Application of the MLPG Mixed Collocation Method for Solving Inverse Problems of Linear Isotropic/Anisotropic Elasticity with Simply/Multiply-Connected Domains

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Abstract: In this paper, a novel Meshless Local Petrov-Galerkin (MLPG) Mixed Collocation Method is developed for solving the inverse Cauchy problem of linear elasticity, wherein both the tractions as well as displacements are prescribed/ measured at a small portion of the boundary of an elastic body. The elastic body may be isotropic/anisotropic and simply connected or multiply-connected. In the MLPG mixed collocation method, the same meshless basis function is used to interpolate both the displacement as well as the stress fields. The nodal stresses are expressed in terms of nodal displacements by enforcing the constitutive relation between stress and the displacement gradient tensor at each nodal point. The equations of linear momentum balance are satisfied at each node using collocation method. The displacement as well as traction boundary conditions are also enforced at each measurement location along the boundary where the conditions are over specified on displacement as well as tractions. The current method is very simple because the inverse problem is directly solved in a fashion similar to a direct problem, without resorting to any iterative optimization. The current method is also very general because it can be applied to arbitrary simply/multiply connected bodies composed of arbitrary isotropic/anisotropic material, and it can also be adapted to solve inverse problems of other physics such as heat transfer, electro-magnetics, etc. Several numerical examples demonstrate the effectiveness and robustness of the current method, even when the prescribed displacement/tractions are corrupted with measurement noises. The extension of the current method to solve nonlinear inverse problems will be straightforward within the framework of incremental

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loading, which will be explored in future studies.

1   Introduction

Computational modeling of solid/fluid mechanics, heat transfer, electromagnetics, and other physical, chemical & biological sciences have experienced an intense development in the past several decades. Tremendous efforts have been devoted to solving the so-called direct problems, where the boundary conditions are generally of the Dirichlet, Neumann, or Robin type. Existence, uniqueness, and stability of the solutions have been established for many of these direct problems. Numerical methods such as finite elements, boundary elements, finite volume, meshless methods etc., have been successfully developed and available in many off-the shelf commercial softwares, see [Atluri (2005)]. On the other hand, inverse problems, although being more difficult to tackle and being less studied, have equal, if not greater importance in the applications of engineering and sciences, such as in structural health monitoring, electrocardiography, etc.

One of the many types of inverse problems is to identify the unknown boundary fields when conditions are over-specified on only a part of the boundary, i.e. the Cauchy problem. Take elasto-static solid mechanics as an example. For the domain Ω which describes the solid under deformation, the governing differential equations can be expressed in terms of the primitive variable-displacements:

$$\left(C_{ijkl}u_{k,l}\right)_{,i} + f_j = 0 \ \text{ in } \ \Omega \tag{1}$$

For direct problems, displacements $u_i = \bar{u}_i$ are prescribed on a part of the boundary S_u, and tractions $t_i = \bar{t}_i$ are prescribed on the other part of the boundary S_t. S_u and S_t should be a complete division of $\partial\Omega$, which means $S_u \cup S_t = \partial\Omega, S_u \cap S_t = \emptyset$. On the other hand, if both the tractions as well as displacements are specified or known only on a small portion of the boundary S_C, the inverse Cauchy problem is to determine the stresses, strains and displacements in the domain as well as on the other part of the boundary. More generalizations can further be made, to include measurements of the strains, measurements in the domain, or the measurement at several time steps for a vibrating solid.

In spite of the popularity of FEM for direct problems, it is essentially very unsuitable for solving inverse problems. This is because the traditional primal FEM are based on the global Symmetric Galerkin Weak Form of equation(1):

$$\int_{\Omega}\left(C_{ijkl}u_{i,j}v_{k,l} - f_i v_i\right)d\Omega - \int_{\partial\Omega}C_{ijkl}u_{i,j}n_k v_l dS = 0 \tag{2}$$

where v_k are test functions, and both the trial functions u_i and the test functions v_k are required to be continuous and differentiable. It is immediately apparent from equation (2) that the symmetric weak form [on which the primal finite element methods are based] does not allow for the simultaneous prescription of both the tractions t_l [$\equiv C_{ijkl}u_{i,j}n_k$] as well as displacements u_l [in which case v_l are set to zero, for convenience] at the same segment of the boundary, $\partial\Omega$. Therefore, in order to solve the inverse problem using FEM, one has to first ignore the over-specified boundary conditions, guess the missing boundary conditions, so that one can iteratively solve a direct problem, and minimize the difference between the solution and over-prescribed boundary conditions by adjusting the guessed boundary fields, see [Kozlov, Maz'ya and Fomin (1991); Cimetiere, Delvare, Jaoua and Pons (2001)] for example. This procedure is cumbersome and expensive, and in many cases highly-dependent on the initial guess of the boundary fields.

Recently, simple non-iterative methods have been under development for solving inverse problems without using the primal symmetric weak-form: with global RBF as the trial function, collocation of the differential equation and boundary conditions leads to the global primal RBF collocation method [Cheng and Cabral (2005)]; with Kelvin's solutions as trial function, collocation of the boundary conditions leads to the method of fundamental solutions [Marin and Lesnic (2004)]; with non-singular general solutions as trial function, collocation of the boundary conditions leads to the boundary particle method [Chen and Fu (2009)]; with Trefftz trial functions, collocation of the boundary conditions leads to Trefftz collocation method [Yeih, Liu, Kuo and Atluri (2010); Dong and Atluri (2012)]. The common idea they share is that the collocation method is used to satisfy either the differential equations and/or the boundary conditions at discrete points. Moreover, collocation method is also more suitable for inverse problems because measurements are most often made at discrete locations.

However, the above-mentioned direct collocation methods are mostly limited to simple geometries, simple constitutive relations, and text-book problems, because: (1) these method are based on global trial functions, and lead to a fully-populated coefficient matrix of the system of equations; (2) the general solutions and particular solutions cannot be easily found for general anisotropic problems, nonlinear problems, and problems with arbitrary body force; (3) it is difficult to derive general solutions that are complete for arbitrarily shaped domains, within a reasonable computational burden. With this understanding, more suitable ways of constructing the trial functions should be explored.

One of the most simple and flexible way is to construct the trial functions through meshless interpolations. Meshless interpolations have been combined with the global Symmetric Galerkin Weak Form to develop the so-called Element-Free Gal-

erkin (EFG) method, see [Belytschko, Lu, and Gu (1994)]. However, as shown in the Weak Form (2), because displacements and tractions cannot be prescribed at the same location, cumbersome iterative guessing and optimization will also be necessary if EFG is used to solve inverse problems. Thus EFG is not suitable for solving inverse problems, for the same reason why FEM is not suitable for solving inverse problems.

Instead of using the global Symmetric Galerkin Weak-Form, the Meshless Local Petrov-Galerkin (MLPG) method by [Atluri and Zhu (1998)] proposed to construct both the trial and test functions in a local subdomain, and write local weak-forms instead of global ones. Various versions of MLPG method have been developed in [Atluri and Shen (2002a, b)], with different trial functions (Moving Least Squares, Local Radial Basis Function, Shepard Function, Partition of Unity methods, etc.), and different test functions (Weight Function, Shape Function, Heaviside Function, Delta Function, Fundamental Solution, etc.). These methods are primal methods, in the sense that all the local weak forms are developed from the Navier's equation (1) with displacement as primary variables. For this reason, the primal MLPG collocation method, which involves direct second-order differentiation of the displacement fields, as shown in equation (1), requires higher-order continuous basis functions, and is reported to be very sensitive to collocation points.

