 CMES, vol.93, no.3, pp.203-220, 2013

Cauchy Problem for the Laplace Equation in 2D and 3D Doubly Connected Domains

Ji-Chuan Liu[1], **Quan-Guo Zhang**[2]

Abstract: In this paper, we propose an algorithm to solve a Cauchy problem of the Laplace equation in doubly connected domains for 2D and 3D cases in which the Cauchy data are given on the outer boundary. We want to seek a solution in the form of the single-layer potential and discrete it by parametrization to yield an ill-conditioned system of algebraic equations. Then we apply the Tikhonov regularization method to solve this ill-posed problem and obtain a stable numerical solution. Based on the regularization parameter chosen suitably by GCV criterion, the proposed method can get the approximate temperature and heat flux on the inner boundary. Numerical examples illustrate that the proposed method is reasonable and feasible.

Keywords: Cauchy problem, Laplace equation, Integral equations, Tikhonov regularization method, GCV.

1 Introduction

In many industrial applications one wants to determine the temperature and heat flux on the inner surface of a body in a doubly connected domain where the inner surface itself is inaccessible for measurements. In this paper, our goal is to recover the temperature and heat flux on the inner boundary from the Cauchy data given on the outer boundary in two-dimensional (2D) and three-dimensional (3D) domains. The Cauchy problem for the Laplace equation arises from many physical and engineering problems such as nondestructive testing techniques [Alessandrini (1993); Cheng, Prößdorf, and Yamamoto (1998)], geophysics [Lavrentev, Romanov, and Shishatskii (1986)], semiology [Vani and Avudainayagam (2002)] and cardiology [Colli-Franzone, Guerri, Tentoni, Viganotti, Baruffi, Spaggiari, and Taccardi

[1] Corresponding author. Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, PR China. Email: liujichuan2003@126.com
[2] School of mathematical sciences, Luoyang Normal University, Luoyang, Henan 471000, PR China. Email: zhangqg07@163.com

(1985)]. This kind of Cauchy problem is severely ill-posed [Belgacem (2007)]. That is the solution (if it exists) does not depend continuously on the Cauchy data. Any small errors in the given data might induce large errors in the solution. In order to obtain a stable numerical solution, several numerical methods have been developed such as the quasi-reversibility method [Klibanov and Santosa (1991)], the Tikhonov regularization method [Ang, Nghia, and Tam (1998)], moment method [Cheng, Hon, Wei, and Yamamoto (2001)], Fourier regularization method [Berntsson and Eldén (2001)], Trefftz Method [Chan, Fan, and Yeih (2011); Dong and Atluri (2012)], etc.

Liu [Liu (2008)] considered an inverse Cauchy problem of Laplace equation in simply and doubly connected plane domains by recovering the unknown boundary value on an inaccessible part of a non-circular contour from over-specified data. Háo and Lesnic [Háo and Lesnic (2000)] applied the conjugate gradient method to solve a Cauchy problem for the Laplace equation in doubly-connected domain. Berntsson and Eldén [Berntsson and Eldén (2001)] considered a standard approach that is to discretize the differential equation by finite differences to obtain the temperature and heat flux on the interior boundary in doubly-connected domain. Based on boundary integral equation, Chapko and Johansson [Chapko and Johansson (2008a); Chapko and Johansson (2009)] proposed some alternating iterative methods to determine the temperature field on the boundary of the inclusion in a quadrant or a semi-infinite regions. Chi, Yeih and Liu [Chi, Yeih, and Liu (2009)] employed the fictitious time integration method to solve the Cauchy problem of Laplace equation. Chapko and Johansson [Chapko and Johansson (2008b)] used an alternating iterative method which involves solving direct mixed problems for the Laplace operator to reconstruct the solution on the cut in a simply connected domain. Marin [Marin (2009)] proposed the iterative MFS algorithm to solve the Cauchy problem for the Laplace equation.

There are only very few works for 3D domains, and high-dimensional problems for the Cauchy problem are far more difficult. In this paper, we propose an algorithm to reconstruct the temperature and heat flux on the interior boundary in doubly connected domains for 2D and 3D cases. We seek the solution in the form of a single-layer potential and obtain a system of integral equations in terms of Cauchy data. Then we parameterize the boundary of the domain, discrete the integral equations and transform the Cauchy problem into an ill-posed system of algebraic equations. Based on the Tikhonov regularization method and Generalized Cross-Validation (GCV), the algebraic equations can be solved stably. Finally, we can recover the temperature and heat flux on the interior boundary.

The outline of the paper is as follows. In Section 2, we introduce the Cauchy problem and integral equations. We introduce parameterized integral equations and Tikhonov regularization method in Section 3. Several numerical experiments are presented in Section 4 to illustrate the efficiency of the proposed method. In Section 5 we give some concluding remarks.

2 Formulation of the Cauchy problem and integral equations

Let Ω be a doubly connected domain in $\mathbb{R}^2$ and $\mathbb{R}^3$, and $\partial\Omega$ is a smooth boundary which consists of two disjoint closed curves or curved surfaces Γ_0 and Γ_1. The Γ_0 is contained in the interior of Γ_1 and ν is the outward unit normal vector of the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$.

We consider the Cauchy problem for the Laplace equation in the following

$$\triangle u = 0, \quad \text{in} \quad \Omega, \tag{1}$$

$$u = f, \quad \text{on} \quad \Gamma_1, \tag{2}$$

$$\frac{\partial u}{\partial \nu} = g, \quad \text{on} \quad \Gamma_1, \tag{3}$$

where u is the temperature distribution, $f \in H^{1/2}(\Gamma_1)$ and $g \in H^{-1/2}(\Gamma_1)$ are the given functions.

