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A New Modified Adomian Decomposition Method for Higher-Order Nonlinear Dynamical Systems

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Abstract: In this paper, we propose a new modification of the Adomian decomposition method for solution of higher-order nonlinear initial value problems with variable system coefficients and solutions of systems of coupled nonlinear initial value problems. We consider various algorithms for the Adomian decomposition series and the series of Adomian polynomials to calculate the solutions of canonical first- and second-order nonlinear initial value problems in order to derive a systematic algorithm for the general case of higher-order nonlinear initial value problems and systems of coupled higher-order nonlinear initial value problems. Our new modified recursion scheme is designed to decelerate the Adomian decomposition series so as to always calculate the solution's Taylor expansion series using easy-to-integrate terms. The corresponding nonlinear recurrence relations for the solution coefficients are deduced. Next we consider convergence acceleration and error analysis for the sequence of solution approximations. Multistage decomposition and numeric algorithms are designed and we debut efficient MATHEMATICA routines PSSOL and NSOL that implement our new algorithms. Finally we investigate several expository examples in order to demonstrate the rapid convergence of our new approach.

Keywords: Adomian decomposition method, Modified decomposition method, Multistage decomposition, Adomian polynomials, Nonlinear dynamical systems.

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1 Introduction

The Adomian decomposition method (ADM) [Adomian and Rach (1983); Adomian (1983); Bellman and Adomian (1985); Adomian (1986, 1989, 1994); Wazwaz (1997, 2002a, 2009, 2011); Serrano (2010, 2011); Duan, Rach, Baleanu, and Wazwaz (2012); Rach (2012)] and its modifications [Adomian and Rach (1993a,b); Wazwaz (1999); Wazwaz and El-Sayed (2001); Duan and Rach (2011a); Duan, Rach, and Wang (2013)] provide efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. The ADM is an analytic approximation method, which does not require accepting a priori assumptions in our models that drastically alter the outcomes so that they do not faithfully replicate reality. We retain the full nonlinear features in our solution algorithms using the ADM. It permits us to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) [Rach, Wazwaz, and Duan (2013); Wazwaz, Rach, and Duan (2013a); Duan, Wang, Fu, and Chaolu (2013); Duan, Rach, and Wazwaz (2013)] without unphysical restrictive assumptions such as required by linearization, perturbation, ad hoc assumptions, guessing the initial term or a set of basis functions, and so forth; most of which would significantly alter the physical behavior of the problem.

Furthermore the ADM and its modifications do not require the use of Green's functions, which would complicate such analytic calculations as Green's functions are not easily determined in most cases. The accuracy of the analytic approximate solutions obtained by the ADM can be readily verified by direct substitution. Advantages of the ADM over Picard's iterated method were demonstrated in [Rach (1987)]. A key notion is the Adomian polynomials [Adomian and Rach (1983); Adomian (1983); Rach (1984, 2008); Azreg-Aïnou (2009); Duan (2010a); Duan, Rach, Baleanu, and Wazwaz (2012)], which are tailored to the particular nonlinearity to solve nonlinear operator equations, and for which new, more efficient and convenient algorithms have been proposed by Duan (2010a,b, 2011a) and Duan and Guo (2010). For a comprehensive bibliography featuring many new engineering applications and a modern review of the ADM, see [Rach (2012); Duan, Rach, Baleanu, and Wazwaz (2012)].

Let us first recall the basic principles of the ADM. Consider an ordinary differential equation in Adomian's operator-theoretic form

$$Lu + Ru + Nu = g(x), \tag{1}$$

where g is the system input and u is the system output, and where L is the linear operator to be inverted, which usually is just the highest order differential operator, R is the linear remainder operator, and N is the nonlinear operator, which is assumed to be analytic. We remark that this choice of the linear operator is designed to yield an easily invertible operator with resulting trivial integrations. Furthermore, we emphasize that the choice for L and concomitantly its inverse L^{-1} are determined by the particular equation to be solved, hence the choice is nonunique, e.g. for cases of differential equations with singular coefficients, we choose a different form for the linear operator [Adomian, Rach, and Shawagfeh (1995); Wazwaz (2001, 2002b)]; see also [Wazwaz, Rach, and Duan (2013a,b); Rach, Duan, and Wazwaz (2013)] for another alternative for the inverse linear operator. Generally, we choose $L = \frac{d^p}{dx^p}(\cdot)$ for pth-order differential equations and thus its inverse L^{-1} follows as the p-fold definite integration operator from x_0 to x. We have $L^{-1}Lu = u - \Phi$, where Φ incorporates the initial values.

Applying the inverse linear operator L^{-1} to both sides of Eq. (1) gives

$$u = \gamma(x) - L^{-1}[Ru + Nu], \tag{2}$$

where $\gamma(x) = \Phi + L^{-1}g$. The ADM defines the unknown function representing the solution by an infinite decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3}$$

and defines the nonlinear term by an infinite decomposition series of the Adomian polynomials

$$Nu\,(x) = \sum_{n=0}^{\infty} A_n\,(u_0\,(x)\,, \ldots, u_n\,(x)), \tag{4}$$

where the solution components $u_n\,(x)\,,\ n \geq 0$, of the solution $u\,(x)$ are to be determined recursively, and the Adomian polynomials $A_n\,(x)\,,\ n \geq 0$, are easily generated for any analytic nonlinearity $Nu = f(u)$ by the definitional formula [Adomian and Rach (1983)]

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[f\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\right]_{\lambda=0},\ n = 0, 1, 2, \cdots, \tag{5}$$

where λ is nothing more and nothing less than a grouping parameter of convenience. We list the formulas of the first several Adomian polynomials for the one-variable simple analytic nonlinearity $Nu = f(u(x))$ from A_0 through A_4, inclusively, for convenient reference as

$$\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= f'(u_0)u_1, \\
A_2 &= f'(u_0)u_2 + f''(u_0)\tfrac{u_1^2}{2!}, \\
A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\tfrac{u_1^3}{3!}, \\
A_4 &= f'(u_0)u_4 + f''(u_0)\left(\tfrac{u_2^2}{2!} + u_1u_3\right) + f^{(3)}(u_0)\tfrac{u_1^2u_2}{2!} + f^{(4)}(u_0)\tfrac{u_1^4}{4!}.
\end{aligned}$$

For the case of the one-variable Adomian polynomials, we prefer Duan's Corollary 3 algorithm [Duan (2011a)] since it does not involve the differentiation operator for the coefficients C_n^k [Duan (2010a, 2011a)], but only requires the elementary operations of addition and multiplication, and is thus eminently convenient for computer algebra systems such as MATHEMATICA, MAPLE or MATLAB:

$$A_0 = f(u_0), \quad A_n = \sum_{k=1}^{n} C_n^k f^{(k)}(u_0), \text{ for } n \geq 1, \tag{6}$$

where the coefficients C_n^k are defined recursively as [Duan (2011a)]

$$\begin{aligned} &C_n^1 = u_n, \; n \geq 1, \\ &C_n^k = \frac{1}{n}\sum_{j=0}^{n-k}(j+1)\,u_{j+1}C_{n-1-j}^{k-1}, \; 2 \leq k \leq n. \end{aligned} \tag{7}$$

We remark that it has been timed in speed tests to be one of the fastest on record using a commercially available laptop computer [Duan (2011a)].

Upon substitution of the Adomian decomposition series for the solution $u(x)$ and the series of Adomian polynomials tailored to the nonlinearity Nu from Eqs. (3) and (4) into Eq. (2), we obtain

$$\sum_{n=0}^{\infty} u_n = \gamma(x) - L^{-1}\left[R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right]. \tag{8}$$

The solution components $u_n(x)$ may be determined by one of several advantageous recursion schemes, which differ from one another by the choice of the initial solution component $u_0(x)$, beginning with the classic Adomian recursion scheme

$$\begin{aligned} &u_0(x) = \gamma(x), \\ &u_{n+1}(x) = -L^{-1}[Ru_n + A_n], \; n \geq 0. \end{aligned} \tag{9}$$

The n-term approximation of the solution is

$$\varphi_n(x) = \sum_{k=0}^{n-1} u_k(x). \tag{10}$$

By various partitions of the original initial term and then delaying the contribution of its remainder by various algorithms, we can design alternate recursion schemes to achieve different computational advantages, such as the modified recursion schemes proposed by Adomian and Rach (1993a,b), Wazwaz (1999), Wazwaz and El-Sayed (2001), Duan (2010a), Duan and Rach (2011a), Rach, Wazwaz, and Duan (2013) and Duan, Rach, and Wang (2013).

Several investigators including Cherruault and co-workers [Cherruault (1989); Cherruault and Adomian (1993); Abbaoui and Cherruault (1994)], among others [Rèpaci (1990); Gabet (1994); Rach (2008); Abdelrazec and Pelinovsky (2011)], have previously proved convergence of the Adomian decomposition series and the series of the Adomian polynomials. For example, Cherruault and Adomian (1993) have proved convergence of the decomposition series without appealing to the fixed point theorem, which is patently too restrictive for most physical and engineering applications. Furthermore, Abdelrazec and Pelinovsky (2011) have recently published a rigorous proof of convergence for the ADM under the aegis of the Cauchy-Kovalevskaya theorem for IVPs. A key concept is that the Adomian decomposition series is a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function, which permits solution by recursion. A remarkable measure of success of the ADM is demonstrated by its widespread adoption and many adaptations to enhance computability for specific purposes [Duan, Rach, Baleanu, and Wazwaz (2012); Rach (2012)], such as the various modified recursion schemes. The choice of decomposition is nonunique, which provides a valuable advantage to the analyst, permitting the freedom to design modified recursion schemes for ease of computation in realistic systems.

Adomian (1989) introduced the concept of the accelerated Adomian polynomials $\hat{A}_n$. Adomian and Rach (1996) presented two new kinds of modified Adomian polynomials $\bar{A}_n$ and $\bar{\bar{A}}_n$. Rach (2008) gave a new definition of the Adomian polynomials, in which different classes of the Adomian polynomials were defined within the same premise. Duan (2011b) presented new recurrence algorithms for these nonclassic Adomian polynomials. Generalized forms of the Adomian polynomials were also proposed by Duan (2011c).

Rach, Adomian, and Meyers (1992) proposed a modified decomposition method (MDM) based on the nonlinear transformation of series by the Adomian-Rach theorem [Adomian and Rach (1991, 1992a)]:

$$\text{If} \quad u(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{then} \quad f(u(x)) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \tag{11}$$

where the $A_n = A_n(a_0, a_1, \ldots, a_n)$ are the Adomian polynomials in terms of the solution coefficients. The Rach-Adomian-Meyers MDM combines the power series solution and the Adomian-Rach theorem, and has been efficiently applied to solve various nonlinear models [Adomian and Rach (1992b,c, 1993c, 1994); Adomian (1994); Adomian, Rach, and Shawagfeh (1995); Lai, Chen, and Hsu (2008)].

Higher-order numeric one-step methods based on the Rach-Adomian-Meyers MDM were developed by Adomian, Rach, and Meyers (1997) and Duan and Rach (2011b, 2012). We observe that one of the difficulties in applying explicit Runge-Kutta one-step methods is that there is no general procedure to generate higher-order numeric methods. In [Duan and Rach (2011b)], our numeric scheme permits a straightforward universal procedure to generate higher-order numeric methods at will such as a 12th-order or 24th-order one-step method, if need be. Another key advantage is that we can easily estimate the maximum step-size prior to computing data sets representing the discretized solution, because we can approximate the radius of convergence from the solution approximants unlike the Runge-Kutta approach with its intrinsic linearization between computed data points. Also Duan and Rach (2011b) proposed new variable step-size, variable order and variable step-size, variable order algorithms for automatic step-size control to increase the computational efficiency and reduce the computational costs even further for critical engineering models.

