)}(0)\frac{t_1^k}{k!} \quad g(t_2) - \gamma_r(t_2)\sum_{k=0}^{n-1} u^{(k)}(0)\frac{t_2^k}{k!} \cdots g(t_{m'}) - \gamma_r(t_{m'})\sum_{k=0}^{n-1} u^{(k)}(0)\frac{t_{m'}^k}{k!}]^T$$

Therefore, Eq.(37) can be written as

$$C^T(\Phi_{m'\times m'} + \sum_{i=1}^{r} P_{m'\times m'}^{\nu-\beta_i}\Phi_{m'\times m'}R_i) = g^T \tag{39}$$

where $\Phi_{m'\times m'} = [\Psi(t_1),\, \Psi(t_2), \cdots, \Psi(t_{m'})]$. Eq.(39) is a linear system of algebraic equations.

5 Numerical examples

In this section, three examples are considered aiming to illustrate how one can apply the proposed algorithm presented in the previous section.

Example 1. Consider the equation, see [Deng and Ma (2010)]

$$aD_*^2u(t) + b(t)D_*^{\nu_2}u(t) + c(t)Du(t) + e(t)D_*^{\nu_1}u(t) + k(t)u(t) = f(t), \quad 0 < \nu_1 < 1, 1 < \nu_2 < 2 \tag{40}$$

where $f(t) = -a - \frac{b(t)}{\Gamma(3-\nu_2)}t^{2-\nu_2} - c(t)t - \frac{e(t)}{\Gamma(3-\nu_1)}t^{2-\nu_1} + k(t)(2 - \frac{1}{2}t^2)$, and $u(0) = 2$, $u'(0) = 0$.

The analytic solution of this problem is $u(t) = 2 - \frac{1}{2}t^2$. The maximum absolute error achieved in [Deng and Ma (2010)], with 1000 steps, is 4.39×10^{-5}, while the maximum absolute error using the second Chebyshev wavelet method is0and the comparison between the second Chebyshev wavelet method and the exact solution is presented in **Fig.1-Fig.4** for $M = 3$ and different values of k.

Example 2. Consider the equation

$$D^2u(t) + \sin(t)D_*^{\frac{1}{2}}u(t) + tu(t) = f(t), u(0) = u'(0) = 0$$

and

$$f(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin(t)(\frac{32768}{6435}t^{\frac{15}{2}} - \frac{2048}{429}t^{\frac{13}{2}})$$

One can easily check that $u(t) = t^8 - t^7$ is the unique analytical solution.

In **Table 1**, we list the absolute errors for $M = 3$ and different values of k and Ref.[Li and Zhao (2010)]. From Tables 1, we can achieve a better approximation with the exact solution by making use of second Chebyshev wavelets method then Ref.[Li and Zhao (2010)]. We may also see that the error is smaller and smaller when