Instead of the primal methods, MLPG mixed finite volume and collocation method were developed in [Atluri, Han and Rajendran (2004); Atluri, Liu and Han (2006)]. The mixed MLPG approaches independently interpolate the primary and secondary fields, such as displacements and stresses, using the same meshless basis function. The compatibility between primary and secondary fields is enforced through a collocation method at each node. Through these efforts, the continuity requirement on the trial functions is reduced by one order, and the complicated second derivatives of the shape function are avoided. Successful applications of the MLPG mixed finite volume and collocation methods are applied in nonlinear and large deformation problems [Han, Rajendran and Atluri (2005)]; impact and penetration problems [Han, Liu, Rajendran and Atluri (2006); Liu, Han, Rajendran and Atluri (2006)], topology optimization problems [Li and Atluri (2008a,b)]. A thorough review of the applications of MLPG method is given in [Sladek, Stanak, Han, Sladek, Atluri (2013)].

In this study, we apply the MLPG mixed collocation method to solve inverse problem of linear elasticity *for simply or multiply connected domains, with isotropic as well as anisotropic material properties.* Similar to [Atluri, Liu and Han (2006)], the same meshless basis functions are used to interpolate both the displacement as well as the stress fields. The nodal stresses are expressed in terms of nodal displacements by enforcing the relation between stress and displacement-gradient at

each nodal point. The equations of linear momentum balance are satisfied at each node using collocation method. The displacement and traction boundary conditions are also enforced at each measurement location along the boundary. The currently developed method seems to be more promising than any other existing method of solving inverse problems, because:

1. it is very simple since the inverse problem is directly solved in a similar fashion to a direct problem, without resorting to any iterative optimization;

2. it can be applied to arbitrary simply/multiply connected bodies;

3. it can be applied to arbitrary isotropic and anisotropic materials;

4. it can be extended to solve nonlinear problems, by using incremental linearization;

5. it can also be adapted to solve inverse problems of other physiccal problems such as heat transfer, electro-magnetics, etc.

6. it can be easily combined with any existing regularization techniques, to reduce the effects of measurement noises.

The rest of this study is organized as follows. In section 2, the meshless interpolation method is briefly introduced with emphasis on the Moving Least Squares interpolation. In section 3, the detailed algorithm of the MLPG mixed collocation method for inverse problem is given. In section 4, we demonstrate the effectiveness of the current method with several numerical examples ***involving simply/multiply connected domains ,and isotropic as well as anisotropic materials***. In section 5, we complete this paper with some concluding remarks.

2 Meshless Interpolation

Among the available meshless approximation schemes, the Moving Least Squares (MLS) is generally considered to be one of the best methods to interpolate random data with a reasonable accuracy, because of its completeness, robustness and continuity. The MLS is adopted in the current MLPG collocation formulation, while the implementation of other meshless interpolation schemes is straightforward within the present framework. For completeness, the MLS formulation is briefly reviewed here, while more detailed discussions on the MLS can be found in [Atluri (2004)].

The MLS method starts by expressing the variable $u(\mathbf{x})$ as polynomials:

$$u(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{3}$$

where $\mathbf{p}^T(\mathbf{x})$ is the monomial basis. In this study, we use first-order interpolation, so that $\mathbf{p}^T(\mathbf{x}) = [1, x_1, x_2]$ for two-dimensional problems. $\mathbf{a}(\mathbf{x})$ is a vector containing the coefficients of each monomial basis, which can be determined by minimizing the following weighted least square objective function, defined as:

$$J(\mathbf{a}(\mathbf{x})) = \sum_{I=1}^{m} w^I(\mathbf{x})[\mathbf{p}^T(\mathbf{x}^I)\mathbf{a}(\mathbf{x}) - \hat{u}^I]^2$$
$$= [\mathbf{Pa}(\mathbf{x}) - \hat{\mathbf{u}}]^T \mathbf{W}[\mathbf{Pa}(\mathbf{x}) - \hat{\mathbf{u}}] \tag{4}$$

where, $\mathbf{x}^I, I = 1, 2, \cdots, m$ is a group of discrete nodes within the influence range of node $\mathbf{x}$, $\hat{u}^I$ is the fictitious nodal value, $w^I(\mathbf{x})$ is the weight function. A fourth order spline weight function is used here:

$$w^I(\mathbf{x}) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & r \le 1 \\ 0 & r > 1 \end{cases} \tag{5}$$

$$r = \frac{\|\mathbf{x} - \mathbf{x}^I\|}{r^I}$$

where, r^I stands for the radius of the support domain $\Omega_{\mathbf{x}}$.

Substituting $\mathbf{a}(\mathbf{x})$ into equation (3), we can obtain the approximate expression as:

$$u(x) = \mathbf{p}^T \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \hat{\mathbf{u}} = \mathbf{\Phi}^T(\mathbf{x}) \hat{\mathbf{u}} = \sum_{I=1}^{m} \Phi^I(\mathbf{x}) \hat{u}^I \tag{6}$$

where, matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are defined by:

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^T \mathbf{W} \mathbf{P} \quad \mathbf{B}(\mathbf{x}) = \mathbf{P}^T \mathbf{W} \tag{7}$$

$\Phi^I(\mathbf{x})$ is named as the MLS basis function for node I, and it is used to interpolated both displacements and stresses, as discussed in section 3.2.

3 MLPG Mixed Collocation Method for Inverse Problem of Elasticity

3.1 Inverse Problem of Linear Elasticity

Considering a linear elastic solid undergoing infinitesimal elasto-static deformation, equations of linear momentum balance, constitutive equations, and kinematic equations can be written as:

$$\begin{aligned} &\sigma_{ij,i} + f_i = 0 \\ &\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad \text{[isotropic or anisotropic material]} \\ &\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \equiv u_{(i,j)} \end{aligned} \tag{8}$$

For inverse problems, we consider that both the displacements as well as tractions are prescribed at a portion of the boundary, denoted as S_C:

$$u_i = \bar{u}_i \quad \text{at} \quad S_C$$
$$t_i = \sigma_{ij} n_j = \bar{t}_i \quad \text{at} \quad S_C \tag{9}$$

where, $\bar{u}_i$ and $\bar{t}_i$ are the prescribed displacements and tractions.

The inverse problem is thus defined as, with the measured displacements and strains [and thus tractions] at S_C, which is a small portion of the boundary of the whole domain, can we determine the displacements, strains, and stress in the whole domain? A MLPG mixed collocation method is developed to solve this problem, and is discussed in detail in the following two subsections.

3.2 MLPG Mixed Collocation Method

We start by interpolating the displacements as well as the stresses, using the same MLS shape function, as discussed in section 2:

$$u_i(\mathbf{x}) = \sum_{J=1}^{m} \Phi^J(\mathbf{x}) \hat{u}_i^J \tag{10}$$

$$\sigma_{ij}(\mathbf{x}) = \sum_{J=1}^{m} \Phi^J(\mathbf{x}) \hat{\sigma}_{ij}^J \tag{11}$$

where, $\hat{u}_i^J$ and $\hat{\sigma}_{ij}^J$ are the fictitious displacements and stresses at node J.