Theorem 2.1 Assume that $\Omega \subset \mathbb{R}^n$ is C^2 and $n \geq 2$, then the problem (1)-(3) exists a unique solution in $H^1(\Omega)$ for (f,g) in a dense subset of $H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1)$.

The proof of uniqueness is based on Holmgren's theorem refer to [John (1982)] and see for the classical solution [Calderon (1958)]. The proof of existence is technical and follows that of [Belgacem (2007); Belgacem, Du, and Jelassi (2011); Belgacem and Fekih (2005)].

In this paper, we want to recover the temperature and heat flux on the interior boundary Γ_0 from the Cauchy data f and g given on the exterior boundary Γ_1. In practical applications, we can get the measurement data $f^\delta \in L^2(\Gamma_1)$ and $g^\delta \in L^2(\Gamma_1)$ which are approximate functions of f and g, satisfying

$$||f^\delta - f||_{L^2(\Gamma_1)} \leq \delta, \quad ||g^\delta - g||_{L^2(\Gamma_1)} \leq \delta, \tag{4}$$

where $||\cdot||_{L^2(\Gamma_1)}$ denotes L^2 norm on the exterior boundary and the constant $\delta > 0$ represents a noisy level.

We seek the solution of the Cauchy problem (1)-(3) in form of a single-layer potential

$$u(x) = \int_{\partial\Omega} \Phi(x,y)\varphi(y)ds(y), \quad x \in \overline{\Omega}, \tag{5}$$

where Φ is the fundament solution for the Laplace equation as follows

$$\Phi(x,y) = \begin{cases} \frac{1}{2\pi}\ln\frac{1}{|x-y|}, & x \neq y, \quad x,y \in \mathbb{R}^2, \\ \frac{1}{4\pi|x-y|}, & x \neq y, \quad x,y \in \mathbb{R}^3, \end{cases}$$

and $\varphi \in L^2(\partial\Omega)$ is an unknown density. Eq.(5) is equivalent to the following form

$$u(x) = \int_{\Gamma_0} \Phi(x,y)\varphi_0(y)ds(y) + \int_{\Gamma_1} \Phi(x,y)\varphi_1(y)ds(y), \quad x \in \overline{\Omega}, \tag{6}$$

where φ_0 and φ_1 are the unknown density on the boundary Γ_0 and Γ_1, respectively. In terms of boundary conditions (2)-(3), the harmonic function u solves the Cauchy problem provided that the density φ_0 and φ_1 are the solutions of the following system of integral equations

$$\int_{\Gamma_0} \Phi(x,y)\varphi_0(y)ds(y) + \int_{\Gamma_1} \Phi(x,y)\varphi_1(y)ds(y) = f(x), \quad x \in \Gamma_1, \tag{7}$$

$$\int_{\Gamma_0} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi_0(y)ds(y) + \int_{\Gamma_1} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi_1(y)ds(y) + \frac{1}{2}\varphi_1(x) = g(x), \quad x \in \Gamma_1. \tag{8}$$

Refer to [Ivanyshyn and Kress (2006)], we know that integral equations (7) and (8) are ill-posed due to their weak singular kernel and smooth kernel. Therefore, we should apply a regularization method to solve the system of integral equations to determine the density φ_0 and φ_1.

According to the density φ_0 and φ_1, we can get the temperature and heat flux on the interior boundary Γ_0 from (6), namely

$$u(x) = \int_{\Gamma_0} \Phi(x,y)\varphi_0(y)ds(y) + \int_{\Gamma_1} \Phi(x,y)\varphi_1(y)ds(y), \quad x \in \Gamma_0, \tag{9}$$

$$\frac{\partial u}{\partial\nu}(x) = \int_{\Gamma_0} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi_0(y)ds(y) + \frac{1}{2}\varphi_0(x) + \int_{\Gamma_1} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi_1(y)ds(y), \quad x \in \Gamma_0. \tag{10}$$

For simplicity, we use $u(\Gamma_0)$ and $u^*(\Gamma_0)$ to denote the temperature and heat flux on the interior boundary Γ_0, respectively.

3 Parameterized integral equations and Tikhonov regularization method

For the numerical solution a parametrization is required. There is the similar parameterization formula of integral equations in $\mathbb{R}^2$ and $\mathbb{R}^3$, we take $\mathbb{R}^2$ as an example for statement in the following. Assume that the boundary curves can be parameterized in the form

$$\Gamma_k = \{z_k(t) : t \in [0, 2\pi)\}, \quad k = 0, 1, \tag{11}$$

where $z_k : \mathbb{R} \to \mathbb{R}^2$ are twice continuously differentiable, injective and 2π periodic functions. Furthermore, we assume that the orientations of Γ_k $(k = 0, 1)$ are counter clockwise. Thus exterior normal vectors to Γ_k $(k = 0, 1)$ are give by $\nu(z_k(t)) = (-1)^k(-z'_{k,2}, z'_{k,1})$ $(k = 0, 1)$. For convenience, we restrict to starlike boundary curves with parametrization

$$z_k(t) = r_k(t)(\cos t, \sin t), \quad k = 0, 1, \tag{12}$$

where $r_k : \mathbb{R} \to (0, \infty)$ are 2π periodic smooth functions.