In order to solve nonlinear fractional differential equations, Duan, Chaolu, and Rach (2012) developed the generalized Adomian-Rach theorem,

$$f(\sum_{n=0}^{\infty} a_n (x-x_0)^{n\lambda}) = \sum_{n=0}^{\infty} A_n (x-x_0)^{n\lambda}, \tag{12}$$

where $A_n = A_n(a_0, a_1, \ldots, a_n)$ are the Adomian polynomials in terms of the solution coefficients.

The multivariable Adomian polynomials are used for decomposing multivariable nonlinear functions occurring in either single nonlinear nth-order differential equations with multi-order differential nonlinearities $f(u, u', u'', ..., u^{(n-1)})$ or in systems of coupled nonlinear differential equations with multivariable nonlinearities. We suppose that f is an m-ary analytic function $f(u_1, \cdots, u_m)$, where the u_k, for $1 \le k \le m$, are the unknown functions to be determined. We decompose the solutions $u_i, i = 1, 2, \cdots, m$, and the nonlinear function $f(u_1, \cdots, u_m)$ as

$$u_i = \sum_{j=0}^{\infty} u_{i,j},\ i = 1, 2, \cdots, m, \text{ and } f(u_1, \cdots, u_m) = \sum_{n=0}^{\infty} A_n, \tag{13}$$

where the multivariable Adomian polynomials A_n depend on the $m(n+1)$ solution components $u_{1,0}, u_{1,1} \ldots, u_{1,n}, u_{2,0}, u_{2,1}, \ldots, u_{2,n}, \ldots, u_{m,0}, u_{m,1}, \ldots, u_{m,n}$, and are defined by the analytic parametrization [Adomian and Rach (1983)]

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} f\left(\sum_{j=0}^{\infty} u_{1,j} \lambda^j, \cdots, \sum_{j=0}^{\infty} u_{m,j} \lambda^j \right) \Bigg|_{\lambda=0}, \tag{14}$$

where λ is again simply a grouping parameter of convenience. The first m-variable Adomian polynomial A_0 is $A_0 = f(\mathbf{u_0})$, where $\mathbf{u_0} = (u_{1,0}, \cdots, u_{m,0})$.

Another expression for the multivariable Adomian polynomials [Abbaoui, Cherruault, and Seng (1995)] is

$$A_n = \sum_{\sum_{j=1}^{n} j \sum_{i=1}^{m} p_{ij} = n} f^{(p_{1*},\cdots,p_{m*})}(\mathbf{u_0}) \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{u_{i,j}{}^{p_{ij}}}{p_{ij}!}, \tag{15}$$

where $p_{i*} = \sum_{j=1}^{n} p_{ij}$ and $f^{(p_{1*},\cdots,p_{m*})}(\mathbf{u_0}) = \frac{\partial^{p_{1*}+\cdots+p_{m*}} f(\mathbf{u_0})}{\partial u_{1,0}^{p_{1*}} \cdots \partial u_{m,0}^{p_{m*}}}$ denotes the mixed partial derivatives. Equivalently, we have, in [Duan (2010b)],

$$A_n = \sum_{k=1}^{n} \sum_{P \in \mathscr{S}_{m,n}^{k}} f^{(p_{1*},\cdots,p_{m*})}(\mathbf{u_0}) \prod_{i=1}^{m} \prod_{j=1}^{n-k+1} \frac{u_{i,j}{}^{p_{ij}}}{p_{ij}!}, \tag{16}$$

where every matrix in $\mathscr{S}_{m,n}^{k}$ is $m \times (n-k+1)$ and $P = (p_{ij}) \in \mathscr{S}_{m,n}^{k}$ if and only if

$$\sum_{j=1}^{n-k+1} j \sum_{i=1}^{m} p_{ij} = n \quad \text{and} \quad \sum_{j=1}^{n-k+1} \sum_{i=1}^{m} p_{ij} = k, \tag{17}$$

and where $p_{i*} = \sum_{j=1}^{n-k+1} p_{ij}$. New, fast algorithms and their MATHEMATICA subroutines for the multivariable Adomian polynomials were crafted by Duan (2010b, 2011a).

We also mention other approximate analytic methods for nonlinear differential equations such as the collocation method [Dai, Schnoor, and Atluri (2012)], variational iteration method [Wazwaz (2009)] and perturbation method [Hinch (1991)], etc.

Our purpose in this work is to introduce a new reliable modification of the ADM, which builds upon the previous modified recursion schemes [Wazwaz (1999); Wazwaz and El-Sayed (2001); Rach, Wazwaz, and Duan (2013)]. We shall examine higher-order inhomogeneous nonlinear differential equations with variable system coefficients in order to compute the solution's Taylor expansion series about the initial point without using differentiation operators in the modified recursion schemes. In the next section, we derive a new modified ADM for higher-order nonlinear dynamical systems. In Section 3, we review the concepts of convergence acceleration, multistage decomposition and the philosophically correct error analysis when the exact solution is unknown in advance, and design numeric algorithms and automated calculation by MATHEMATICA, then investigate several numeric examples. Section 4 summarizes our conclusions.

2 Derivation of a new modified ADM for higher-order nonlinear dynamical systems

We illustrate the steps of our new modification by considering the sequence of the canonical first-order nonlinear IVP and the canonical second-order nonlinear IVP with a variable input and variable system coefficients in order to extend this pattern to the case of the canonical pth-order nonlinear IVP with a variable input and variable system coefficients, and finally to the canonical $p_{\max}$th-order system of q-coupled p_kth-order inhomogeneous nonlinear IVPs with variable inputs and variable system coefficients.

2.1 The canonical first-order inhomogeneous nonlinear IVP with a variable input $g(x)$ and variable system coefficients $\alpha_0(x)$ and $\beta(x)$

We consider the following nonlinear first-order differential equation with an analytic input function and analytic system coefficient functions subject to a bounded initial condition.

$$u'(x) + \alpha_0(x)u(x) + \beta(x) f(u(x)) = g(x), \tag{18}$$

$$u(0) = C_0, \tag{19}$$

or equivalently in Adomian's operator-theoretic notation

$$Lu(x) + Ru(x) + Nu(x) = g(x), \tag{20}$$

where L is the linear operator, R is the linear remainder operator and N is the nonlinear operator such that

$$L = \frac{d}{dx}(\cdot), \quad Ru(x) = \alpha_0(x)u(x), \quad Nu(x) = \beta(x) f(u(x)).$$

We assume that $\alpha_0(x)$, $\beta(x)$ and $g(x)$ are analytic, and hence have the respective Taylor expansion series

$$\alpha_0(x) = \sum_{n=0}^{\infty} \alpha_{0,n}x^n, \quad \alpha_{0,n} = \frac{\alpha_0^{(n)}(0)}{n!},$$
$$\beta(x) = \sum_{n=0}^{\infty} \beta_n x^n, \quad \beta_n = \frac{\beta^{(n)}(0)}{n!},$$
$$g(x) = \sum_{n=0}^{\infty} g_n x^n, \quad g_n = \frac{g^{(n)}(0)}{n!}.$$

Applying the Adomian decomposition series and the series of Adomian polynomials, we have

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad Nu(x) = \sum_{n=0}^{\infty} A_n(x), \quad f(u(x)) = \sum_{n=0}^{\infty} B_n(x), \tag{21}$$

where the simple nonlinearity $f(u)$ can be any analytic function in u and the respective one-variable Adomian polynomials $B_n(x)$ have the standard formula [Adomian and Rach (1983); Duan (2010a,b, 2011a)] $B_n(x) = B_n(u_0(x), \ldots, u_n(x))$.

Calculating the respective Cauchy products for $\alpha_0(x)u(x)$ and $\beta(x)f(u(x))$, we obtain

$$Ru(x) = \alpha_0(x)u(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{0,n-m}x^{n-m}u_m(x), \tag{22}$$

$$Nu(x) = \beta(x)f(u(x)) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \beta_{n-m}x^{n-m}B_m(x), \tag{23}$$

hence we have also determined that

$$A_n(x) = \sum_{m=0}^{n} \beta_{n-m}x^{n-m}B_m(x). \tag{24}$$

Next we solve Eq. (20) for $Lu(x)$ as

$$Lu(x) = g(x) - Ru(x) - Nu(x). \tag{25}$$

Applying the one-fold definite integral operator $L^{-1} = \int_0^x (\cdot)\,dx$ to both sides of Eq. (25), we obtain

$$L^{-1}Lu(x) = L^{-1}g(x) - L^{-1}Ru(x) - L^{-1}Nu(x). \tag{26}$$

Next we integrate the left hand side of Eq. (26) and substitute the value specified in Eq. (19) to obtain

$$L^{-1}Lu(x) = u(x) - u(0) = u(x) - C_0. \tag{27}$$

Upon substitution of the right hand side of Eq. (27) in place of the left hand side of Eq. (26), we obtain

$$u(x) = C_0 + L^{-1}g(x) - L^{-1}Ru(x) - L^{-1}Nu(x), \tag{28}$$

which is the equivalent nonlinear Volterra integral equation for the solution $u(x)$ as defined by Eqs. (18) and (19). Evaluating the respective integrations $L^{-1}g(x)$, $L^{-1}Ru(x)$ and $L^{-1}Nu(x)$, we calculate

$$L^{-1}g(x) = \sum_{n=0}^{\infty} \frac{g_n x^{n+1}}{(n+1)}, \quad L^{-1}Ru(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{0,n-m}\int_0^x x^{n-m}u_m(x)\,dx, \tag{29}$$

$$L^{-1}Nu(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \beta_{n-m} \int_0^x x^{n-m} B_m(x)\,dx. \tag{30}$$

Upon substitution of Eqs. (21), (29) and (30) into Eq. (28), we obtain the Adomian decomposition series as

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x) &= C_0 + \sum_{n=0}^{\infty} \frac{g_n x^{n+1}}{(n+1)} - \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{0,n-m} \int_0^x x^{n-m} u_m(x)\,dx \\ &\quad - \sum_{n=0}^{\infty}\sum_{m=0}^{n} \beta_{n-m} \int_0^x x^{n-m} B_m(x)\,dx.\end{aligned}$$

Consequently, we set the following modified recursion scheme

$$\begin{aligned} u_0(x) &= C_0, \\ u_{n+1}(x) &= \tfrac{g_n x^{n+1}}{(n+1)} - \sum_{m=0}^{n} \int_0^x x^{n-m} \left[\alpha_{0,n-m} u_m(x) + \beta_{n-m} B_m(x)\right] dx, \end{aligned} \tag{31}$$

for $n \geq 0$, where the one-variable Adomian polynomials $B_m(x)$ are

$$B_m(x) = B_m(u_0(x), \ldots, u_m(x)). \tag{32}$$

In view of this pattern, the $(m+1)$th-stage solution approximant is given by $\phi_{m+1}(x) = \sum_{n=0}^{m} u_n(x)$, for $m \geq 0$, which, in the limit, yields $\lim_{m\to\infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x)$, i.e. the exact solution. Calculating the first several solution components using Eqs. (31) and (32), we deduce the following sequence

$$\begin{aligned} &u_0(x) = C_0 = c_0,\ u_1(x) = \tfrac{g_0 - \alpha_{0,0}c_0 - \beta_0 B_0}{(1)}\, x = c_1 x, \\ &u_2(x) = \tfrac{g_1 - \alpha_{0,1}c_0 - \alpha_{0,0}c_1 - \beta_1 B_0 - \beta_0 B_1}{(2)}\, x^2 = c_2 x^2,\ \ldots. \end{aligned}$$

By induction, we determine for $n \geq 0$ that

$$u_{n+1}(x) = \frac{g_n - \sum_{m=0}^{n} \left(\alpha_{0,n-m} c_m + \beta_{n-m} B_m\right)}{(n+1)}\, x^{n+1} = c_{n+1} x^{n+1}, \tag{33}$$

where the one-variable Adomian polynomials B_m now depend just on the solution coefficients c_j, for $0 \leq j \leq m$, and are determined concomitantly within Eq. (33) as

$$B_0 = B_0(c_0),\ B_1 = B_1(c_0, c_1),\ B_2 = B_2(c_0, c_1, c_2),\ \ldots,\ B_m = B_m(c_0, \ldots, c_m) \tag{34}$$

instead of the solution components $u_j(x)$ for $0 \leq j \leq m$. Thus we have derived the desired Taylor expansion series for the solution $u(x)$ as $u(x) = \sum_{n=0}^{\infty} c_n x^n$, where the solution coefficients c_n are given by the nonlinear recurrence relation, obtained from inspection of Eq. (33), as

$$c_0 = C_0,\ c_{n+1} = \frac{g_n - \sum\limits_{m=0}^{n} \left(\alpha_{0,n-m} c_m + \beta_{n-m} B_m\right)}{n+1},\ n \geq 0, \tag{35}$$

where the one-variable Adomian polynomials B_m are the same as shown in Eq. (34). Thus we have realized the rule of recursion for the solution coefficients of the first-order canonical inhomogeneous nonlinear IVP with a variable input and variable system coefficients as $c_{n+1} = c_{n+1}(c_0, \ldots, c_n)$, for $n \geq 0$.