With the constitutive and kinematic equation as shown in (8), the stress components at node I can also be expressed as:

$$\sigma_{ij}(\mathbf{x}^I) = \frac{1}{2} C_{ijkl} \sum_{J=1}^{m} \left[\Phi_{,l}^J(\mathbf{x}^I) \hat{u}_k^J + \Phi_{,k}^J(\mathbf{x}^I) \hat{u}_l^J \right]; \quad I = 1, 2, \cdots, N \tag{12}$$

where N is the total number of nodes in the domain.

This allows us to relate nodal stresses to nodal displacements, which is written here in matrix-vector form:

$$\mathbf{S} = \mathbf{Tu} \tag{13}$$

And the equations of linear momentum balance are independently enforced at each node, as:

$$\sum_{J=1}^{m} \Phi_{,j}^J(\mathbf{x}^I) \hat{\sigma}_{ij}^J + f_i(\mathbf{x}^I) = 0; \quad I = 1, 2, \cdots, N \tag{14}$$

or in an equivalent Matrix-Vector from:

$$\mathbf{K}_S \mathbf{S} = \mathbf{f} \tag{15}$$

By substituting equation (13) into equation(15), we can obtain a discretized system of equations in term of nodal displacements:

$$\mathbf{K}_{\text{eq}} \mathbf{u} = \mathbf{f}_{\text{eq}} \tag{16}$$

From equation (12) and (14), we see that both the equation of linear momentum balance, and the stress displacement-gradient relation are enforced by a collocation method, at each node of the MLS interpolation. In the following subsection, the same collocation method will be carried out to enforce the boundary conditions of the inverse problem.

3.3 Over-Specified Boundary conditions in a Cauchy Inverse Problem of Isotropic/Anisotropic Linear Elasticity in Simply/Multiply Connected Domains

In most applications of inverse problems, the measurements are only available at discrete locations at a small portion of the boundary. In this study, we consider that both displacements $\bar{u}_i$ as well as tractions $\bar{t}_i$ are prescribed at discrete points $\mathbf{x}^I$, $I = 1,2,3...,M$, on the same segment of the boundary. We use collocation method to enforce such boundary conditions:

$$\sum_{J=1}^{m} \Phi^J(\mathbf{x}^I) \hat{u}_i^J = \bar{u}_i(\mathbf{x}^I)$$
$$n_j \frac{1}{2} C_{ijkl} \sum_{J=1}^{m} \left[\Phi_{,l}^J(\mathbf{x}^I) \hat{u}_k^J + \Phi_{,k}^J(\mathbf{x}^I) \hat{u}_l^J \right] = \bar{t}_i(\mathbf{x}^I) \tag{17}$$

or, in matrix-vector form:

$$\mathbf{K}_u \mathbf{u} = \mathbf{f}_u$$
$$\mathbf{K}_t \mathbf{u} = \mathbf{f}_t \tag{18}$$

3.4 Regularization for Noisy Measurements

Equation (16) and (18) can rewritten as:

$$\mathbf{K}\mathbf{u} = \mathbf{f}, \quad \mathbf{K} = \begin{pmatrix} \mathbf{K}_{\text{eq}} \\ \mathbf{K}_u \\ \mathbf{K}_t \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}_{\text{eq}} \\ \mathbf{f}_u \\ \mathbf{f}_t \end{pmatrix} \tag{19}$$

This gives a complete discretized system of equations of the governing differential equations as well as the over-specified boundary conditions. It can be directly solved using least square method without iterative optimization.

However, it is well-known that the inverse problems are ill-posed. A very small perturbation of the measured data can lead to a significant change of the solution. In order to mitigate the ill-posedness of the inverse problem, regularization techniques can be used. For example, following the work of Tikhonov and Arsenin (1977), many regularization techniques were developed. Hansen and O'Leary (1993) has given an explanation that the Tikhonov regularization of ill-posed linear algebra equations is a trade-off between the size of the regularized solution, and the quality to fit the given data. With a positive regularization parameter, the solution is found by:

$$\min \left(\|\mathbf{Ku} - \mathbf{f}\|^2 + \gamma \|\mathbf{f}\|^2 \right) \tag{20}$$

This leads to the regularized solution:

$$\mathbf{u} = \left(\mathbf{K}^{\mathbf{T}} \mathbf{K} + \gamma \mathbf{I} \right)^{-1} \mathbf{K}^{\mathbf{T}} \mathbf{f} \tag{21}$$

4   Numerical Examples

In this section, several 2D numerical examples are given to demonstrate the effectiveness of the MLPG mixed collocation method for solving inverse problems in isotropic/anisotropic linear elastic, simply connected/multiply-connected domains.

4.1   Case 1: Cantilever beam
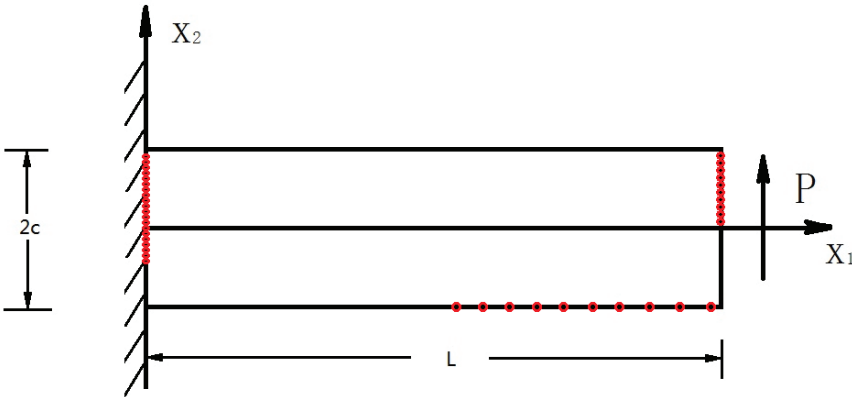
Figure 1: A cantilever beam under a shear load at the free end, with displacements as well as tractions prescribed at discrete locations over only a part of the boundary (denoted by red points)

In the first case, a beam-shaped simply-connected domain is considered, as shown in figure 1. The geometrical parameters are: $L = 24, c = 2, P = 1$. Arbitrarily we consider the Young's modulus $E = 1$, and the Poisson's ratio $v = 0.25$. The analytical solution of a cantilever beam subjected to shear load at the free end is given in [Timoshenko and Goodier (1970)]:

$$u_1 = -\frac{Px_2}{6EI}\left(3x_1\left(2L - x_1\right) + \left(2 + v\right)\left(x_2^2 - c^2\right)\right)$$

$$u_2 = \frac{Py}{6EI}\left(3vx_2^2\left(L - x_1\right) + \left(4 + 5v\right)c^2x_1 + x_1^2\left(3L - x_1\right)\right) \tag{22}$$

$$\sigma_{11} = -\frac{Px_2}{I}\left(L - x_1\right), \sigma_{22} = 0, \sigma_{12} = \frac{P}{2I}\left(c^2 - x_2^2\right)$$

where the moment of inertia is $I = c^3/3$.

So we consider a problem like this: if the beam is subject to such a deformation, can we identify the displacements, strains, and stresses in the whole domain, if the data of displacements and tractions are only available at the discrete locations over a small portion of the boundary shown in figure 1.

The inverse problem is posed as follows: if the displacements as well as tractions corresponding to the analytical solution(21) are prescribed at all the "red" points in figure 1, can we use the MLPG Mixed Collocation Method to determine the displacements and tractions elsewhere on the boundary and inside the domain?