Inserting (12) into the system of integral equations (7)-(8), then we have

$$\begin{aligned} f(t) &= \int_0^{2\pi} \Phi(z_1(t), z_0(\tau))\varphi_0(\tau)|z'_0(\tau)|d\tau \\ &\quad + \int_0^{2\pi} \Phi(z_1(t), z_1(\tau))\varphi_1(\tau)|z'_1(\tau)|d\tau, \end{aligned} \tag{13}$$

$$\begin{aligned} g(t) &= \int_0^{2\pi} \frac{\partial \Phi(z_1(t), z_0(\tau))}{\partial \nu(z_1(t))}\varphi_0(\tau)|z'_0(\tau)|d\tau \\ &\quad + \int_0^{2\pi} \frac{\partial \Phi(z_1(t), z_1(\tau))}{\partial \nu(z_1(t))}\varphi_1(\tau)|z'_1(\tau)|d\tau + \frac{1}{2}\varphi_1(t), \end{aligned} \tag{14}$$

where $\varphi_0(\tau) = \varphi_0(z_0(\tau))$ and $\varphi_1(\tau) = \varphi_1(z_1(\tau))$. For the discretization of the integral equations, we note that the first term on the right hand side of (13) is smooth that the trapezoidal rule can be employed for numerical approximation. However, the second term on the right hand side of (13) has a logarithmic singularity, we deal with the logarithmic singularity as follows

$$2\pi\Phi(z_1(t), z_1(\tau)) = -\ln|\sin\frac{t-\tau}{2}| + \ln\frac{|\sin\frac{t-\tau}{2}|}{|z_1(t) - z_1(\tau)|}, \tag{15}$$

where the second term on the right hand side of (15) is smooth with diagonal values

$$\lim_{\tau \to t} \ln\frac{|\sin\frac{t-\tau}{2}|}{|z_1(t) - z_1(\tau)|} = -\ln 2|z'_1(t)|. \tag{16}$$

Therefore, the well-estimated quadrature rules for logarithmic singularities are available. We can use the Nyström method to approximate integral equations with weakly singular kernels in [Kress (1999)].

For (14), we know that the second term on the right hand side is smooth with the diagonal values given through the limit

$$\lim_{\tau\to t} 2\pi \frac{\partial \Phi(z_1(t),z_1(\tau))}{\partial \nu(z_1(t))} = -\lim_{\tau\to t} \frac{\nu(z_1(t))\cdot[z_1(t)-z_1(\tau)]}{|z_1(t)-z_1(\tau)|^2} = -\frac{\nu(z_1(t))\cdot z_1''(t)}{2|z_1'(t)|^2}. \tag{17}$$

Thus the trapezoidal rule can be employed to the integral equation (14) for numerical approximation.

The interval $[0,2\pi]$ is partitioned as $0=\tau_0<\tau_1<\cdots<\tau_m=2\pi$ and $0=t_0<t_1<\cdots<t_n=2\pi$ where $\tau_i=ih_\tau$ $(i=0,1,\cdots,m)$, $t_j=jh_t$ $(j=0,1,\cdots,n)$ and $h_\tau=\frac{2\pi}{m}$, $h_t=\frac{2\pi}{n}$ are the step sizes. Denoting the discrete vector of $\varphi_k(\tau)$ $(k=1,2)$ as

$$\Psi_k=[\varphi_k(\tau_0),\varphi_k(\tau_1),\cdots,\varphi_k(\tau_{m-1})]^T, \tag{18}$$

and the discrete vectors of $f(t)$ and $g(t)$ as follows

$$F=[f^\delta(t_0),f(t_1),\cdots,f(t_{n-1})]^T, G=[g(t_0),g(t_1),\cdots,g(t_{n-1})]^T. \tag{19}$$

Therefore, we can use the well-estimated quadrature rules and the trapezoidal rule to obtain the system of algebraic equations from the system of integral equations (13) and (14) as follows

$$A_{11}\Psi_1+A_{12}\Psi_2=F, \quad A_{21}\Psi_1+A_{22}\Psi_2=G, \tag{20}$$

where $A_{k,\ell}$ $(k,\ell=1,2)$ are $n\times m$ matrices. We rewrite the system of algebraic equations (20) in a matrix equation

$$A\Psi=b, \tag{21}$$

where

$$A=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \Psi=[\Psi_1,\Psi_2]^T, b=[F,G]^T.$$

Similarly, for 3D case, we have

$$f(\theta,t) = \int_0^\pi\int_0^{2\pi} \Phi(z_1(\theta,t),z_0(\phi,\tau))\varphi_0(\phi,\tau)|z_0'(\phi,\tau)|d\phi d\tau + \int_0^\pi\int_0^{2\pi} \Phi(z_1(\theta,t),z_1(\phi,\tau))\varphi_1(\phi,\tau)|z_1'(\phi,\tau)|d\phi d\tau, \tag{22}$$

$$
\begin{aligned}
g(\theta,t) &= \int_0^{\pi}\int_0^{2\pi} \frac{\partial\Phi(z_1(\theta,t),z_0(\phi,\tau))}{\partial\nu(z_1(\theta,t))}\varphi_0(\phi,\tau)|z_0'(\phi,\tau)|d\phi d\tau \\
&+\int_0^{\pi}\int_0^{2\pi} \frac{\partial\Phi(z_1(\theta,t),z_1(\phi,\tau))}{\partial\nu(z_1(\theta,t))}\varphi_1(\phi,\tau)|z_1'(\phi,\tau)|d\phi d\tau+\frac{1}{2}\varphi_1(\theta,t) \\
&\theta\in[0,\pi],t\in[0,2\pi]. \qquad (23)
\end{aligned}
$$