2.2 The canonical second-order inhomogeneous nonlinear IVP with a variable input $g(x)$ and variable system coefficients $\alpha_1(x)$, $\alpha_0(x)$ and $\beta(x)$

We consider the following nonlinear second-order differential equation with an analytic input function and analytic system coefficient functions subject to two bounded initial conditions.

$$u''(x) + \alpha_1(x) u'(x) + \alpha_0(x) u(x) + \beta(x) f(u(x), u'(x)) = g(x), \tag{36}$$

$$u(0) = C_0,\ u'(0) = C_1, \tag{37}$$

or equivalently in Adomian's operator-theoretic notation

$$Lu(x) + Ru(x) + Nu(x) = g(x), \tag{38}$$

where $L = \frac{d^2}{dx^2}(\cdot)$, $Ru(x) = \alpha_1(x) u'(x) + \alpha_0(x) u(x)$, $Nu(x) = \beta(x) f(u(x), u'(x))$. We assume that $\alpha_1(x)$, $\alpha_0(x)$, $\beta(x)$ and $g(x)$ are analytic, and hence have the respective Taylor expansion series

$$\alpha_1(x) = \sum_{n=0}^{\infty} \alpha_{1,n} x^n, \quad \alpha_{1,n} = \frac{\alpha_1^{(n)}(0)}{n!},$$
$$\alpha_0(x) = \sum_{n=0}^{\infty} \alpha_{0,n} x^n, \quad \alpha_{0,n} = \frac{\alpha_0^{(n)}(0)}{n!},$$
$$\beta(x) = \sum_{n=0}^{\infty} \beta_n x^n, \quad \beta_n = \frac{\beta^{(n)}(0)}{n!},$$
$$g(x) = \sum_{n=0}^{\infty} g_n x^n, \quad g_n = \frac{g^{(n)}(0)}{n!}.$$

For the second-order differential operator L, the inverse operator L^{-1} exists and is given by $L^{-1} = \int_0^x \int_0^x (\cdot)\, dx dx$, where the initial conditions are specified at the origin.

We now recall that our stated goal is to calculate the Taylor expansion series of the solution $u(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ by decelerating the rate of convergence of the Adomian decomposition series of the solution $u(x) = \sum_{n=0}^{\infty} u_n(x)$, conclusively demonstrating that the Adomian decomposition series, any rearrangement of the Banach-space analog of the Taylor expansion series about a function that permits solution by recursion, subsumes the classic Taylor expansion series and that the ADM subsumes the classic power series method and additionally extends the class of amenable nonlinearities from only polynomial nonlinearities to any analytic nonlinearity. In order to achieve this result, we also adopt a new paradigm for the derivatives of the solution components $u_n(x)$ within the appropriate modified recursion scheme. We thus make the compatible assignment

$$u_n^{(j)}(x) := \frac{d^j}{dx^j} u_{n+j}(x),\ 0 \leq j \leq p-1,\ n \geq 0, \tag{39}$$

for a pth-order nonlinear differential equation, since there is always choice in the decomposition for benefit of computational advantage [Rach and Adomian (1990)], i.e. the decomposition is not unique, within the formulas for the respective components of $L^{-1}Ru(x)$ and $L^{-1}Nu(x)$ in the various modified recursion schemes. Our motivation is to have the first term of the Taylor expansion series of the various derivatives of the solution $u^{(j)}(x) = \sum_{n=0}^{\infty} c_n^{(j)} x^n$, i.e. a constant labeled $c_0^{(j)}$, to each be generally nonzero in order to facilitate subsequent calculations. Note that we reserve the notation $c_n^{(j)}$, $0 \leq j \leq p$, for coefficients of the Taylor expansion series of the solution derivatives in a pth-order nonlinear differential equation, and the similar, but distinct, notation $c_{k,n}$, $0 \leq k \leq q$, is reserved for coefficients of the Taylor expansion series of solutions in a system of q-coupled nonlinear differential equations.

Applying the Adomian decomposition series and the series of Adomian polynomials, we have

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad Nu(x) = \sum_{n=0}^{\infty} A_n(x), \quad f\left(u(x), u'(x)\right) = \sum_{n=0}^{\infty} B_n(x), \tag{40}$$

where the multi-order differential nonlinearity $f(u, u')$ can be any analytic function in u and u' and the respective two-variable Adomian polynomials have the standard formula [Adomian and Rach (1983); Duan (2010a,b, 2011a)]

$$\begin{aligned} B_n(x) &= B_n\left(u_0(x), \ldots, u_n(x);\ u_0'(x), \ldots, u_n'(x)\right) \\ &= B_n\left(u_0(x), \ldots, u_n(x);\ \frac{d}{dx}u_1(x), \ldots, \frac{d}{dx}u_{n+1}(x)\right). \end{aligned}$$

Calculating the respective Cauchy products, we obtain

$$\alpha_1(x)u'(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{1,n-m}x^{n-m}\frac{d}{dx}u_{m+1}(x),$$

$$\alpha_0(x)u(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{0,n-m}x^{n-m}u_m(x),$$

$$Nu(x) = \beta(x)f\left(u(x),u'(x)\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \beta_{n-m}x^{n-m}B_m(x),$$

hence we have also determined that $A_n(x) = \sum_{m=0}^{n} \beta_{n-m}x^{n-m}B_m(x)$. Next we solve Eq. (38) for $Lu(x)$ as

$$Lu(x) = g(x) - Ru(x) - Nu(x). \tag{41}$$

Applying the two-fold integral operator L^{-1} to both sides of Eq. (41), calculating integrations on the left hand side and using the values specified in Eq. (37), we obtain

$$u(x) = C_0 + xC_1 + L^{-1}g(x) - L^{-1}Ru(x) - L^{-1}Nu(x), \tag{42}$$

which is the equivalent nonlinear Volterra integral equation for the solution $u(x)$ as defined by Eqs. (36) and (37). Evaluating the respective integrations $L^{-1}g(x)$, $L^{-1}Ru(x)$ and $L^{-1}Nu(x)$, we calculate

$$L^{-1}g(x) = \sum_{n=0}^{\infty} \frac{g_n x^{n+2}}{(n+1)(n+2)}, \tag{43}$$

$$L^{-1}Ru(x) = L^{-1}\alpha_1(x)u'(x) + L^{-1}\alpha_0(x)u(x), \tag{44}$$

where

$$L^{-1}\alpha_1(x)u'(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{1,n-m}\int_0^x\int_0^x x^{n-m}\frac{d}{dx}u_{m+1}(x)\,dxdx, \tag{45}$$

$$L^{-1}\alpha_0(x)u(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{0,n-m}\int_0^x\int_0^x x^{n-m}u_m(x)\,dxdx, \tag{46}$$

$$L^{-1}Nu(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \beta_{n-m}\int_0^x\int_0^x x^{n-m}B_m(x)\,dxdx. \tag{47}$$

Upon substitution of Eqs. (40), (43), (44), (45), (46) and (47) into Eq. (42), we can consequently set the following modified recursion scheme

$$\begin{aligned}
&u_0(x) = C_0,\ u_1(x) = xC_1,\\
&u_{n+2}(x) = \frac{g_n x^{n+2}}{(n+1)(n+2)} - \sum_{m=0}^{n}\Big[\alpha_{1,n-m}L^{-1}x^{n-m}\frac{d}{dx}u_{m+1}(x)\\
&\qquad +\alpha_{0,n-m}L^{-1}x^{n-m}u_m(x) + \beta_{n-m}L^{-1}x^{n-m}B_m(x)\Big]
\end{aligned} \tag{48}$$

for $n \geq 0$, where the two-variable Adomian polynomials $B_m(x)$ are

$$B_m(x) = B_m\left(u_0(x),\ldots,u_m(x);\ \frac{d}{dx}u_1(x),\ldots,\frac{d}{dx}u_{m+1}(x)\right). \tag{49}$$

Calculating the first several solution components $u_n(x)$ using Eqs. (48) and (49), we deduce the following sequence

$$\begin{aligned}
&u_0(x) = C_0 = c_0,\ u_1(x) = xC_1 = c_1 x,\\
&u_2(x) = \tfrac{g_0-\alpha_{1,0}c_1-\alpha_{0,0}c_0-\beta_0 B_0}{(1)(2)}\,x^2 = c_2x^2,\\
&u_3(x) = \tfrac{g_1-\alpha_{1,1}c_1-2\alpha_{1,0}c_2-\alpha_{0,1}c_0-\alpha_{0,0}c_1-\beta_1 B_0-\beta_0 B_1}{(2)(3)}\,x^3 = c_3x^3,\ \ldots.
\end{aligned} \tag{50}$$

By induction, we determine for $n \geq 0$ that

$$u_{n+2}(x) = \frac{g_n - \sum\limits_{m=0}^{n}\left[(m+1)\alpha_{1,n-m}c_{m+1}+\alpha_{0,n-m}c_m+\beta_{n-m}B_m\right]}{(n+1)(n+2)}\,x^{n+2} = c_{n+2}x^{n+2}, \tag{51}$$

where the two-variable Adomian polynomials B_m are now constants depending just on the solution coefficients and solution derivative coefficients $c_j^{(k)}$, for $0 \leq j \leq m$ and $0 \leq k \leq 1$, and are determined concomitantly within Eqs. (50) and (51) as

$$\begin{aligned}
&B_0 = B_0\left(c_0; c_0^{(1)}\right) = B_0\left(c_0;\ (1)c_1\right),\\
&B_1 = B_1\left(c_0,c_1; c_0^{(1)},c_1^{(1)}\right) = B_1\left(c_0,c_1;\ (1)c_1,(2)c_2\right),\\
&B_2 = B_2\left(c_0,c_1,c_2; c_0^{(1)},c_1^{(1)},c_2^{(1)}\right) = B_2\left(c_0,c_1,c_2;\ (1)c_1,(2)c_2,(3)c_3\right),\ \ldots.
\end{aligned}$$

By induction, we also determine for $n \geq 0$ that

$$\begin{aligned}
B_m &= B_m(c_0,\ldots,c_m; c_0^{(1)},\ldots,c_m^{(1)})\\
&= B_m(c_0,\ldots,c_m;\ (1)c_1,\ldots,(m+1)c_{m+1})
\end{aligned} \tag{52}$$

instead of the solution components $u_j(x)$ and solution derivative components $u_j^{(k)}(x)$ for $0 \leq j \leq m$ and $0 \leq k \leq 1$. Thus we have derived the desired Taylor expansion series for the solution as $u(x) = \sum_{n=0}^{\infty} c_n x^n$, where the solution coefficients c_n are given by the nonlinear recurrence relation, obtained from inspection of Eqs. (50) and (51), as

$$c_0 = C_0,\ c_1 = C_1,$$
$$c_{n+2} = \frac{g_n - \sum_{m=0}^{n} \left[(m+1)\,\alpha_{1,n-m} c_{m+1} + \alpha_{0,n-m} c_m + \beta_{n-m} B_m\right]}{(n+1)(n+2)},\ n \geq 0, \tag{53}$$

where the two-variable Adomian polynomials B_m are the same as shown in Eq. (52). Thus we have realized the rule of recursion for the solution coefficients of the second-order canonical inhomogeneous nonlinear IVP with a variable input and variable system coefficients as $c_{n+2} = c_{n+2}(c_0, \ldots, c_{n+1})$, for $n \geq 0$.