We first solve this problem with125 uniformly distributed nodes as shown in figure 2. As discussed in section 3, collocations are made for equilibrium equations, stress displacement-gradient relations, displacement and traction boundary conditions. The stress component σ_{11} and vertical displacement v at the upper side of the beam is normalized to their maximum values, and are shown in figure 3. The computed displacements and stresses agree with the analytical solution very well.
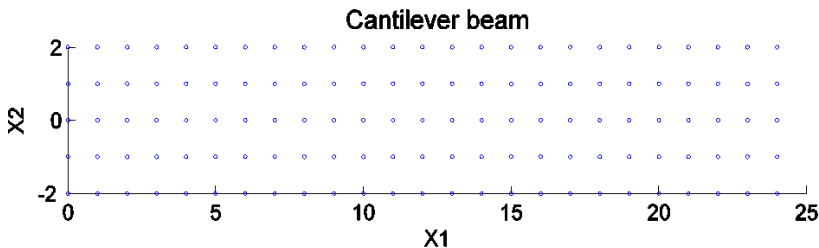
Figure 2: The nodal configuration of the cantilever beam

Moreover, in order to study the convergence of the current method, three different nodal configurations with 21, 52 and 297 nodes, respectively, are used to solve

the inverse problem as shown in figure 1. The nodal configurations are uniform, as shown in figure 4. The computed normalized stress σ_{11} and normalized vertical displacement v along the top edge ($x_2 = c$) with the 3 nodal configurations are shown in figure 5 and 6. Compared with the analytical solution, each of the 3 nodal configurations can give acceptable results. And finer nodal configurations will give more accurate computations.
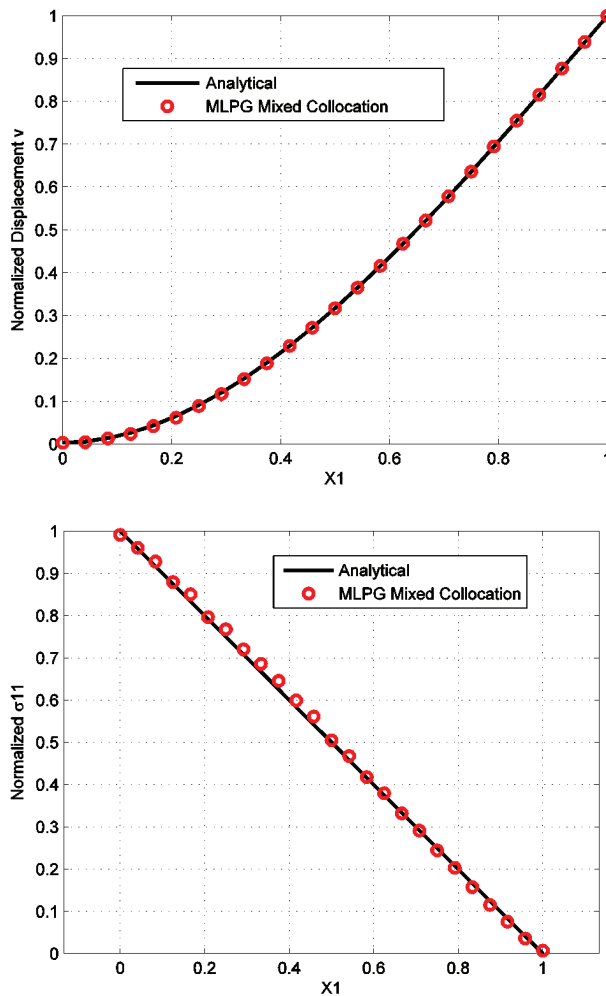

Figure 3: Computed normalized stress σ_{11} and normalized vertical displacement v at the upper side of the beam, using nodal configuration in figure 2.
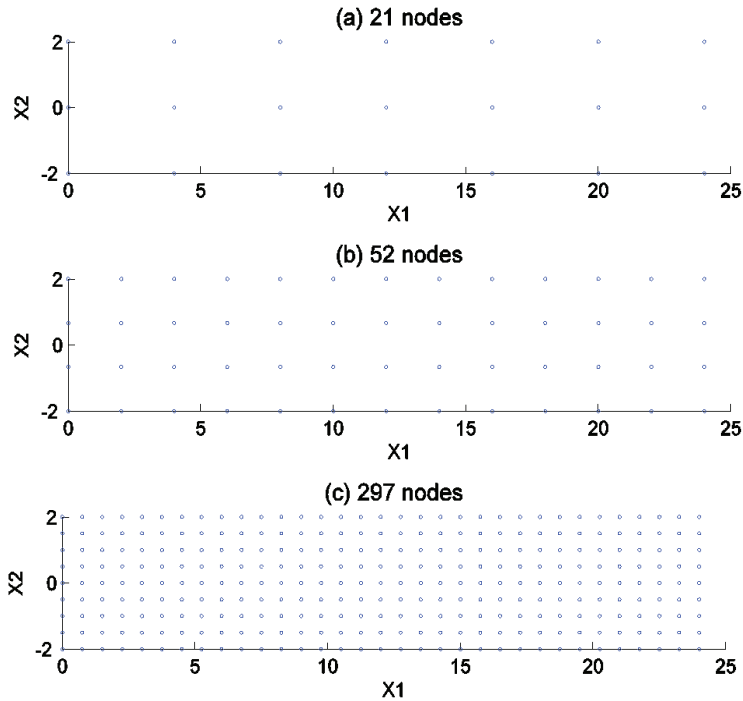
Figure 4: Three nodal configurations of a cantilever beam
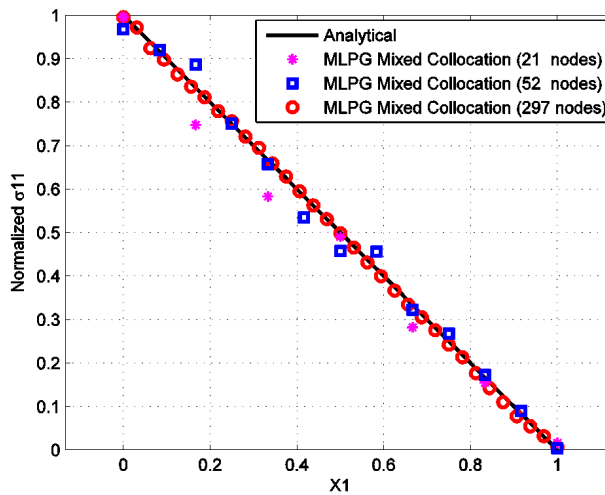
Figure 5: Computed normalized stress σ_{11} using 3 different nodal configurations
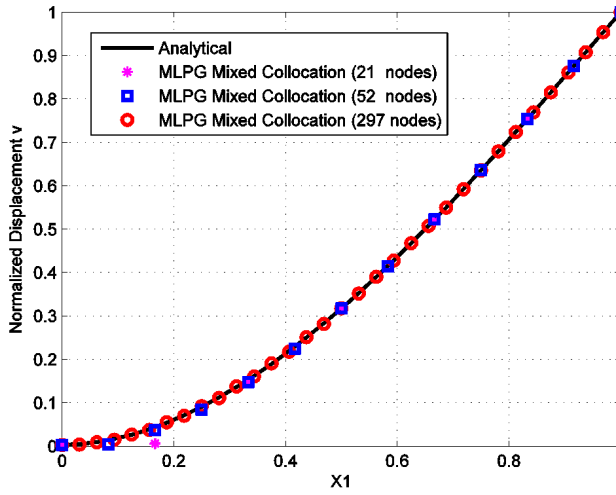
Figure 6: Computed normalized displacement *v* using 3 different nodal configuration