For the discretization of integral equations, we note that the second terms on the right hand side of (22) and (23) have a singularity in 3D domain. Refer to [Beale (2004)], we replace the singular kernel in (22) with

$$
\begin{aligned}
k(\theta,t,\phi,\tau) &= \Phi(z_1(\theta,t),z_1(\phi,\tau))(\varphi_1(\phi,\tau)-\varphi_1(\theta,t)) \\
&+\Phi(z_1(\theta,t),z_1(\phi,\tau))\varphi_1(\theta,t). \qquad (24)
\end{aligned}
$$

For (23), we modify the singular kernel as follows

$$
k(\theta,t,\phi,\tau)=\nu(z_1(\theta,t))\cdot\nabla\Phi(z_1(\theta,t),z_1(\phi,\tau))s(|z_1(\theta,t)-z_1(\phi,\tau)|/\beta), \qquad (25)
$$

where s is a shape factor and β is a smoothing parameter. In terms of [Beale (2004)], we have

$$
s(r)=erf(r)-(2\sqrt{\pi})re^{-r^2},
$$

where erf is the usual error function

$$
erf(r)=\frac{2}{\sqrt{\pi}}\int_0^r e^{-t^2}dt.
$$

For discretization of integral equations (22) and (23), we can get the same the system of algebraic equations as (21) for 3D case.

For noisy Cauchy data, the corresponding linear system of equations (21) is

$$
A\Psi=b^{\delta}, \qquad (26)
$$

where b^{δ} is generated by the noisy data f^{δ} and g^{δ}.

The matrix A in Eq.(26), however, is severely ill-conditioned. Most numerical methods can not directly employ to solve the matrix Eq.(26). In fact, the condition number of the matrix A increases dramatically with respect to the increase of n and m. For illustration, we take $m=n$ so that the matrix A is $2n\times 2n$ square matrix. We show the relation between n and the condition number of the matrix A in the first Example given in the next section in Figure 1.

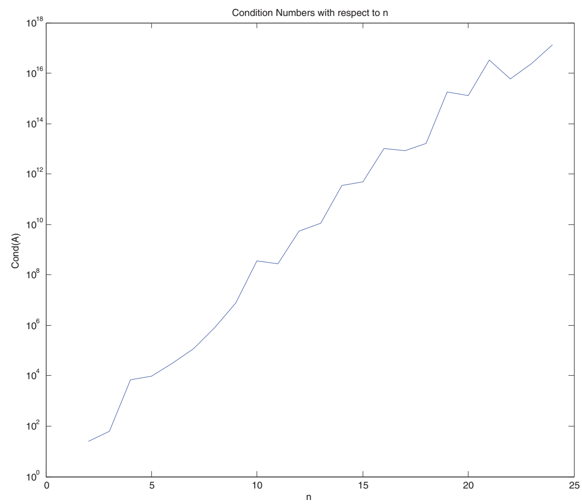

Figure 1: Condition numbers with respect to n.

Thus we should employ a regularization method to solve (26) to eliminate the ill-posedness. The Tikhonov regularization is a popular approach to remedy this difficulty. Instead of solving the linear system (26), we want to seek the solution of the minimization problem

$$\min_{\Psi}\{||A\Psi - b^{\delta}||^2 + \lambda||\Psi||^2\}, \tag{27}$$

where $\lambda \geq 0$ is a regularization parameter.

The determination of a suitable value of the regularization parameter λ is very important and is still under intensive research. In our computations, we use the generalized cross-validation (GCV) [Gene, Heath, and Wahba (1979)] criterion to choose the regularization parameter λ. The basic idea is to find the parameter λ that minimizes the GCV functional

$$G(\lambda) = \frac{||A\Psi_{\lambda}^{\delta} - b^{\delta}||^2}{(trace(I_n - AA^T))^2}, \tag{28}$$

where A^T satisfies $\Psi_{\lambda}^{\delta} = A^T b^{\delta}$.

Denoting the discrete vectors of $u(\Gamma_0)$ and $u^*(\Gamma_0)$ as follows

$$U = [u(t_0), u(t_1), \cdots, u(t_{n-1})]^T, \quad U^* = [\frac{\partial u}{\partial \nu}(t_0), \frac{\partial u}{\partial \nu}(t_1), \cdots, \frac{\partial u}{\partial \nu}(t_{n-1})]^T. \tag{29}$$

By analyzing, we know that we can deal with the weak singular and smooth kernel in (9) and (10) with the same method as (7) and (8), respectively. Using the well-estimated quadrature rules and the trapezoidal rule to the system of integral equations (9) and (10), we have

$$U = B_{11}\Psi_1 + B_{12}\Psi_2, \tag{30}$$

$$U^* = B_{21}\Psi_1 + B_{22}\Psi_2, \tag{31}$$

where $B_{k,\ell}$ $(k,\ell = 1,2)$ are $n \times m$ matrices. Therefore, we can get the computed temperature and heat flux on the interior boundary Γ_0 from (30) and (31) based on the regularization solution Ψ_1 and Ψ_2 given by (27). For 3D case, we can compute the temperature and heat flux on the interior boundary with same expressions as (30) and (31).