2.3 The canonical pth-order, for the first- or higher-order, inhomogeneous nonlinear IVP with a variable input $g(x)$ and variable system coefficients $\alpha_{p-1}(x), \ldots, \alpha_0(x)$ and $\beta(x)$

We consider the following nonlinear pth-order differential equation for $p \geq 1$ with an analytic input function and analytic system coefficient functions subject to p bounded initial conditions.

$$u^{(p)}(x) + \sum_{\nu=0}^{p-1} \alpha_{p-1-\nu}(x)\, u^{(p-1-\nu)}(x) + \beta(x)\, f\left(u(x), \ldots, u^{(p-1)}(x)\right) = g(x), \tag{54}$$

$$u^{(j)}(0) = C_j, \quad 0 \leq j \leq p-1, \quad p \geq 1, \tag{55}$$

or equivalently in Adomian's operator-theoretic notation

$$Lu(x) + Ru(x) + Nu(x) = g(x), \tag{56}$$

where L is the linear operator, R is the linear remainder operator, i.e. generally a sequential-order differential operator, and N is the nonlinear operator such that

$$L = \frac{d^p}{dx^p}(\cdot),\ Ru(x) = \sum_{\nu=0}^{p-1} \alpha_{p-1-\nu}(x)\, u^{(p-1-\nu)}(x), \text{ and}$$

$$Nu(x) = \beta(x)\, f\left(u(x), \ldots, u^{(p-1)}(x)\right).$$

We assume that $\alpha_{p-1}(x),\ldots,\alpha_0(x)$, $\beta(x)$, and $g(x)$ are analytic, and hence have the respective Taylor expansion series

$$\alpha_j(x)=\sum_{n=0}^{\infty}\alpha_{j,n}x^n,\quad \alpha_{j,n}=\frac{\alpha_j^{(n)}(0)}{n!},\quad 0\le j\le p-1,$$
$$\beta(x)=\sum_{n=0}^{\infty}\beta_n x^n,\quad \beta_n=\frac{\beta^{(n)}(0)}{n!},$$
$$g(x)=\sum_{n=0}^{\infty}g_n x^n,\quad g_n=\frac{g^{(n)}(0)}{n!}.$$

The inverse operator L^{-1} of the pth-order differential operator L exists and is given by the p-fold definite integral

$$L^{-1}=\int_0^x\cdots\int_0^x(\cdot)\,dx\cdots dx,$$

where the initial conditions are specified at the origin.

Applying the Adomian decomposition series and the series of the Adomian polynomials, we have

$$u(x)=\sum_{n=0}^{\infty}u_n(x),\ Nu(x)=\sum_{n=0}^{\infty}A_n(x),\ f\left(u(x),\ldots,u^{(p-1)}(x)\right)=\sum_{n=0}^{\infty}B_n(x),\quad (57)$$

where the multi-order differential nonlinearity $f\left(u,\ldots,u^{(p-1)}\right)$ can be any analytic function in $u,\ldots,u^{(p-1)}$ and the respective p-variable Adomian polynomials $B_n(x)$ have the standard formula [Adomian and Rach (1983); Duan (2010a,b, 2011a)]

$$\begin{aligned}B_n(x)&=B_n\left(u_0(x),\ldots,u_n(x);\ldots;u_0^{(p-1)}(x),\ldots,u_n^{(p-1)}(x)\right)\\&=B_n\left(u_0(x),\ldots,u_n(x);\ldots;\tfrac{d^{p-1}}{dx^{p-1}}u_{p-1}(x),\ldots,\tfrac{d^{p-1}}{dx^{p-1}}u_{n+p-1}(x)\right).\end{aligned}$$

Calculating the respective Cauchy products for $\alpha_{p-1}(x)u^{(p-1)}(x),\ldots,\alpha_0(x)u(x)$ and $\beta(x)f\left(u(x),\ldots,u^{(p-1)}(x)\right)$, we obtain $Ru(x)=\sum\limits_{\nu=0}^{p-1}\alpha_{p-1-\nu}(x)u^{(p-1-\nu)}(x)$, where

$$\alpha_j(x)u^{(j)}(x)=\sum_{n=0}^{\infty}\alpha_{j,n}x^n\sum_{m=0}^{\infty}u_m^{(j)}(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\alpha_{j,n-m}x^{n-m}\frac{d^j}{dx^j}u_{m+j}(x),$$

where we have made the substitution $j=p-1-\nu$ for convenience,

$$Nu(x)=\beta(x)f\left(u(x),\ldots,u^{(p-1)}(x)\right)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\beta_{n-m}x^{n-m}B_m(x),$$

hence we have also determined that $A_n(x) = \sum_{m=0}^{n} \beta_{n-m} x^{n-m} B_m(x)$.

Next we solve Eq. (56) for $Lu(x)$ as

$$Lu(x) = g(x) - Ru(x) - Nu(x). \tag{58}$$

Applying the p-fold integral operator L^{-1} to both sides of Eq. (58), we obtain

$$L^{-1}Lu(x) = L^{-1}g(x) - L^{-1}Ru(x) - L^{-1}Nu(x). \tag{59}$$

Next we integrate the left hand side of Eq. (59) and substitute the values specified in Eq. (55) to obtain

$$L^{-1}Lu(x) = u(x) - \sum_{\nu=0}^{p-1} \frac{x^\nu}{\nu!} C_\nu. \tag{60}$$

Upon substitution of the right hand side of Eq. (60) in place of the left hand side of Eq. (59), we obtain

$$u(x) = \sum_{\nu=0}^{p-1} \frac{x^\nu}{\nu!} C_\nu + L^{-1}g(x) - L^{-1}Ru(x) - L^{-1}Nu(x), \tag{61}$$

which is the equivalent nonlinear Volterra integral equation for the solution $u(x)$ as defined by Eqs. (54) and (55). Evaluating the respective integrations $L^{-1}g(x)$, $L^{-1}Ru(x)$ and $L^{-1}Nu(x)$, we calculate

$$L^{-1}g(x) = \sum_{n=0}^{\infty} \frac{g_n x^{n+p}}{(n+1)\cdots(n+p)}, \tag{62}$$

$$L^{-1}Ru(x) = \sum_{\nu=0}^{p-1} L^{-1}\alpha_{p-1-\nu}(x)\, u^{(p-1-\nu)}(x), \tag{63}$$

where

$$L^{-1}\alpha_j(x)\, u^{(j)}(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \alpha_{j,n-m} L^{-1} x^{n-m} \frac{d^j}{dx^j} u_{m+j}(x), \tag{64}$$

where $j = p-1-\nu$, and

$$L^{-1}Nu(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \beta_{n-m} L^{-1} x^{n-m} B_m(x). \tag{65}$$

Upon substitution of Eqs. (57), (62), (63), (64) and (65) into Eq. (61), we can consequently set the following modified recursion scheme

$$u_0(x) = C_0,\ u_1(x) = xC_1,\ u_2(x) = \frac{x^2}{2!}C_2,\ \ldots,\ u_{p-1}(x) = \frac{x^{p-1}}{(p-1)!}C_{p-1},$$

$$\begin{aligned} u_{n+p}(x) &= \frac{g_n x^{n+p}}{\prod\limits_{i=1}^{p}(n+i)} - \sum_{m=0}^{n}\sum_{\nu=0}^{p-1}\alpha_{p-1-\nu,n-m}L^{-1}x^{n-m}\frac{d^{p-1-\nu}}{dx^{p-1-\nu}}u_{m+p-1-\nu}(x) \\ &\quad - \sum_{m=0}^{n}\beta_{n-m}L^{-1}x^{n-m}B_m(x), \end{aligned}$$

for $n \geq 0$, where the p-variable Adomian polynomials are

$$B_m(x) = B_m\left(u_0(x), \ldots, u_m(x); \ldots; \frac{d^{p-1}}{dx^{p-1}}u_{p-1}(x), \ldots, \frac{d^{p-1}}{dx^{p-1}}u_{m+p-1}(x)\right).$$

In view of this pattern, the $(m+1)$th-stage solution approximant $\phi_{m+1}(x)$ is given by $\phi_{m+1}(x) = \sum\limits_{n=0}^{m} u_n(x)$, for $m \geq 0$.

Calculating the first several solution components, we deduce the following sequence

$$u_0(x) = C_0 = c_0,\ u_1(x) = xC_1 = c_1 x,\ \ldots,\ u_{p-1}(x) = \frac{x^{p-1}}{(p-1)!}C_{p-1} = c_{p-1}x^{p-1},$$

and for $n \geq 0$, we determine, by induction, that

$$\begin{aligned} u_{n+p}(x) &= \frac{g_n - \sum\limits_{m=0}^{n}\left\{\beta_{n-m}B_m + \sum\limits_{\nu=0}^{p-1}\left[\prod\limits_{j=1}^{p-1-\nu}(m+j)\right]\alpha_{p-1-\nu,n-m}\,c_{m+p-1-\nu}\right\}}{\prod\limits_{i=1}^{p}(n+i)}x^{n+p} \\ &= c_{n+p}x^{n+p}, \end{aligned} \tag{66}$$

when we define the product $\prod\limits_{j=1}^{0}(m+j) = 1$, i.e. when $\nu = p-1$, in analogy to the commonly accepted convention of zero factorial, i.e. $0! = \prod\limits_{j=1}^{0}(j) = 1$, where $n! = \prod\limits_{j=1}^{n}(j) = (1)\cdots(n)$, and where the p-variable Adomian polynomials B_m are now constants depending just on the solution coefficients c_j, for $0 \leq j \leq m+p-1$, and are determined concomitantly as

$$\begin{aligned}
B_0 &= B_0\left(c_0;\ldots;c_0^{(p-1)}\right) = B_0\left(c_0;\ldots;(1)\cdots(p-1)\,c_{p-1}\right),\\
B_1 &= B_1\left(c_0,c_1;\ldots;c_0^{(p-1)},c_1^{(p-1)}\right)\\
&= B_1\left(c_0,c_1;\ldots;(1)\cdots(p-1)\,c_{p-1},(2)\cdots(p)\,c_p\right),\\
B_2 &= B_2\left(c_0,c_1,c_2;\ldots;c_0^{(p-1)},c_1^{(p-1)},c_2^{(p-1)}\right)\\
&= B_2\left(c_0,c_1,c_2;\ldots;(1)\cdots(p-1)\,c_{p-1},(2)\cdots(p)\,c_p,(3)\cdots(p+1)\,c_{p+1}\right).
\end{aligned}$$

By induction, we also determine for $m \geq 0$ that

$$\begin{aligned}
B_m &= B_m\left(c_0,\ldots,c_m;\ldots;c_0^{(p-1)},\ldots,c_m^{(p-1)}\right)\\
&= B_m\left(c_0,\ldots,c_m;\ldots;\left\{\prod_{j=1}^{p-1}(j)\right\}c_{p-1},\ldots,\left\{\prod_{j=1}^{p-1}(m+j)\right\}c_{m+p-1}\right)
\end{aligned} \tag{67}$$

instead of the solution components $u_j(x)$ and solution derivative components $u_j^{(k)}(x)$ for $0 \leq j \leq m$ and $0 \leq k \leq p-1$. Thus we have derived the desired Taylor expansion series for the solution as $u(x) = \sum_{n=0}^{\infty} c_n x^n$, where the solution coefficients c_n are given by the nonlinear recurrence relation, obtained from inspection of Eq. (66), as

$$\begin{aligned}
c_j &= \frac{C_j}{j!},\ 0 \leq j \leq p-1,\\
c_{n+p} &= \frac{g_n - \sum\limits_{m=0}^{n}\left\{\beta_{n-m}B_m + \sum\limits_{\nu=0}^{p-1}\left\{\prod\limits_{j=1}^{p-1-\nu}(m+j)\right\}\alpha_{p-1-\nu,n-m}c_{m+p-1-\nu}\right\}}{\prod\limits_{i=1}^{p}(n+i)},\ n \geq 0,
\end{aligned} \tag{68}$$

where the $p-$variable Adomian polynomials B_m depend on the solution coefficients c_j, for $0 \leq j \leq m+p-1$, as in Eq. (67).