4.2 Case 2: An isotropic plate with a circular hole
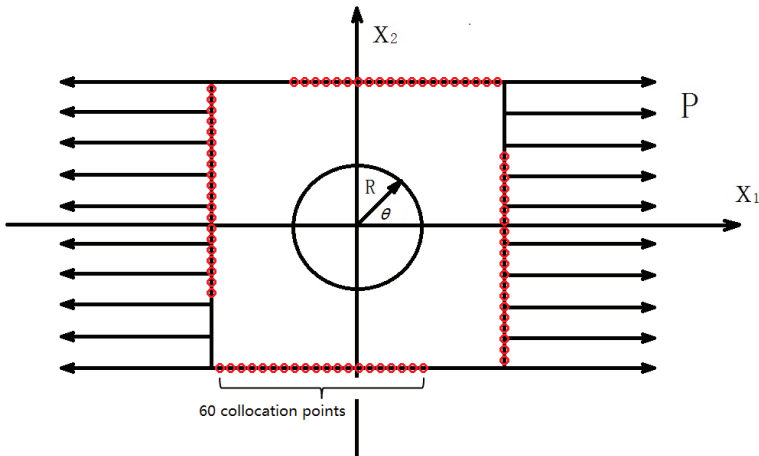
Figure 7: a plate with a circular hole under uniform tension

We also consider a doubly connected domain as shown in figure 7. A 20 by 20 square plate is considered and *R*=5 denotes the radius of the circular hole. Arbitrar-

ily we consider the Young's modulus $E = 1$, and the Poisson's ratio $v = 0.25$. The analytical solution for this problem is given in [Timoshenko and Goodier (1970)]:

$$u_r = \frac{p}{4G} \left\{ r \left[\frac{\kappa - 1}{2} + \cos(2\theta) \right] + \frac{a^2}{r} \left[1 + (1 + \kappa)\cos(2\theta) \right] - \frac{a^4}{r^3}\cos(2\theta) \right\}$$

$$u_\theta = \frac{p}{4G} \left\{ (1 - \kappa)\frac{a^2}{r} - r - \frac{a^4}{r^3} \right\} \sin(2\theta)$$

$$\sigma_x = p \left\{ 1 - \frac{a^2}{r^2} \left[\frac{2}{3}\cos(2\theta) + \cos(4\theta) \right] + \frac{3a^4}{2r^4}\cos(4\theta) \right\} \tag{23}$$

$$\sigma_y = -p \left\{ \frac{a^2}{r^2} \left[\frac{1}{2}\cos(2\theta) - \cos(4\theta) \right] + \frac{3a^4}{2r^4}\cos(4\theta) \right\}$$

$$\sigma_{xy} = -p \left\{ \frac{a^2}{r^2} \left[\frac{1}{2}\sin(2\theta) + \sin(4\theta) \right] - \frac{3a^4}{2r^4}\sin(4\theta) \right\}$$

wherein, G is the shear modulus and $\kappa = (3 - v) / (1 + v)$.

We prescribe both the displacements as well as tractions corresponding to the analytical solution (23) at only a part of boundary as shown in red points in figure 7, and use two different nodal configurations to solve this problem. The two nodal configurations have 620 nodes and 2416 nodes respectively, as shown in figure 8. Computed displacements and stresses are given in figure 8-12, showing the convergence of the solution.
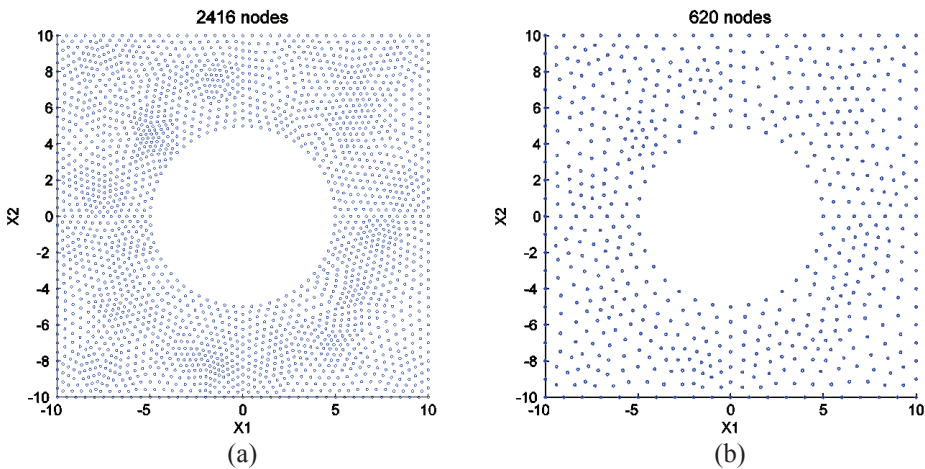
Figure 8: Two nodal configurations of the plate with a circular hole, with 2416 and 628 nodes respectively

It expected that, the smaller S_C is, i.e. when boundary conditions are over-specified at a smaller portion of the boundary, the inverse problem itself becomes more ill-posed, and more difficult to tackle. How to improve the accuracy and stability of the computed displacements, strains, and stresses with MLPG Mixed Collocation Method, when boundary conditions are over-specified at a smaller S_C, will be explored in future studies.
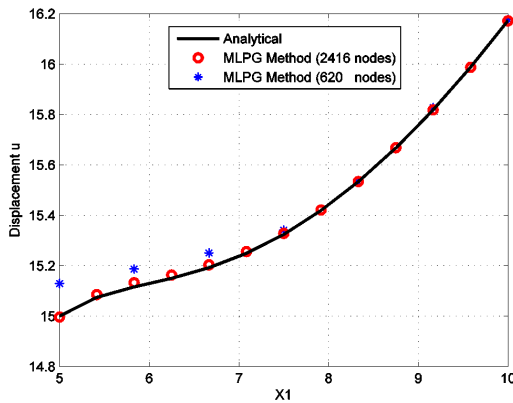

Figure 9: Computed horizontal displacement *u* along the positive x_1-axis for an isotropic plate with a circular hole with two nodal configurations
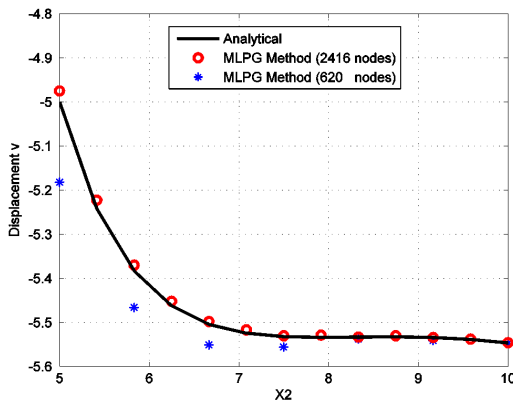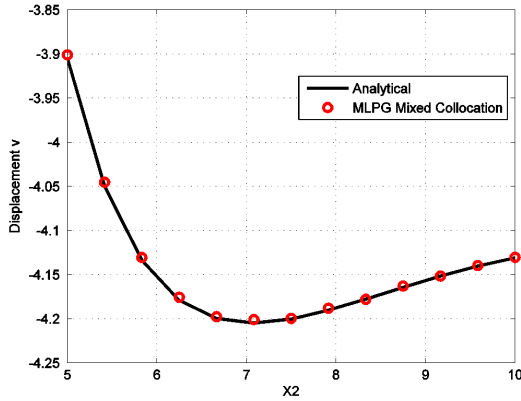