4 Numerical experiments

In this section, we test numerical examples to demonstrate the feasible of the proposed approach. In order to check the effect of numerical computations, we compute the root mean square error in the following

$$\varepsilon(u) = \left(\frac{1}{n}\sum_{j=0}^{n-1}(u(t_j) - U(t_j))^2\right)^{\frac{1}{2}}, \quad \varepsilon(u^*) = \left(\frac{1}{n}\sum_{j=0}^{n-1}(u^*(t_j) - U^*(t_j))^2\right)^{\frac{1}{2}},$$

in 2D domain, and

$$\varepsilon(u) = \Big(\frac{1}{n^2}\sum_{i,j=0}^{n-1}(u(\theta_i,t_j) - U(\theta_i,t_j))^2\Big)^{\frac{1}{2}}, \varepsilon(u^*) = \Big(\frac{1}{n^2}\sum_{i,j=0}^{n-1}(u^*(\theta_i,t_j) - U^*(\theta_i,t_j))^2\Big)^{\frac{1}{2}}$$

in 3D domain, where $\{t_j\}$ and $\{(\theta_i,t_j)\}$ are the set of test points on the internal boundary. The noisy Cauchy data are generated by

$$f^\delta = f(1 + \delta \cdot rand(size(f))), \quad g^\delta = g(1 + \delta \cdot rand(size(g))),$$

where f and g are the exact data, $rand(size(f)))$ and $rand(size(g)))$ are a random number uniformly distributed in $[-1,1]$ and the magnitude δ indicates a relative noise level.

For the sake of simplicity, the exterior boundary Γ_1 is chosen to be the unit circle $z_1 = (\cos t, \sin t)$ for 2D case and the unit ball $z_1 = (\sin\theta\cos t, \sin\theta\sin t, \cos\theta)$ for 3D case, $\theta \in [0,\pi], t \in [0,2\pi]$.

Example 1: Let the exact solution for the problem (1)-(3) be

$$u(x,y) = \ln\sqrt{(x-x_p)^2 + (y-y_p)^2} \tag{32}$$

where (x_p, y_p) is a singularity. The interior boundary Γ_0 is a Kite-shaped curve with the parametrization

$$z_0(t) = (0.6\cos t + 0.3\cos 2t - 0.3, 0.6\sin t), \quad t \in [0, 2\pi]. \tag{33}$$

The Cauchy data can be calculated as

$$f(t) = \ln\sqrt{(\cos t - x_p)^2 + (\sin t - y_p)^2},$$
$$g(t) = \frac{1 - x_p \cos t - y_p \sin t}{(\cos t - x_p)^2 + (\sin t - y_p)^2}.$$

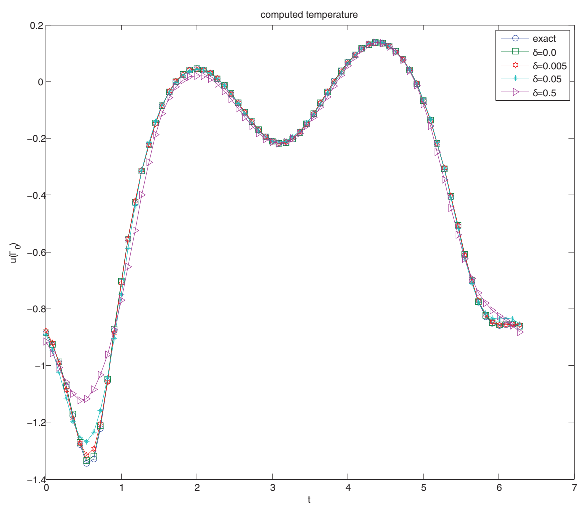

Figure 2: Ex.1. Regularization parameters $\lambda = 6.3151e-9, 0.0014, 0.0064, 0.0384$ for the cases of $\delta = 0, 0.005, 0.05, 0.5$,respectively.

In this example the step size for t is $2\pi/50$ and for τ is $2\pi/50$, and the singularity is $(0.2, 0.1)$. Numerical results for various levels δ of relative noises are shown for the temperature and heat flux on the interior boundary in Figure 2 and Figure 3, respectively. The root mean square errors are $\varepsilon(u) = 0.0027, 0.0067, 0.0206, 0.0592$ and $\varepsilon(u^*) = 7.3e-5, 0.0663, 0.1645, 0.3162$ for the cases of $\delta = 0, 0.005, 0.05, 0.5$, respectively. From Figure 2 and Figure 3, we can see that numerical results are close to the exact one for noise data, and the smaller the relative noise is, the better the approximate solution is.

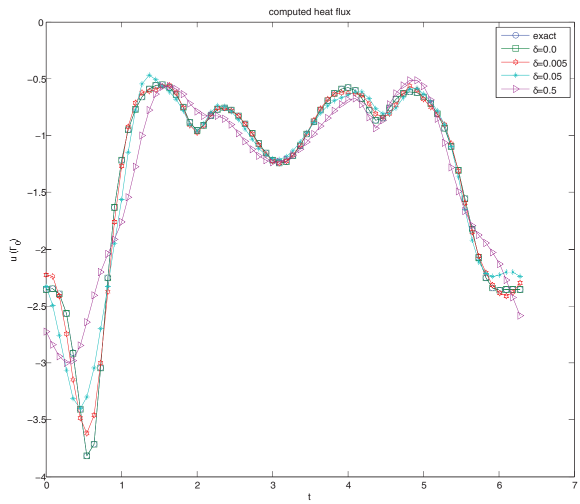

Figure 3: Ex.1. Regularization parameters $\lambda = 6.3151e-9, 0.0014, 0.0064, 0.0384$ for the cases of $\delta = 0, 0.005, 0.05, 0.5$,respectively.