Thus we have realized the rule of recursion for the solution coefficients of the pth-order canonical inhomogeneous nonlinear IVP with a variable input and variable system coefficients as $c_{n+p} = c_{n+p}(c_0,\ldots,c_{n+p-1})$, for $n \geq 0$.

2.4 *The canonical $p_{\max}$th-order system of q-coupled p_kth-order inhomogeneous nonlinear IVPs with variable inputs $g_k(x)$ and variable system coefficients $\alpha_{k,s,j}(x)$ and $\beta_k(x)$ for $1 \leq k,s \leq q$, $0 \leq j \leq p_s - 1$, $p_k, p_s \geq 1$, $p_{\max} = \max(p_1,\ldots,p_q)$, $q \geq 1$*

We consider the following canonical system of coupled nonlinear IVPs consisting of q-coupled nonlinear p_kth-order differential equations for $p_k \geq 1$, $q \geq 1$ with analytic input functions and analytic system coefficient functions subject to appropriate ($\sum\limits_{s=1}^{q} p_s$) bounded initial conditions, i.e. p_k bounded initial conditions for each p_kth-order nonlinear differential equation, $1 \le k \le q$.

$$\begin{aligned} &u_k^{(p_k)}(x) + \sum_{s=1}^{q}\sum_{\nu=0}^{p_s-1} \alpha_{k,s,p_s-1-\nu}(x)\, u_s^{(p_s-1-\nu)}(x) + \\ &\beta_k(x)\, f_k\left(u_1(x), \ldots, u_1^{(p_1-1)}(x); \ldots; u_q(x), \ldots, u_q^{(p_q-1)}(x)\right) = g_k(x), \end{aligned} \tag{69}$$

$$u_k^{(j)}(0) = C_{k,j}, \quad 1 \le k \le q, \quad 0 \le j \le p_k - 1, \quad q \ge 1, \quad p_k \ge 1, \tag{70}$$

or equivalently in Adomian's operator-theoretic notation

$$L_k u_k(x) + R_k u_1(x), \ldots, u_q(x) + N_k u_1(x), \ldots, u_q(x) = g_k(x), \tag{71}$$

where L_k represents the linear operators, R_k represents the linear remainder operators, i.e. generally sequential-order differential operators, and N_k represents the nonlinear operators such that

$$\begin{aligned} &L_k = \tfrac{d^{p_k}}{dx^{p_k}}(\cdot),\ R_k u_1(x), \ldots, u_q(x) = \sum_{s=1}^{q}\sum_{\nu=0}^{p_s-1} \alpha_{k,s,p_s-1-\nu}(x)\, u_s^{(p_s-1-\nu)}(x), \\ &N_k u_1(x), \ldots, u_q(x) = \beta_k(x)\, f_k\left(u_1(x), \ldots, u_1^{(p_1-1)}(x); \ldots; u_q(x), \ldots, u_q^{(p_q-1)}(x)\right). \end{aligned}$$

For a particular p_kth-order nonlinear differential equation in the system represented by Eq. (69) or Eq. (71), we designate the corresponding solution $u_k(x)$ as the primary solution and the solutions $u_s(x)$, for $s \ne k$, as the secondary solutions with respect to this same p_kth-order differential equation. We assume that $\alpha_{k,s,p_s-1}(x), \ldots, \alpha_{k,s,0}(x)$, $\beta_k(x)$, and $g_k(x)$ are analytic, and hence have the respective Taylor expansion series

$$\begin{aligned} &\alpha_{k,s,j}(x) = \sum_{n=0}^{\infty} \alpha_{k,s,j,n} x^n, \quad \alpha_{k,s,j,n} = \tfrac{\alpha_{k,s,j}^{(n)}(0)}{n!}, \quad 1 \le k, s \le q, \quad 0 \le j \le p_s - 1, \\ &\beta_k(x) = \sum_{n=0}^{\infty} \beta_{k,n} x^n, \quad \beta_{k,n} = \tfrac{\beta_k^{(n)}(0)}{n!}, \\ &g_k(x) = \sum_{n=0}^{\infty} g_{k,n} x^n, \quad g_{k,n} = \tfrac{g_k^{(n)}(0)}{n!}. \end{aligned}$$

The linear differential operators L_k are invertible, and their inverse operators L_k^{-1} are given by the p_k-fold integral

$$L_k^{-1} = \int_0^x \cdots \int_0^x (\cdot)\, dx \cdots dx,$$

for the case of a system of q-coupled p_kth-order IVPs, where the initial conditions are all specified at the origin.

Applying the Adomian decomposition series and the series of the Adomian polynomials, we have

$$u_k(x) = \sum_{n=0}^{\infty} u_{k,n}(x), \quad N_k u_1(x), \ldots, u_q(x) = \sum_{n=0}^{\infty} A_{k,n}(x),$$
$$f_k\left(u_1(x), \ldots, u_1^{(p_1-1)}(x); \ldots; u_q(x), \ldots, u_q^{(p_q-1)}(x)\right) = \sum_{n=0}^{\infty} B_{k,n}(x), \tag{72}$$

where the multi-variable, multi-order differential nonlinearity

$$f_k\left(u_1(x), \ldots, u_1^{(p_1-1)}(x); \ldots; u_q(x), \ldots, u_q^{(p_q-1)}(x)\right)$$

can be any analytic function in $u_1(x), \ldots, u_1^{(p_1-1)}(x)$, $\ldots$, $u_q(x), \ldots, u_q^{(p_q-1)}(x)$ and the respective $(\sum_{s=1}^{q} p_s)$-variable Adomian polynomials $B_{k,n}(x)$ have the standard formula [Adomian and Rach (1983); Duan (2010a,b, 2011a)],

$$\begin{aligned} B_{k,n}(x) \;&=\; B_{k,n}\Big(u_{1,0}(x), \ldots, u_{1,n}(x); \ldots; u_{1,0}^{(p_1-1)}(x), \ldots, u_{1,n}^{(p_1-1)}(x); \ldots; \\ &\qquad u_{q,0}(x), \ldots, u_{q,n}(x); \ldots; u_{q,0}^{(p_q-1)}(x), \ldots, u_{q,n}^{(p_q-1)}(x)\Big), \end{aligned}$$

or equivalently,

$$\begin{aligned} &B_{k,n}(x) = B_{k,n}\Big(u_{1,0}(x), \ldots, u_{1,n}(x); \ldots; \tfrac{d^{p_1-1}}{dx^{p_1-1}} u_{1,p_1-1}(x), \ldots, \tfrac{d^{p_1-1}}{dx^{p_1-1}} u_{1,n+p_1-1}(x); \\ &\ldots; u_{q,0}(x), \ldots, u_{q,n}(x); \ldots; \tfrac{d^{p_q-1}}{dx^{p_q-1}} u_{q,p_q-1}(x), \ldots, \tfrac{d^{p_q-1}}{dx^{p_q-1}} u_{q,n+p_q-1}(x)\Big). \end{aligned} \tag{73}$$

Calculating the respective Cauchy products, we obtain

$$R_k u_1(x), \ldots, u_q(x) = \sum_{s=1}^{q} \sum_{\nu=0}^{p_s-1} \alpha_{k,s,p_s-1-\nu}(x)\, u_s^{(p_s-1-\nu)}(x),$$

where

$$\alpha_{k,s,j}(x)\, u_s^{(j)}(x) = \sum_{n=0}^{\infty} \alpha_{k,s,j,n} x^n \sum_{m=0}^{\infty} u_{s,m}^{(j)}(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha_{k,s,j,n-m} x^{n-m} \tfrac{d^j}{dx^j} u_{s,m+j}(x),$$

where we have made the substitution $j = p_s - 1 - \nu$ for convenience,

$$N_k u_1(x), \ldots, u_q(x) = \sum_{n=0}^{\infty} \beta_{k,n} x^n \sum_{m=0}^{\infty} B_{k,m}(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \beta_{k,n-m} x^{n-m} B_{k,m}(x),$$

hence we have also determined that

$$A_{k,n}(x) = \sum_{m=0}^{n} \beta_{k,n-m} x^{n-m} B_{k,m}(x).$$

Next we solve Eq. (71) for $L_k u(x)$ as

$$L_k u_k(x) = g_k(x) - R_k u_1(x), \ldots, u_q(x) - N_k u_1(x), \ldots, u_q(x). \tag{74}$$

Applying the p_k-fold integral operator L_k^{-1} to both sides of Eq. (74), we obtain

$$L_k^{-1} L_k u_k(x) = L_k^{-1} g_k(x) - L_k^{-1} R_k u_1(x), \ldots, u_q(x) - L_k^{-1} N_k u_1(x), \ldots, u_q(x). \tag{75}$$

Next we integrate the left hand side of Eq. (75) and substitute the values specified in Eq. (70) to obtain

$$L_k^{-1} L_k u_k(x) = u_k(x) - \sum_{\nu=0}^{p_k-1} \frac{x^\nu}{\nu!} C_{k,\nu}. \tag{76}$$

Upon substitution of the right hand side of Eq. (76) in place of the left hand side of Eq. (75), we obtain

$$u_k(x) = \sum_{\nu=0}^{p_k-1} \frac{x^\nu}{\nu!} C_{k,\nu} + L_k^{-1} g_k(x) - L_k^{-1} R_k u_1(x), \ldots, u_q(x) - L_k^{-1} N_k u_1(x), \ldots, u_q(x), \tag{77}$$

which is the equivalent system of q-coupled nonlinear Volterra integral equations for the q solutions $u_k(x)$, $1 \le k \le q$, as defined by Eqs. (69) and (70) representing the system of q-coupled nonlinear IVPs. Evaluating the respective integrations, we obtain

$$L_k^{-1} g_k(x) = \sum_{n=0}^{\infty} \frac{g_{k,n} x^{n+p_k}}{(n+1)\cdots(n+p_k)}, \tag{78}$$

$$L_k^{-1} R_k u_1(x), \ldots, u_q(x) = \sum_{s=1}^{q} \sum_{\nu=0}^{p_s-1} L_k^{-1} \alpha_{k,s,p_s-1-\nu}(x)\, u_s^{(p_s-1-\nu)}(x), \tag{79}$$

where

$$L_k^{-1} \alpha_{k,s,j}(x)\, u_s^{(j)}(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha_{k,s,j,n-m} L_k^{-1} x^{n-m} \frac{d^j}{dx^j} u_{s,m+j}(x), \tag{80}$$

where we made the substitution $j = p_s - 1 - \nu$ for convenience.

$$L_k^{-1} N_k u_1(x), \ldots, u_q(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \beta_{k,n-m} L_k^{-1} x^{n-m} B_{k,m}(x). \tag{81}$$

Upon substitution of Eqs. (72), (78), (79), (80) and (81) into Eq. (77), we can consequently set the following system of q-coupled modified recursion schemes

$$u_{k,0}(x) = C_{k,0},\ u_{k,1}(x) = xC_{k,1},\ u_{k,2}(x) = \frac{x^2}{2!}C_{k,2},\ \ldots,\ u_{k,p_k-1}(x) = \frac{x^{p_k-1}}{(p_k-1)!}C_{k,p_k-1},$$

$$\begin{aligned} u_{k,n+p_k}(x) = {} & \frac{g_{k,n}x^{n+p_k}}{\prod\limits_{i=1}^{p_k}(n+i)} - \sum_{m=0}^{n} \beta_{k,n-m} L_k^{-1} x^{n-m} B_{k,m}(x) \\ & - \sum_{m=0}^{n}\sum_{s=1}^{q}\sum_{\nu=0}^{p_s-1} \alpha_{k,s,p_s-1-\nu,n-m} L_k^{-1} x^{n-m} \frac{d^{p_s-1-\nu}}{dx^{p_s-1-\nu}} u_{s,m+p_s-1-\nu}(x), \end{aligned}$$

for $1 \le k \le q$, $p_k, p_s \ge 1$, $n \ge 0$. In view of this pattern, the $(m+1)$th-stage solution approximants $\phi_{k,m+1}(x)$ are given by $\phi_{k,m+1}(x) = \sum\limits_{n=0}^{m} u_{k,n}(x)$, for $m \ge 0$.