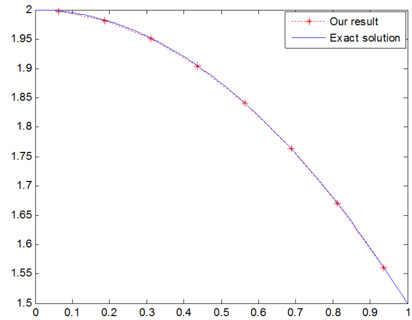

Figure 1: Comparison of Num. sol. and Exa. Sol. of $k = 3, M = 3$

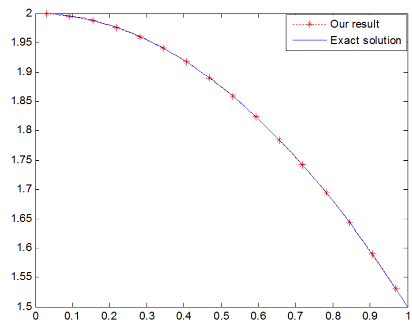

Figure 2: Comparison of Num. sol. and Exa. Sol. of $k = 4, M = 3$

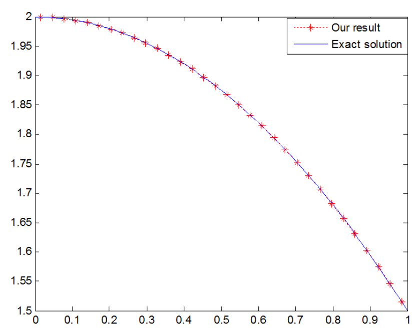

Figure 3: Comparison of Num. sol. and Exa. Sol. of $k = 5, M = 3$

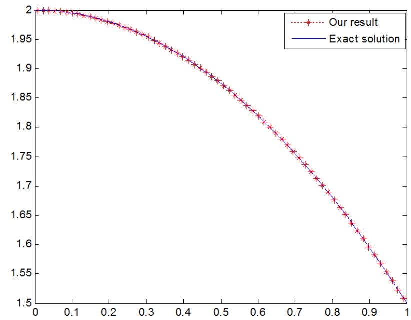

Figure 4: Comparison of Num. sol. and Exa. Sol. of $k = 6, M = 3$

k increases. Therefore for better results, using a larger k is recommended. The computational results show that the method in this article can be effectively used in numerical calculus for fractional differential equation with variable coefficient, and the method is also feasibility to the realistic fractional differential equation.

Table 1: The absolute errors for$M = 3$and different values of$k$

t	$k=3,M=3$		$k=4,M=3$		$k=5,M=3$	
	ours	Ref. [Li and Zhao(2010)]	ours	Ref. [Li and Zhao2010]	ours	Ref. [Li and Zhao2010]
0.0625	6.8166e-008	9.2040e-008	5.5745e-010	7.4471e-010	4.4517e-012	5.9848e-012
0.1875	6.2025e-006	1.2422e-005	6.4280e-007	8.3627e-007	8.3173e-008	5.9025e-007
0.3125	5.7883e-005	4.6183e-007	8.1457e-006	3.9354e-007	1.5545e-006	1.8749e-007
0.4375	1.9796e-004	8.8140e-004	3.8396e-005	8.1461e-004	9.6917e-006	8.0265e-004
0.5625	4.1838e-004	5.7552e-003	1.1637e-004	5.1642e-003	4.4897e-005	4.5698e-003
0.6875	5.2079e-004	2.0001e-002	2.9048e-004	1.7360e-002	1.8488e-004	1.5707e-002
0.8125	1.1363e-004	4.5151e-002	6.4027e-004	3.8760e-002	5.2730e-004	3.6057e-002
0.9375	3.1379e-003	5.9717e-002	1.0678e-003	5.0103e-002	8.3468e-004	4.9362e-002

Example 3. Consider this equation

$$D^{\alpha}u(t) + (e^t + t)u(t) = e^{2t} + te^t - t$$

such that $u(0) = 0$. The exact solution of this equation for $\alpha = 1$ is given by $u(t) = e^t - 1$. We applied the second Chebyshev wavelet approach to solve this problem with $k = 5$, $M = 3$ for various values of α. It is evident from the Fig. 5 that, as α close to 1, the numerical solution by the second Chebyshev wavelet, converge to the exact solution, i.e. the solution of fractional differential equation approaches to the solution of integer order differential equation.

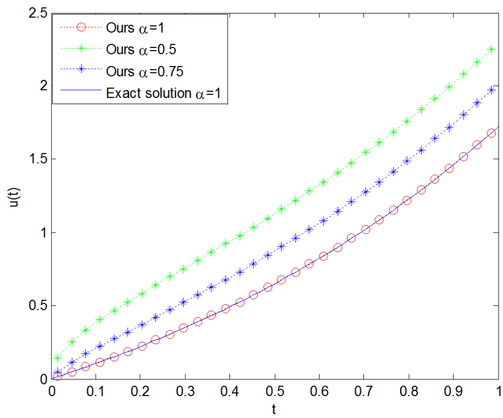

Figure 5: Numerical and exact solution for $\alpha = 1$of Example 3

6 Conclusion

In this paper, we described the second Chebyshev wavelet method for multi-term FDEs with variable coefficients. We derive the SCW operational matrix of fractional order integration and use the wavelet basis together with operational matrix to reduce the factional differential equation to a system of algebraic equations. The matrix elements of the discrete operators are provided explicitly, and this in turn greatly simplifies the steps for obtaining solutions. Three examples are given to demonstrate that the method is effective and accurate for solving multi-term FDEs with variable coefficients.

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