Example 2: Let the exact solution for the problem (1)-(3) be

$$u(x,y) = \ln\sqrt{(x-x_p)^2 + (y-y_p)^2} \tag{34}$$

where (x_p, y_p) is a singularity. The interior boundary Γ_0 is an apple-shaped curve with the parametrization

$$z_0(t) = r_0(t)(\cos t, \sin t), \quad r_0(t) = \frac{0.1\sin 2t + 0.4\cos t + 0.5}{1 + 0.7\cos t}, \quad t \in [0, 2\pi]. \tag{35}$$

The Cauchy data can be calculated as

$$f(t) = \ln\sqrt{(\cos t - x_p)^2 + (\sin t - y_p)^2},$$
$$g(t) = \frac{1 - x_p\cos t - y_p \sin t}{(\cos t - x_p)^2 + (\sin t - y_p)^2}.$$

In this example the step size for t is $2\pi/40$ and for τ is $2\pi/40$, and the singularity is $(0.3, 0.3)$. Numerical results for various levels δ of relative noises are shown for the temperature and heat flux on the interior boundary in Figure 4 and Figure 5, respectively. The root mean square errors are $\varepsilon(u) = 0.0066, 0.0188, 0.0384, 0.0541$ and $\varepsilon(u^*) = 0.41531, 0.2129, 0.3731, 0.4813$ for the cases of $\delta = 0, 0.005, 0.05$, 0.1, respectively. From Figure 4 and Figure 5, we can see that numerical results

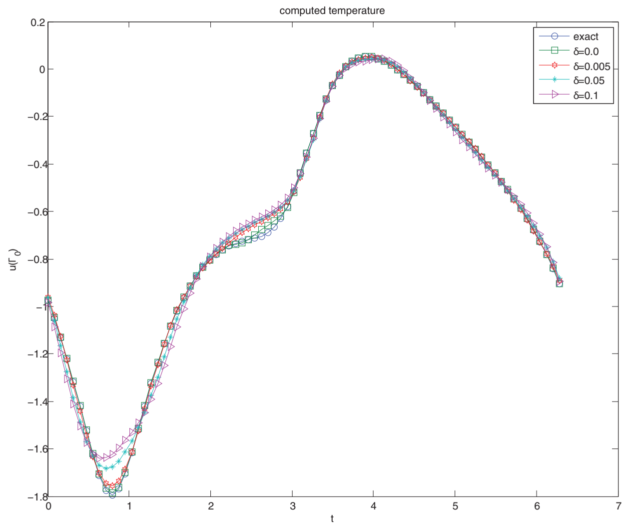

Figure 4: Ex.2. Regularization parameters $\lambda = 2.5e-15, 0.0012, 0.0044, 0.0062$ for the cases of $\delta = 0, 0.005, 0.05, 0.1$,respectively.

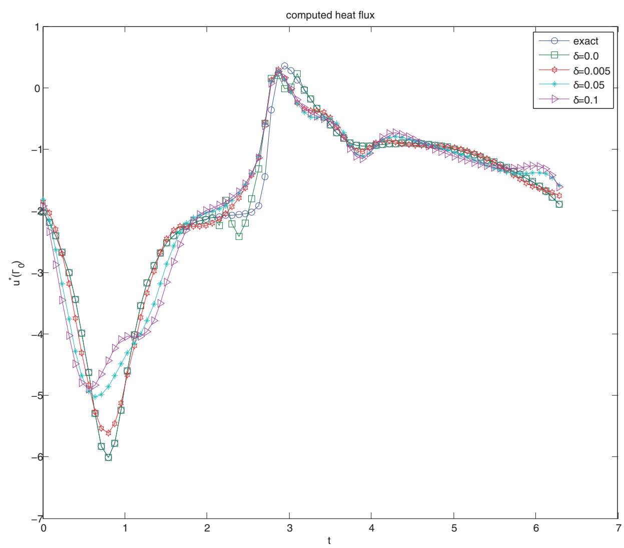

Figure 5: Ex.2. Regularization parameters $\lambda = 2.5e-15, 0.0012, 0.0044, 0.0062$ for the cases of $\delta = 0, 0.005, 0.05, 0.1$,respectively.

match the exact one very well. Numerical results show that the proposed method is stable and effective.

Example 3: In this example, we consider a Cauchy problem in a doubly connected domain for 3D case. Let the exact solution for the problem (1)-(3) be

$$u(x,y,z) = \exp(x) * (\sin(y) + \sin(z)). \quad (36)$$

The interior boundary Γ_0 is an ellipsoid with the parametrization

$$z_0(\theta,t) = (0.1\sin\theta\cos t, 0.2\sin\theta\sin t, 0.3\cos\theta), \quad \theta \in [0,\pi], t \in [0,2\pi]. \quad (37)$$

The Cauchy data can be calculated as

$$\begin{aligned} f(\theta,t) &= \exp(\sin\theta\cos t) * (\sin(\sin\theta\sin t) + \sin(\cos\theta)), \\ g(\theta,t) &= \exp(\sin\theta\cos t) * (\sin\theta\cos t * (\sin(\sin\theta\sin t) + \sin(\cos\theta)) \\ &\quad + \sin\theta\sin t * \cos(\sin\theta\sin t) + \cos\theta * \cos(\cos\theta)). \end{aligned}$$