Calculating the first several solution components, we deduce the following sequence

$$\begin{aligned} & u_{k,0}(x) = C_{k,0} = c_{k,0},\ \ u_{k,1}(x) = xC_{k,1} = c_{k,1}x,\ \ldots, \\ & u_{k,p_k-1}(x) = \tfrac{x^{p_k-1}}{(p_k-1)!}C_{k,p_k-1} = c_{k,p_k-1}x^{p_k-1}, \end{aligned}$$

$$\begin{aligned} u_{k,n+p_k}(x) &= \frac{g_{k,n} - \sum\limits_{m=0}^{n}\left\{\beta_{k,n-m}B_{k,m} + \sum\limits_{s=1}^{q}\sum\limits_{\nu=0}^{p_s-1}\left[\prod\limits_{j=1}^{p_s-1-\nu}(m+j)\right]\alpha_{k,s,p_s-1-\nu,n-m}\,c_{s,m+p_s-1-\nu}\right\}}{\prod\limits_{i=1}^{p_k}(n+i)} x^{n+p_k} \\ &= c_{k,n+p_k}x^{n+p_k},\quad n \ge 0, \end{aligned} \tag{82}$$

where the $(\sum\limits_{s=1}^{q} p_s)$-variable Adomian polynomials $B_{k,m}$ are now constants depending just on the solution coefficients $c_{k,j}$, for $0 \le j \le m+p_s-1$, and are determined through induction as

$$\begin{aligned} B_{k,m} = B_{k,m}\Big(& c_{1,0},\ldots,c_{1,m};\ldots;c_{1,0}^{(p_1-1)},\ldots,c_{1,m}^{(p_1-1)};\ldots;c_{q,0},\ldots,c_{q,m};\ldots; \\ & c_{q,0}^{(p_q-1)},\ldots,c_{q,m}^{(p_q-1)}\Big) \end{aligned} \tag{83}$$

or equivalently,

$$\begin{aligned} B_{k,m} = B_{k,m}\Bigg(& c_{1,0},\ldots,c_{1,m};\ldots;\left\{\prod_{j=1}^{p_1-1}(j)\right\}c_{1,p_1-1},\ldots,\left\{\prod_{j=1}^{p_1-1}(m+j)\right\}c_{1,m+p_1-1}; \\ & \ldots;c_{q,0},\ldots,c_{q,m};\ldots;\left\{\prod_{j=1}^{p_q-1}(j)\right\}c_{q,p_q-1},\ldots,\left\{\prod_{j=1}^{p_q-1}(m+j)\right\}c_{q,m+p_q-1}\Bigg) \end{aligned}$$

(84)

instead of the solution components $u_{k,j}(x)$ and solution derivative components $u_{k,j}^{(i)}(x)$ for $1 \le k \le q$, $0 \le j \le m$ and $0 \le i \le p_s - 1$. Thus we have derived the desired Taylor expansion series for each of the q solutions $u_k(x)$ as $u_k(x) = \sum_{n=0}^{\infty} c_{k,n} x^n$, where the solution coefficients $c_{k,n}$ are given by the system of q-coupled nonlinear recurrence relations, obtained from inspection of Eq. (82), as

$$
\begin{aligned}
c_{k,j} &= \frac{C_{k,j}}{j!},\ 1 \le k \le q,\ 0 \le j \le p_k - 1, \\
c_{k,n+p_k} &= \frac{g_{k,n} - \sum\limits_{m=0}^{n} \left\{ \beta_{k,n-m} B_{k,m} + \sum\limits_{s=1}^{q} \sum\limits_{\nu=0}^{p_s-1} \left\{ \prod\limits_{j=1}^{p_s-1-\nu} (m+j) \right\} \alpha_{k,s,p_s-1-\nu,n-m} c_{s,m+p_s-1-\nu} \right\}}{\prod\limits_{i=1}^{p_k} (n+i)},
\end{aligned}
\tag{85}
$$

$n \ge 0$, $q \ge 1$ and $p_k \ge 1$, where the $(\sum_{s=1}^{q} p_s)$-variable Adomian polynomials $B_{k,m}$ depend on the solution coefficients $c_{k,j}$, for $0 \le j \le m + p_s - 1$, as in Eq. (84).

Thus we have realized the rule of recursion for the solution coefficients of the canonical $p_{\max}$th-order system of q-coupled p_kth-order inhomogeneous nonlinear IVPs with variable inputs and variable system coefficients as

$$c_{k,n+p_k} = c_{k,n+p_k}\left(c_{1,0},\ldots,c_{1,n+p_1-1};\ldots;c_{s,0},\ldots,c_{s,n+p_s-1};\ldots;c_{q,0},\ldots,c_{q,n+p_q-1}\right),$$

for $n \ge 0$, $1 \le k$, $s \le q$.

Although all of our results consisting of Eqs. (34) and (35), (52) and (53), (67) and (68), (84) and (85) can also be directly calculated by the MDM [Rach, Adomian, and Meyers (1992)], which is based on the Adomian-Rach theorem for nonlinear transformation of power series [Adomian and Rach (1991, 1992a)], we do emphasize that we have derived these several nonlinear recurrence relations for the solution coefficients of the solutions' unique Taylor expansion series solely by decelerating the Adomian decomposition series by a judicious choice of a system of appropriate coupled modified recursion schemes along with compatible assignments of the derivatives of the solution components for these several nonlinear IVPs with variable inputs and variable system coefficients.

3 Automated calculation, convergence acceleration, multistage decomposition, error analysis and numeric algorithms

We have designed the MATHEMATICA routine PSSOL for implementing the algorithm in Subsection 2.4, as given by Eqs. (84) and (85). PSSOL requires six inputs, i.e. in the form PSSOL[Init, Alpha, Be, F, G, M], where Init denotes the

2-D table of the initial values $C_{k,j}$, Alpha denotes the 3-D table of the coefficients $\alpha_{k,s,j}$, Be denotes the 1-D table of the coefficients β_k of nonlinearities, F denotes the 1-D table of the nonlinearities f_k, G denotes the 1-D table of the system input $g_k(x)$, and M is a nonnegative integer that determines the degrees of the output Taylor polynomial solutions to be $p_k + M$ for the solutions $u_k(x)$, $1 \leq k \leq q$. The MATHEMATICA routine PSSOL is listed in Appendix A.

We remark that the domain of the convergence for the decomposition series solution, like other series solutions, may not always be sufficiently large for engineering purposes. But we can readily address this issue by means of one of several common convergence acceleration techniques [Sidi (2003)], such as the diagonal Padé approximants [Adomian (1994); Wazwaz (2009); Rach and Duan (2011)] or the iterated Shanks transform [Sen (1988); Adomian (1994); Duan, Chaolu, Rach, and Lu (2013)]. For example, the MATHEMATICA built-in command 'PadeApproximant' can be used to easily generate the Padé approximants.

Rach and Duan (2011) presented the combined solution of the near-field and far-field approximations by the Adomian and asymptotic decomposition methods, where the Padé approximant technique was used in the mid-field region as necessary. In the ADM, Duan's parametrized recursion scheme [Duan (2010a); Duan and Rach (2011a); Duan, Rach, and Wang (2013)] was also proposed in order to obtain decomposition solutions with large effective regions of convergence.

The multistage ADM and its numeric schemes for IVPs were considered in [Adomian, Rach, and Meyers (1991, 1997); Vadasz and Olek (2000); Ghosh, Roy, and Roy (2007); Arenas, González-Parra, Jódar, and Villanueva (2009); Duan and Rach (2011b, 2012); Al-Sawalha, Noorani, and Hashim (2008)]. Duan and Rach (2011b) considered one-step numeric algorithms for IVPs based on the ADM and the Rach-Adomian-Meyers MDM, respectively. In [Duan and Rach (2012)] higher-order numeric schemes based on the Wazwaz-El-Sayed modified ADM were proposed.

New error analysis formulas for the approximate decomposition solutions have been introduced in [Duan and Rach (2011a), Duan, Rach, and Wazwaz (2013); Duan, Wang, Fu, and Chaolu (2013)] when the exact solution is unknown in advance. The ADM and its modifications naturally provide the engineer and applied scientist with a valuable capacity to refine their model equations whenever the computed solution differs from laboratory measurements, allowing, of course, for experimental margins of error. This advantage is unavailable with other solution methodologies that involve a priori restrictive assumptions or otherwise alter the model equations merely for the sake of mathematical tractability over preference to physical fidelity, including linearization or perturbation except in so-called linear regimes, where the nonlinearity becomes irrelevant anyway. Modeling and the solution procedure should be used interactively in the design process. When the exact solution is known in advance, we can use the usual error function, e.g. [Duan, Rach, Wazwaz, Chaolu, and Wang (2013)]. However when the exact solution is unknown in advance, as is most often the case for nonlinear engineering models, we instead compute the following error remainder function:

$$ER_n(x) = L\phi_n(x) + R\phi_n(x) + N\phi_n(x) - g(x), \tag{86}$$

which we recommend as the best objective measure of how well the sequence of solution approximants $\phi_n(x)$ satisfy the original nonlinear differential equation (1). Similarly in our error analysis, we also compute the maximal error remainder parameter:

$$MER_n = \max_{a \le x \le b} |L\phi_n(x) + R\phi_n(x) + N\phi_n(x) - g(x)|, \tag{87}$$

where the logarithmic plots of the maximal error remainder parameter versus the number of solution components per solution approximant characterize the rate of convergence, e.g. a linear relation signifies an exponential rate of convergence. Thus our new approach yields an easily computable, readily verifiable and rapidly convergent sequence of analytic approximations for the solution of the authentic nonlinear differential equation that represents the actual physical process under consideration.

We also designed the MATHEMATICA routine NSOL based on our new algorithm to generate the numeric solution for a nonlinear system in Subsection 2.4. NSOL requires eight inputs, the first six are the same as in PSSOL, and the last two are the end point X and the step-size h. The MATHEMATICA routine NSOL is listed in Appendix B.