Figure 10: Computed vertical displacement *v* along the positive x_2-axis for an isotropic plate with a circular hole with two nodal configurations
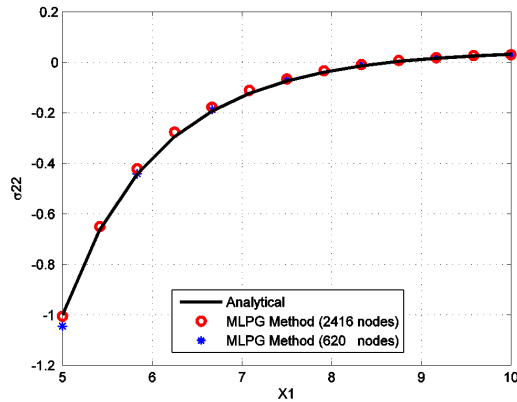
Figure 11: Comparison of the identified stress σ_{22} along the positive x_1-axis for an isotropic plate with a circular hole with two nodal configurations
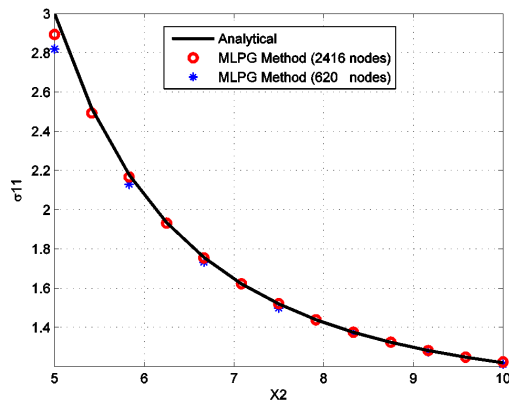
Figure 12: Computed stress σ_{11} along the positive x_2-axis for an isotropic plate with a circular hole with two nodal configurations

4.3   Case 3: An anisotropic plate with a circular hole

In this example, we consider the problem of an anisotropic plate with a circular hole. The same geometry, load, measurement locations as shown in figure 7 are
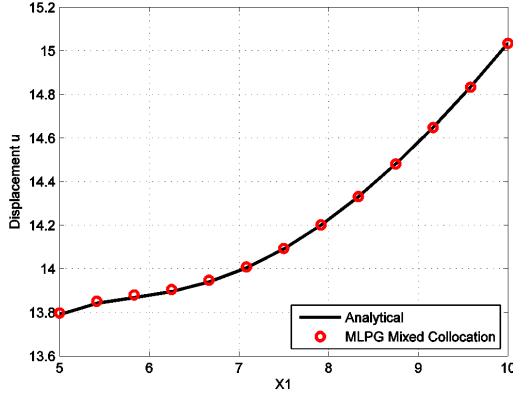
Figure 13: Computed horizontal displacement u along the positive x_1-axis for an anisotropic plate with a circular hole with 2416 nodes

Figure 14: Computed vertical displacement v along the positive x_2-axis for an anisotropic plate with a circular hole with 2416 nodes

used. The orthotropic material properties are arbitrarily taken to be:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{21} & C_{22} & C_{26} \\ C_{61} & C_{62} & C_{66} \end{bmatrix} = \begin{bmatrix} 1.0667 & 0.2667 & 0 \\ 0.2667 & 1.667 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \tag{24}$$

The analytical solution of an infinite plate with a circular hole under uniform tension, given by [Lekhnitskii (1963)], is used at here. And we use the MLPG mixed

collocation method to solve this inverse problem, with the finer nodal configuration (2416 node) as shown in figure 8. The computed stress σ_{22} and horizontal displacement u at the positive x_1-axis, the computed stressσ_{11} and vertical displacement v at the positive x_2-axis are given in figure 13-16 respectively. Compared with the analytical solutions, good agreements are obtained for both the displacements and stress.
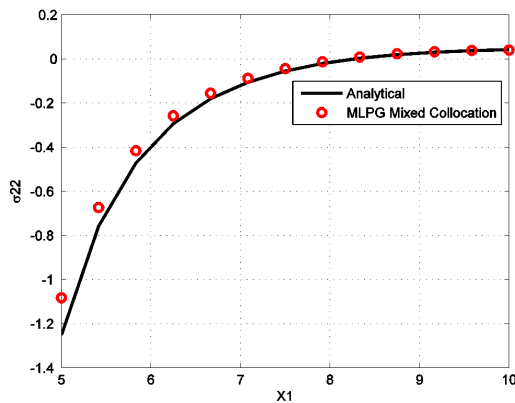
Figure 15: Comparison of the identified stress σ_{22} along the positive x_1-axis for an anisotropic plate with a circular hole with 2416 nodes
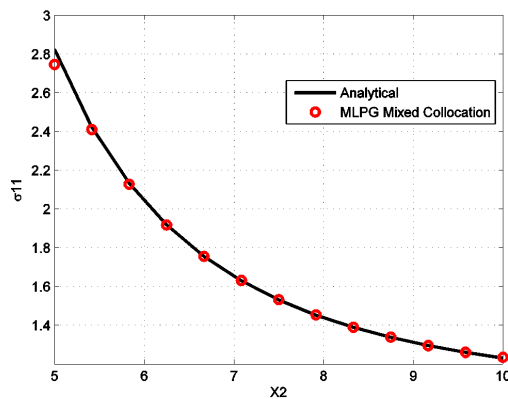
Figure 16: Computed stress σ_{11} along the positive x_2-axis for an anisotropic plate with a circular hole with 2416 nodes

4.4 Case 4: An isotropic plate with a circular hole with noise in the over-specified boundary data

In this example, we consider the same problem of case 2, but with measurement noises in both the displacements and tractions at a boundary where the conditions are over specified. The same material, geometry, load as shown in figure 7 is used. Measurement noises with SNR 40dBW and 30dBW are added to the analytical solution(22), and are measured at locations shown in figure 7. We use the MLPG mixed collocation method to solve this inverse problem, with 2416 node as shown in figure 8. The Tikhonov Regularization as discussed in section 3.4 is used here to mitigate the influence of noise corruption, with a small regularization parameter $\gamma = 10^{-7}$. The computed stress σ_{22} and horizontal displacement u at the positive x_1-axis, the computed stress σ_{11} and vertical displacement v at the positive x_2-axis are given in figure 17-20 respectively. From the numerical results, we see that the current method is robust enough to solve the inverse problems with measurement noises. This is important because there is always measurement noise in real applications.
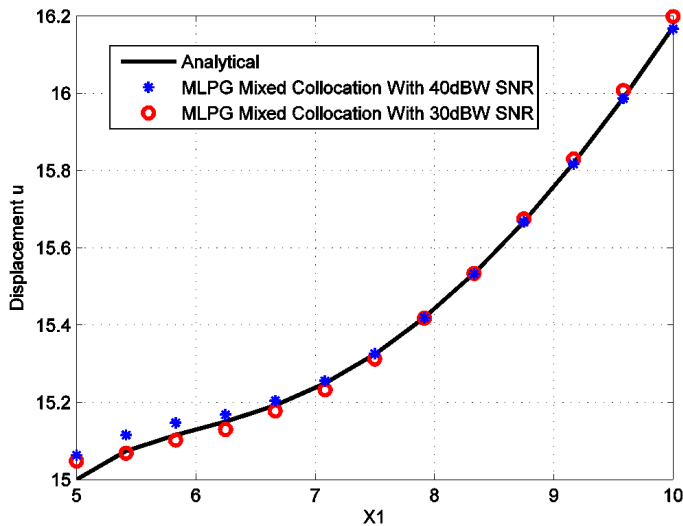