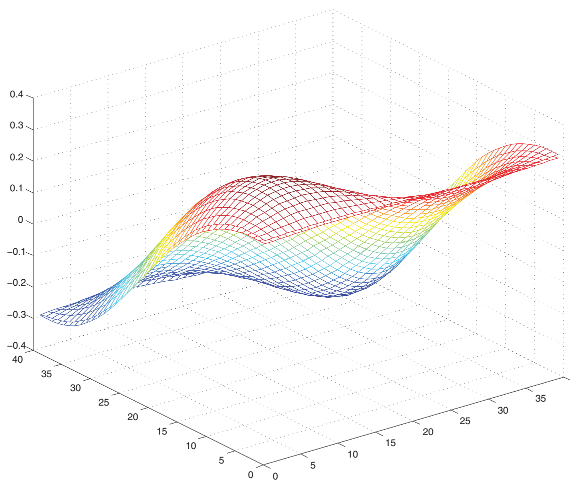

Figure 6: Ex.3. Computed temperature and exact solution, and regularization parameter $\lambda = 0.01$ for $\delta = 0.05$

In this example the step size for θ is $\pi/40$ and for t is $2\pi/40$, the smoothing parameter $\beta = 0.002$. The computed temperature on the interior boundary is shown in Figure 6 and the numerical error of temperature is given in Figure 7 for $\delta = 0.05$ of relative noise. The computed heat flux is shown in Figure 8 and the numerical

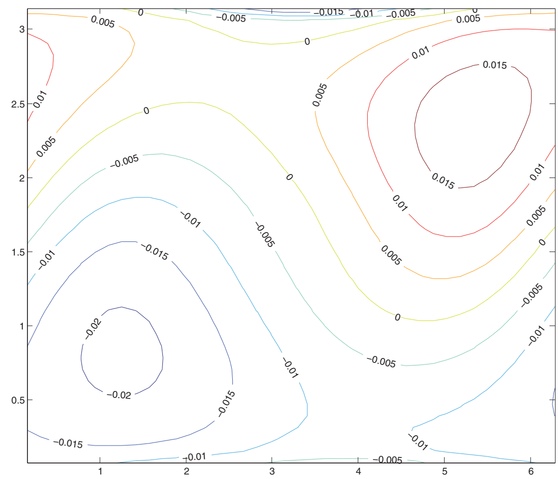

Figure 7: Ex.3. Numerical error for computed temperature and exact solution

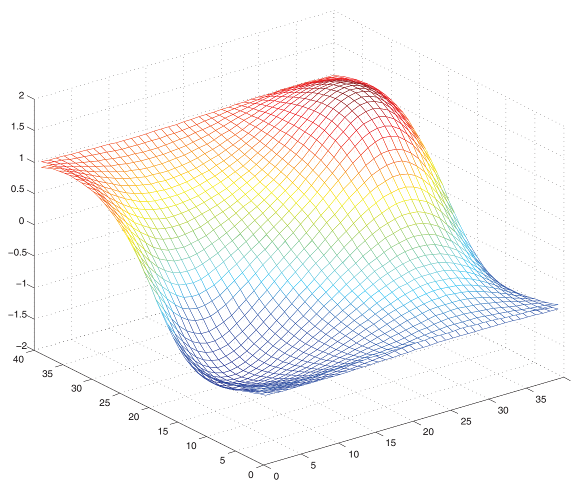

Figure 8: Ex.3. Computed heat flux and exact solution, and regularization parameter $\lambda = 0.01$ for $\delta = 0.05$

error of heat flux is given in Figure 9 for $\delta = 0.05$ of relative noise. The root mean square errors are $\varepsilon(u) = 0.0163$ and $\varepsilon(u^*) = 0.0403$ for the case of $\delta = 0.05$. From Figure 6 and Figure 8, it can be seen that numerical results match the exact one very well for 3D case.

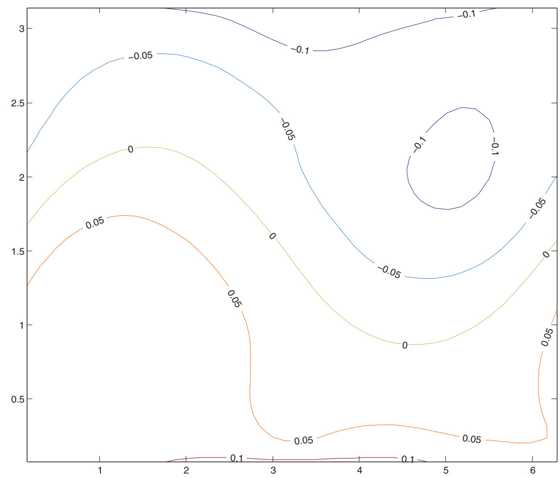

Figure 9: Ex.3. Numerical error for computed heat flux and exact solution

5 Conclusions

In this paper, we consider the Cauchy problem in 2D and 3D doubly connected bounded domains. The problem is ill-posed in the sense that any small errors in the Cauchy data will cause a dramatic change in the solution. Our goal is to transform the system of integral equations into the system of discrete equations with parametrization. Then we employ the Tikhonov regularization method to obtain the regularized solution and regularization parameter is chosen by GCV. From numerical examples, we know that the computed temperature and heat flux match the exact ones very well in doubly connected domains for 2D and 3D cases. Numerical results show that the proposed method is reasonable, feasible and stable.

Acknowledgments

The research of J.C. Liu was supported by the Fundamental Research Funds for the Central Universities (2013XK03).

References

Alessandrini, G. (1993): Stable determination of a crack from boundary measurements. *Proc. Roy. Soc. Edinburgh Sect. A*, vol. 123, no. 3, pp. 497–516.

Ang, D. D.; Nghia, N. H.; Tam, N. C. (1998): Regularized solutions of a Cauchy problem for the Laplace equation in an irregular layer: a three-dimensional model. *Acta Math. Vietnam.*, vol. 23, no. 1, pp. 65–74.