Thus for the system of coupled nonlinear initial value problems with variable coefficients and variable inputs in Subsection 2.4, we present the following equal-step numeric algorithm based on the multistage ADM.
Input: the initial values' table Init, the coefficients' table Alpha and Be, the nonlinearities' table F, the system inputs' table G, a nonnegative integer M such that the decomposition solutions of $u_k(x)$ have the degree $p_k + M$ on each subintervals, the end point X of the interval $[0, X]$ and the step-size h.
Algorithm:

(a) Nsol:=the first column of Init; $N = [X/h]$; $x_0 = 0$.

(b) for $k = 1$ to N do
  $x_k = x_{k-1} + h$;
  dat:=PSSOL[Init,Alpha,Be,F,G,M];
  for $j = 1$ to q do

dat[j, p_j]:=the zeroth-order through $(p_j - 1)$th-order derivative of the jth entry of dat; end for;
Nsol:= Nsol combines values of dat at $x = h$;
Update Init by the values of dat[j, p_j] at $x = h$, Alpha by replacing $x \to h + x$ in Alpha, Be by replacing $x \to h + x$ in Be, G by replacing $x \to h + x$ in G.
end for.

Output: the numeric solution Nsol.

Next we consider seven numeric examples, including single and systems of nonlinear IVPs for variable system coefficients with various analytic nonlinearities, such as quadratic, exponential, product, radical, negative-power and even decimal-power nonlinearities and combinations thereof. We emphasize that our new algorithm in Section 2.4 incorporates all of our previous canonical cases, hence we can write one MATHEMATICA routine that is applicable for all. Then we represent the solutions by their truncated Taylor expansion series and plot these approximate solutions using convergence acceleration techniques as needed for extrapolation of the effective region of convergence, including the diagonal Padé approximants as well as the numeric solution based on a multistage version of our new modified ADM, in other words precisely a numeric version of analytic continuation. Hence we have demonstrated the convenience and efficiency of our new approach.

Example 1. Consider the second-order inhomogeneous nonlinear differential equation with a quadratic nonlinearity

$$u'' - \frac{2+x}{1+x}u' + u^2 = x^2 e^{2x},\ u(0) = 0,\ u'(0) = 1.$$

This IVP has the exact solution $u^*(x) = xe^x$. Within MATHEMATICA, after running the function PSSOL, we input:
$\mathrm{Be} = \{1\};\ \mathrm{F} = \{v_1\hat{}2\};\ \mathrm{G} = \{x\hat{}2 * E\hat{}(2x)\}$; $\mathrm{Init} = \{\{0,1\}\}$; $\mathrm{Alpha} = \{\{\{0, -(2+x)/(1+x)\}\}\}$; then run PSSOL[Init, Alpha, Be, F, G, 10] to output the 12-degree approximant of the solution, or equivalently, the 13-term approximant, $\phi_{13}(x)$.

The MATHEMATICA command PadeApproximant[$\phi_{13}(x)$, {x, 0, {6, 6}}] outputs the $[6/6]$ diagonal Padé approximant of $\phi_{13}(x)$, denoted as $P_6(x) := [6/6]\{\phi_{13}(x)\}$.

In Fig. 1, the curves of the exact solution $u^*(x)$, the 13-term approximant $\phi_{13}(x)$ and the diagonal Padé approximant $[6/6]\{\phi_{13}(x)\}$ are plotted, where the exact solution and the diagonal Padé approximant overlap.

We denote $P_n(x) = [n/n]\{\phi_{2n+1}(x)\}$ and consider the absolute error function

$$|E_n(x)| = |P_n(x) - u^*(x)|$$

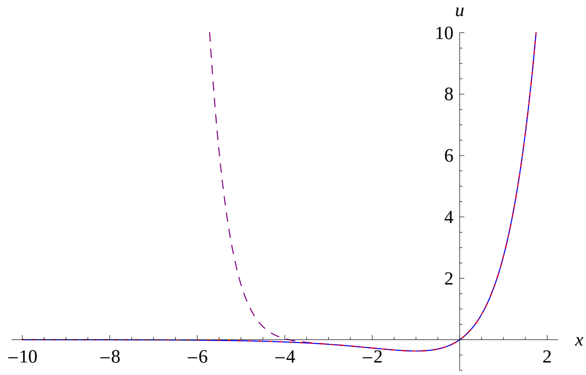


Figure 1: The curves of the exact solution $u^*(x)$ (solid line), the 13-term approximant $\phi_{13}(x)$ (dash line) and the diagonal Padé approximant $[6/6]\{\phi_{13}(x)\}$ (dot line).

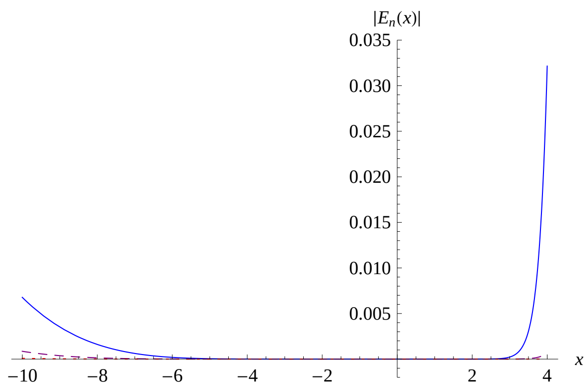


Figure 2: The curves of the absolute error function $|E_n(x)|$ for $n = 6$ (solid line), $n = 7$ (dash line) and $n = 8$ (dot line).

for $P_n(x)$ on the interval $-10 \leq x \leq 4$ and the maximal error parameter

$$ME_n = \max_{-10 \leq x \leq 4} |E_n(x)|.$$

In Fig. 2, the curves of the absolute error function $|E_n(x)|$ for $n = 6, 7$ and 8 are shown. The maximal error parameters ME_n for $n = 6, 7, 8$ and 9 are listed in Table 1. The maximal errors decrease approximately at an exponential rate.

Example 2. Consider the second-order inhomogeneous nonlinear differential equation

Table 1: The maximal error parameters ME_n for $n = 6, 7, 8$ and 9.

n	6	7	8	9
ME_n	0.03214699	0.0008560502	0.00008417801	6.607640×10^{-6}

with an exponential nonlinearity

$$u'' + xu' - 2e^u = 2x\tan x,\ u(0) = 0,\ u'(0) = 0.$$

This IVP has the exact solution $u^*(x) = -2\ln(\cos x)$, which is defined on the interval $-\pi/2 < x < \pi/2$. Within MATHEMATICA, we input:
$\mathrm{Be} = \{-2\}; \mathrm{F} = \{E^{\wedge}(v_1)\}; \mathrm{G} = \{2x * \mathrm{Tan}[x]\};\ \mathrm{Init} = \{\{0,0\}\}; \mathrm{Alpha} = \{\{\{0,x\}\}\};$
then run PSSOL[Init, Alpha, Be, F, G, 18] to output the 21-term approximant of the solution, $\phi_{21}(x)$. The MATHEMATICA command PadeApproximant[$\phi_{21}(x)$, {x, 0, {10, 10}}] outputs the diagonal Padé approximant $[10/10]\{\phi_{21}(x)\}$.

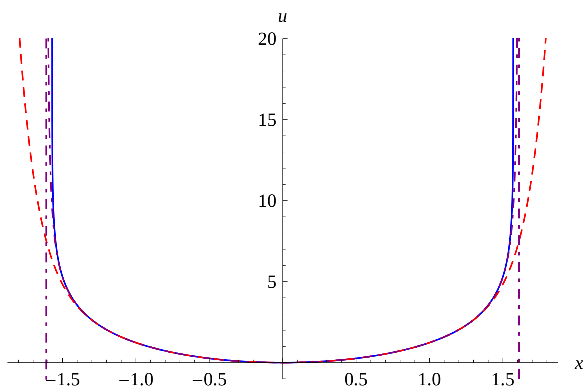


Figure 3: The curves of the exact solution $u^*(x)$ (solid line), the 21-term approximant $\phi_{21}(x)$ (dash line) and the diagonal Padé approximant $[10/10]\{\phi_{21}(x)\}$ (dot-side line).

In Fig. 3, the curves of the exact solution $u^*(x)$, the 21-term approximant $\phi_{21}(x)$ and the diagonal Padé approximant $[10/10]\{\phi_{21}(x)\}$ with two vertical asymptotes are plotted. We also note that the ADM can be combined with the diagonal Padé approximants to estimate the blow-up time [Duan, Rach, and Lin (2013)].
We denote $P_n(x) = [n/n]\{\phi_{2n+1}(x)\}$ and consider the absolute error function

$$|E_n(x)| = |P_n(x) - u^*(x)|$$

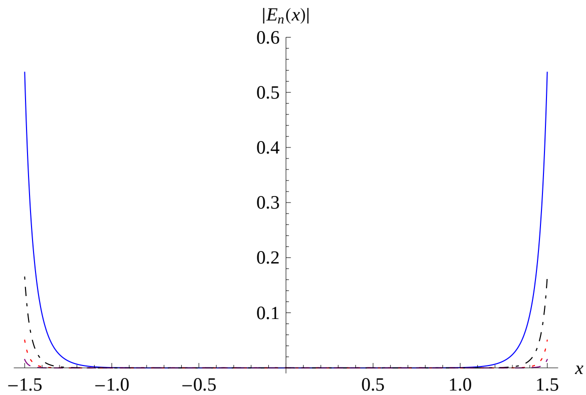


Figure 4: The curves of the absolute error function $|E_n(x)|$ for $n = 4$ (solid line), $n = 6$ (dot-dash line), $n = 8$ (dot line) and $n = 10$ (dash line).

for $P_n(x)$ on the interval $-1.5 \leq x \leq 1.5$ and the maximal error parameter

$$ME_n = \max_{-1.5 \leq x \leq 1.5} |E_n(x)|.$$

In Fig. 4, the curves of the absolute error function $|E_n(x)|$ for $n = 4, 6, 8$ and 10 are shown. The maximal error parameters ME_n for $n = 4, 6, 8, 10$ and 12 are listed in Table 2. The maximal errors decrease approximately at an exponential rate.

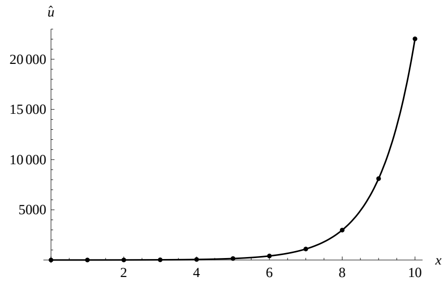
Table 2: The maximal error parameters ME_n for $n = 4, 6, 8, 10$ and 12.

n	4	6	8	10	12
ME_n	0.5364752	0.1648135	0.05029073	0.01521255	0.004528533

Example 3. Consider the second-order system of coupled inhomogeneous nonlinear differential equations with product and quadratic nonlinearities

$$\begin{cases} u'' - u + xu^2 v = xe^x, \\ v'' - v + xuv^2 = xe^{-x}, \end{cases} \quad u(0) = 1,\ u'(0) = 1,\ v(0) = 1,\ v'(0) = -1.$$

The solution of this problem is $u(x) = e^x, v(x) = e^{-x}$. Within MATHEMATICA, we input:
Be $= \{x, x\}$; F $= \{v_1{}^\wedge 2 * v_3, v_1 * v_3{}^\wedge 2\}$; G $= \{x * E^\wedge x, x * E^\wedge(-x)\}$; Init $= \{\{1, 1\}, \{1, -1\}\}$; Alpha $= \{\{\{-1, 0\}, \{0, 0\}\}, \{\{0, 0\}, \{-1, 0\}\}\}$; then run PSSOL[Init,



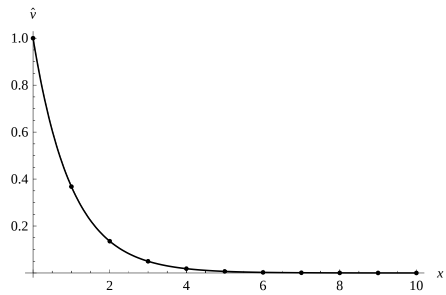


Figure 5: (a) The exact solution $u^*(x)$ (solid line) and the numeric solution (dots). (b) The exact solution $v^*(x)$ (solid line) and the numeric solution (dots).