Figure 17: Computed horizontal displacement u along the positive x_1-axis for an isotropic plate with a circular hole using 2416 nodes, subjected to measurement noises
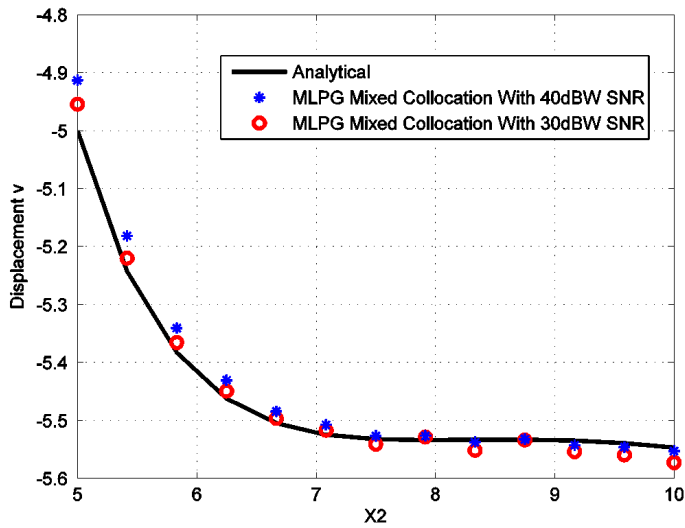
Figure 18: Computed vertical displacement *v* along the positive x_2-axis for an isotropic plate with a circular hole using 2416 nodes, subjected to measurement noises
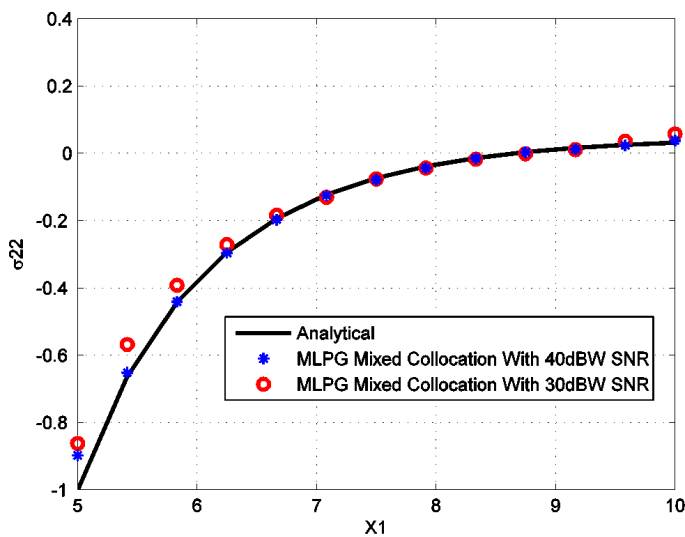

Figure 19: Comparison of the identified stress σ_{22} along the positive x_1-axis for an isotropic plate with a circular hole using 2416 nodes, subjected to measurement noises
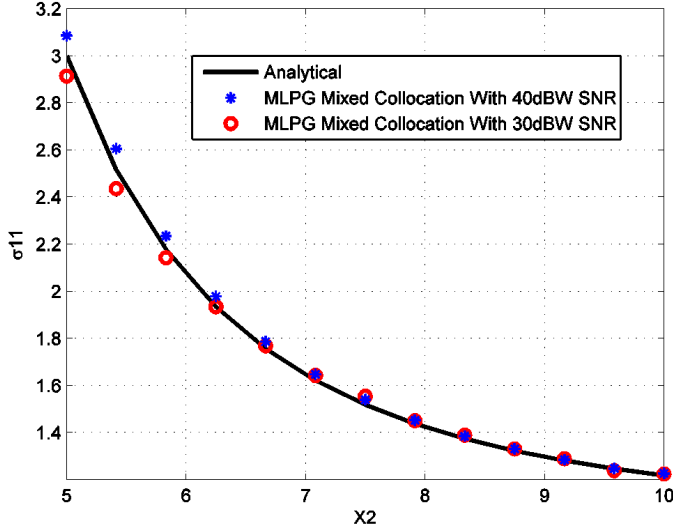
Figure 20: Computed stress σ_{11} along the positive x_2-axis for an isotropic plate with a circular hole using 2416 nodes, subjected to measurement noises

4.5 Case 5: An anisotropic plate with an elliptical hole

In the last example, we consider a 20 by 20 square plate with an elliptical hole, see figure 21. The semi-axes of the elliptical hole is $a = 6, b = 4$. Uniform tension $P = 1$ is applied horizontally to the plate. And arbitrarily we consider the orthotropic material properties:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{21} & C_{22} & C_{26} \\ C_{61} & C_{62} & C_{66} \end{bmatrix} = \begin{bmatrix} 1.0667 & 0.2667 & 0 \\ 0.2667 & 1.667 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \tag{25}$$

where a and b are the major and minor semi- axis of the ellipse. The analytical solution of this problem is given in [Lekhnitskii (1963)].

Randomly generated 2416 nodes are used to solve the problem, with MLPG mixed collocation method. The nodal configuration is shown in figure 22. Both displacements as well as tractions corresponding to the analytical solution [Lekhnitskii (1963)] are prescribed at all the boundary points marked as red in figure 21. In figure 22-25, we plot the computed stress σ_{22} and horizontal displacement u along the positive x_1-axis, and the stress σ_{11} and vertical displacement v along the positive x_2-axis. Good agreement is found between the computed and analytical solutions.
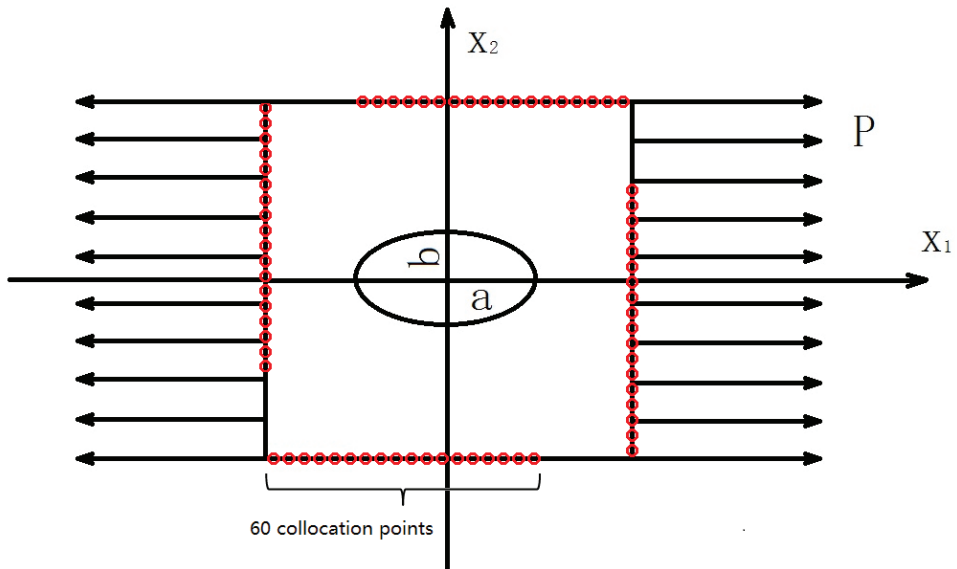
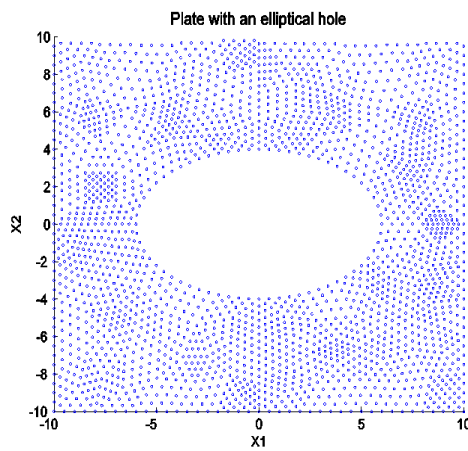Figure 21: A plate with an elliptical hole under uniform tension