Beale, J. T. (2004): A grid-based boundary integral method for elliptic problems in three dimensions. *SIAM J. Numer. Anal.*, vol. 42, no. 2, pp. 599–620.

Belgacem, F. B. (2007): Why is the Cauchy problem severely ill-posed? *Inverse Probl.*, vol. 23, pp. 823–36.

Belgacem, F. B.; Du, D. T.; Jelassi, F. (2011): Extended-domain-Lavrentiev's regularization for the Cauchy problem. *Inverse Probl.*, vol. 27, no. 4, pp. 045005.

Belgacem, F. B.; Fekih, H. E. (2005): On Cauchy's problem: I. A variational Steklov-Poincaré theory. *Inverse Probl.*, vol. 21, no. 6, pp. 1915–36.

Berntsson, F.; Eldén, L. (2001): Numerical solution of a Cauchy problem for the Laplace equation. *Inverse Probl.*, vol. 17, no. 4, pp. 839–853.

Calderon, A. P. (1958): Uniqueness in the Cauchy problem for partial differential equations. *American Journal of Mathematics*, vol. 80, no. 1, pp. 16–36.

Chan, H. F.; Fan, C. M.; Yeih, W. (2011): Solution of inverse boundary optimization problem by Trefftz method and exponentially convergent scalar homotopy algorithm. *Computer Materials and Continua*, vol. 24, no. 2, pp. 125.

Chapko, R.; Johansson, B. T. (2008): An alternating boundary integral based method for a Cauchy problem for the Laplace equation in semi-infinite regions. *Inverse Probl. Imaging*, vol. 2, no. 3, pp. 317–333.

Chapko, R.; Johansson, B. T. (2008): An alternating potential-based approach to the Cauchy problem for the Laplace equation in a planar domain with a cut. *Comput. Methods Appl. Math.*, vol. 8, no. 4, pp. 315–335.

Chapko, R.; Johansson, B. T. (2009): An alternating boundary integral based method for a Cauchy problem for the Laplace equation in a quadrant. *Inverse Probl. Sci. Eng.*, vol. 17, no. 7, pp. 871–883.

Cheng, J.; Hon, Y.; Wei, T.; Yamamoto, M. (2001): Numerical computation of a Cauchy problem for Laplace's equation. *ZAMM Z. Angew. Math. Mech*, vol. 81, no. 10, pp. 665–674.

Cheng, J.; Prößdorf, S.; Yamamoto, M. (1998): Local estimation for an integral equation of first kind with analytic kernel. *J. Inverse Ill-Posed Probl.*, vol. 6, no. 2, pp. 115–126.

Chi, C. C.; Yeih, W.; Liu, C. S. (2009): A novel method for solving the cauchy problem of laplace equation using the fictitious time integration method. *Computer Modeling in Engineering and Sciences (CMES)*, vol. 47, no. 2, pp. 167.

Colli-Franzone, P.; Guerri, L.; Tentoni, S.; Viganotti, C.; Baruffi, S.; Spaggiari, S.; Taccardi, B. (1985): A mathematical procedure for solving the inverse potential problem of electrocardiography. analysis of the time-space accuracy from in vitro experimental data. *Mathematical Biosciences*, vol. 77, no. 1-2, pp. 353 – 396.

Dong, L.; Atluri, S. N. (2012): A Simple Multi-Source-Point Trefftz Method for Solving Direct/Inverse SHM Problems of Plane Elasticity in Arbitrary Multiply-Connected Domains. *Computer Modeling in Engineering and Sciences (CMES)*, vol. 85, no. 1, pp. 1–43.

Gene, H. G.; Heath, M. T.; Wahba, G. (1979): Generalized Cross-Validation as a Method for Choosing a Good Ridge Parameter. *Technometrics*, vol. 21, pp. 215–223.

Háo, D. N.; Lesnic, D. (2000): The Cauchy problem for Laplace's equation via the conjugate gradient method. *IMA J. Appl. Math.*, vol. 65, no. 2, pp. 199–217.

Ivanyshyn, O.; Kress, R. (2006): Nonlinear integral equations for solving inverse boundary value problems for inclusions and cracks. *J. Integral Equations Appl*, vol. 18, no. 1, pp. 13–38.

John, F. (1982): *Partial Differential Equations*. Applied Mathematical Sciences. Springer-Verlag, New York, fourth edition.

Klibanov, M. V.; Santosa, F. (1991): A computational quasi-reversibility method for Cauchy problems for Laplace's equation. *SIAM J Appl Math*, vol. 51, no. 6, pp. 1653–1675.

Kress, R. (1999): *Linear integral equations*, volume 82 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition.

Lavrentev, M. M.; Romanov, V. G.; Shishatskii, S. P. (1986): *Ill-posed problems of mathematical physics and analysis*, volume 64 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI. Translated from the Russian by J. R. Schulenberger, Translation edited by Lev J. Leifman.

Liu, C. S. (2008): A highly accurate mctm for inverse cauchy problems of laplace equation in arbitrary plane domains. *Computer Modeling in Engineering and Sciences (CMES)*, vol. 35, no. 2, pp. 91–112.

Marin, L. (2009): An iterative mfs algorithm for the cauchy problem associated with the laplace equation. *Computer Modeling in Engineering and Sciences (CMES)*, vol. 48, no. 2, pp. 121.

Vani, C.; Avudainayagam, A. (2002): Regularized solution of the Cauchy problem for the Laplace equation using meyer wavelets. *Math. Comput. Model.*, vol. 36, no. 9-10, pp. 1151 – 1159.