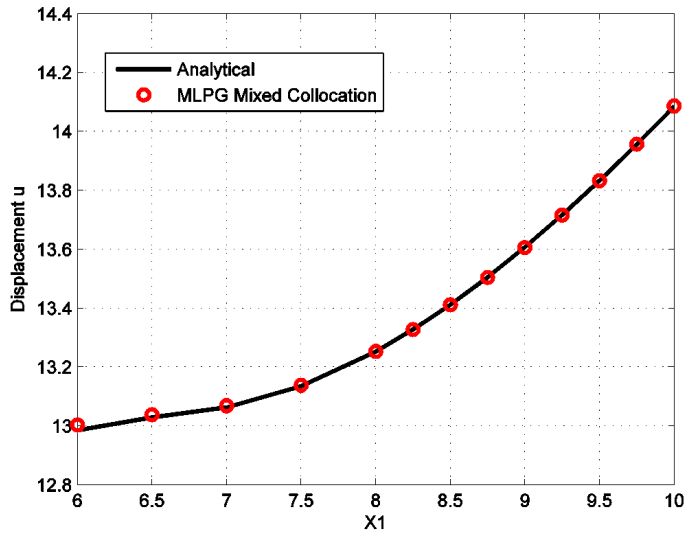Figure 22: The nodal configuration of the plate with an elliptical hole

Figure 23: Computed horizontal displacement *u* along the positive x_1-axis for an anisotropic plate with an elliptical hole using 2416 nodes
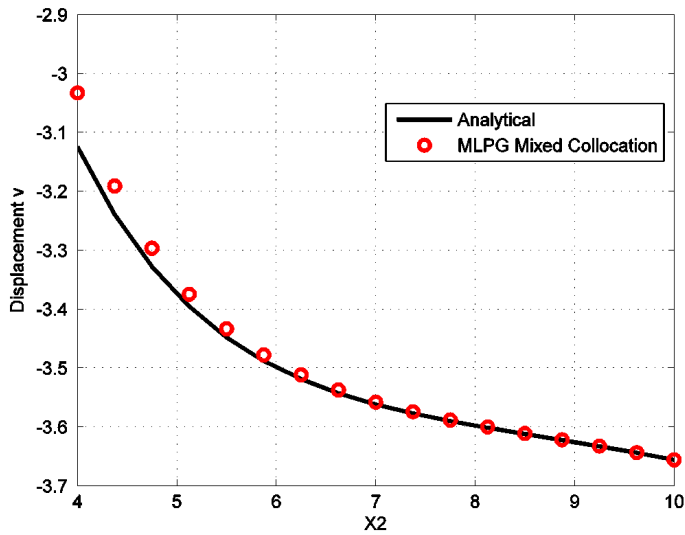

Figure 24: Computed vertical displacement v along the positive x_2-axis for an anisotropic plate with an elliptical hole using 2416 nodes
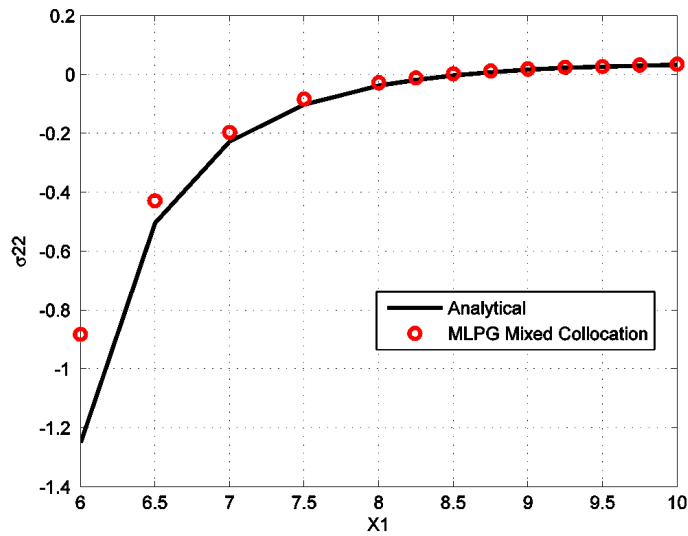
Figure 25: Comparison of the identified stress σ_{22} along the positive x_1-axis for an anisotropic plate with an elliptical hole using 2416 nodes
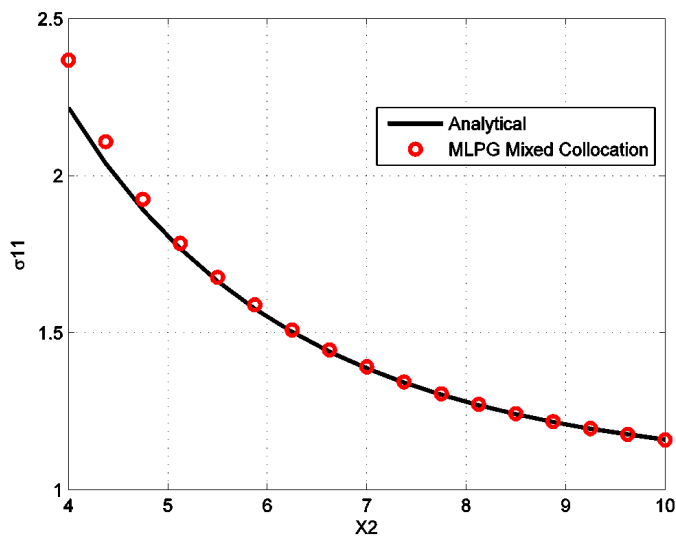

Figure 26: Computed stress σ_{11} along the positive x_2-axis for an anisotropic plate with an elliptical hole using 2416 nodes

5   Conclusions:

The MLPG mixed collocation method is developed to solve the inverse problems of linear elasticity with isotropic or anisotropic material properties, and simply/multiply-connected domains. The MLS is used to construct the MLPG basis functions, and the displacements and stresses are independently interpolated. Equations of equilibrium, stress displacement-gradient relations, and over-specified displacement as well as traction boundary conditions over a small portion of the boundary are all enforced at discrete points using the method of collocation.

The current method is considered to be very promising, and better than the current existing solvers of inverse problems such as FEM, EFG, MFS, etc. Some obvious advantages of using MLPG mixed collocation method to solve inverse problems are:

1. it is very simple since the inverse problem is directly solved in a fashion similar to a direct problem, without resorting to any iterative optimization;

2. it can be applied to arbitrary simply/multiply connected bodies;

3. it can be applied to arbitrary isotropic and anisotropic materials;

4. it can be even extended to solve nonlinear problems, by using incremental linearization;

5. it can also be adapted to solve inverse problems of other physical problems such as heat transfer, electro-magnetics, etc.

6. it can be easily combined with any existing regularization techniques, to reduce the effects of measurement noises.

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