Table 3: The numeric solutions $\hat{u}$ and $\hat{v}$ and the relative errors.

x	e^x	$\hat{u}$	relative error	e^{-x}	$\hat{v}$	relative error
1	2.71828	2.71828	1.11425×10^{-7}	0.367879	0.367879	6.86254×10^{-7}
2	7.38906	7.38907	1.48213×10^{-6}	0.135335	0.135333	1.40494×10^{-5}
3	20.0855	20.0857	8.31431×10^{-6}	0.0497871	0.0497870	1.22021×10^{-6}
4	54.5982	54.5971	1.95132×10^{-5}	0.0183156	0.0183164	4.02425×10^{-5}
5	148.413	148.418	3.05546×10^{-5}	0.00673795	0.00673728	9.86173×10^{-5}
6	403.429	403.408	5.11515×10^{-5}	0.00247875	0.00247918	1.70706×10^{-4}
7	1096.63	1096.77	1.27315×10^{-4}	9.11882×10^{-4}	9.11669×10^{-4}	2.34074×10^{-4}
8	2980.96	2980.05	305622×10^{-4}	3.35463×10^{-4}	3.35526×10^{-4}	1.88827×10^{-4}
9	8103.08	8106.45	4.15323×10^{-4}	1.23410×10^{-4}	1.23394×10^{-4}	1.25396×10^{-4}
10	22026.5	22030.3	1.74325×10^{-4}	4.53999×10^{-5}	4.54032×10^{-5}	7.30879×10^{-5}

Alpha, Be, F, G, 9] to output the 12-term approximants of the two solutions, $\phi_{12}(x)$ and $\psi_{12}(x)$.

We run NSOL[Init, Alpha, Be, F, G, 7, 10, 1] to output the numeric solutions for $u(x)$ and $v(x)$ on the interval $0 \le x \le 10$ with the step-size $h = 1$ and the degree of polynomials 9. So the numeric solutions are of order 9. The numeric solutions for $u(x)$ and $v(x)$ on the interval $0 \le x \le 10$ are plotted in Fig. 5(a) and Fig. 5(b) by dots. More detailed data for the numeric solutions are listed in Table 3.

Example 4. Consider the second-order inhomogeneous nonlinear differential equation with quadratic and product nonlinearities

$$u'' + u - 20\left(\tan^{-1}(x) + 1\right)\left(1 - u^2\right)u' = \frac{1}{x+1},\ u(0) = 1,\ u'(0) = 0.$$

We remark that this nonlinear IVP does not have a known exact solution in terms of special functions. Within MATHEMATICA, we input

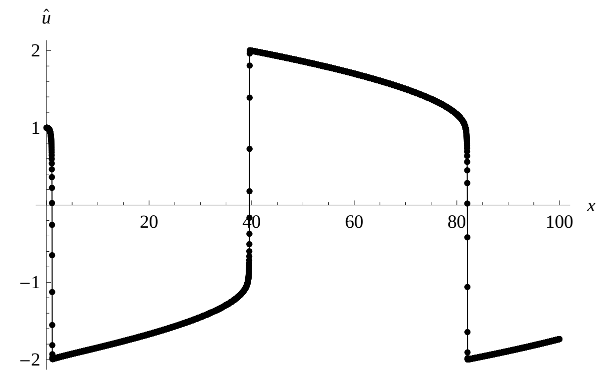


Figure 6: The numeric solutions $\hat{u}$ (dots).

Be $= \{20(\text{ArcTan}[x]+1)\}$; F $= \{v_1\text{^}2 * v_2\}$; G $= \{1/(1+x)\}$; Init $= \{\{1,0\}\}$; Alpha $= \{\{\{1, -20\ (\text{ArcTan}[x]+1)\}\}\}$; then running the function PSSOL[Init, Alpha, Be, F, G, 7] to generate

$$\phi_{10}(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{24} + \frac{11x^6}{360} - \frac{103x^7}{1008} + \frac{61x^8}{2240} - \frac{115x^9}{24192},$$

and running NSOL[Init, Alpha, Be, F, G, 3, 100, 0.01] to generate the numeric solutions $\hat{u}$, which are of order 5. The numeric solutions $\hat{u}$ are plotted in Fig. 6, which we observe as a waveform train of distinctive pulses.

Example 5. Consider the second-order system of coupled inhomogeneous nonlinear differential equations with quadratic, product, radical and negative-power nonlinearities

$$\begin{cases} u'' + u' - v + \frac{xu\left(u^2+v^2-1\right)}{\sqrt{u^2+v^2}} = e^{-x}, \\ v' + u + \frac{xv\left(u^2+v^2-1\right)}{\sqrt{u^2+v^2}} = \sin(x), \end{cases} \quad u(0) = 2,\ u'(0) = 0,\ v(0) = 1.5.$$

We remark that this system of coupled nonlinear IVPs does not have a known pair of exact solutions in terms of special functions. Within MATHEMATICA, we input Be $= \{x,x\}$; F $= \{v_1/\text{Sqrt}[v_1\text{^}2 + v_3\text{^}2] * (v_1\text{^}2 + v_3\text{^}2 - 1), v_3/\text{Sqrt}[v_1\text{^}2 + v_3\text{^}2] * (v_1\text{^}2 + v_3\text{^}2 - 1)\}$; G $= \{E\text{^}(-x), \text{Sin}[x]\}$; Init $= \{\{2,0\},\{3/2\}\}$; Alpha $= \{\{\{0,1\}, \{-1\}\}, \{\{1,0\}, \{0\}\}\}$; then running the function PSSOL[Init, Alpha, Be, F, G, 4] to generate $\phi_7(x)$ and $\psi_6(x)$, and running NSOL[Init, Alpha, Be, F, G, 3, 40, 0.05] to generate the numeric solutions $\hat{u}$ and $\hat{v}$, which are of orders 5 and 4, respectively. In Fig. 7, the numeric solutions $\hat{u}$ and $\hat{v}$ are plotted.

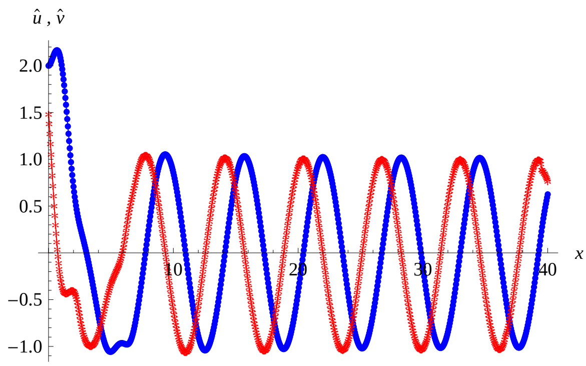


Figure 7: Numeric solutions $\hat{u}$ (dots) and $\hat{v}$ (stars).

Example 6. Consider the third-order system of coupled inhomogeneous nonlinear differential equations with product and exponential nonlinearities

$$\begin{cases} u'' + u' + \sin(x)v'' - v + uv' = xe^{-x} - \log(x+1), \\ v''' - u' + xv'' + \cos(x)v + ue^{-v} = -\sin(x), \end{cases}$$

$u(0) = 1,\ u'(0) = 1,\ v(0) = 3,\ v'(0) = 2,\ v''(0) = 1.$

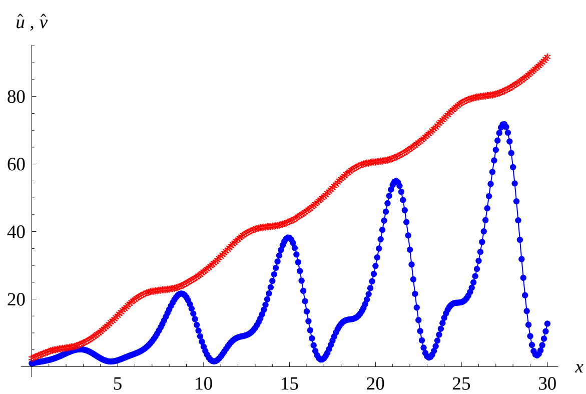


Figure 8: Numeric solutions $\hat{u}$ (dots) and $\hat{v}$ (stars).

We remark that this system of coupled nonlinear IVPs does not have a known pair of exact solutions in terms of special functions. Within MATHEMATICA, we input $\mathrm{Be} = \{1,1\}$; $\mathrm{F} = \{v_1 * v_4, v_1 * E^\wedge(-v_3)\}$; $\mathrm{G} = \{x{*}E^\wedge(-x) - \mathrm{Log}[1+x], -\mathrm{Sin}[x]\}$; $\mathrm{Init} = \{\{1.,1.\},\{3.,2.,1.\}\}$; $\mathrm{Alpha} = \{\{\{0,1\},\{-1,0,\mathrm{Sin}[x]\}\},\{\{0,-1\},\{\mathrm{Cos}[x],$

$0, x\}\}\}$; then running the function PSSOL[Init, Alpha, Be, F, G, 4] to generate $\phi_7(x)$ and $\psi_8(x)$, and running NSOL[Init, Alpha, Be, F, G, 2, 30, 0.1] to generate the numeric solutions $\hat{u}$ and $\hat{v}$, which are of orders 4 and 5, respectively. In Fig. 8, the numeric solutions $\hat{u}$ and $\hat{v}$ are plotted.

Example 7. Consider the third-order system of coupled inhomogeneous nonlinear differential equations with decimal-power and negative-power nonlinearities

$$\begin{cases} u' + \frac{1}{6}\left(e^{-x/2} - x^2\right)(v'')^{0.85} = -1 + 2x, \\ v''' + (1.125 - x + \frac{x^2}{2} + 0.25x^4)u^{-1.15} = 1 + x^2, \end{cases}$$

$$u(0) = 1,\ v(0) = 2,\ v'(0) = 0.5,\ v''(0) = 0.125.$$

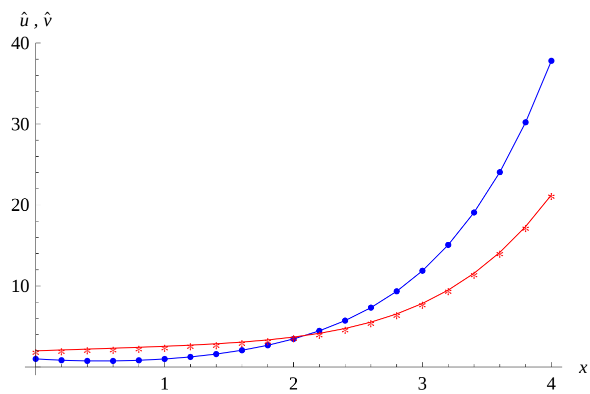


Figure 9: Numeric solutions $\hat{u}$ (dots) and $\hat{v}$ (stars).

We remark that this system of coupled nonlinear IVPs does not have a known pair of exact solutions in terms of special functions. Within MATHEMATICA, we input Be $= \{(E\hat{}(-x/2) - x\hat{}2)/6, 1.125 - x + x\hat{}2/2 + 0.25\ x\hat{}4\}$; F $= \{v_4\hat{}(0.85), v_1\hat{}(-1.15)\}$; G $= \{-1 + 2x,\ 1 + x\hat{}2\}$; Init $= \{\{1\},\ \{2,\ 0.5,\ 0.125\}\}$; Alpha $= \{\{\{0\}, \{0,0,0\}\},\ \{\{0\}, \{0,0,0\}\}\}$; then running the function PSSOL[Init, Alpha, Be, F, G, 4] to generate $\phi_6(x)$ and $\psi_8(x)$, and running NSOL[Init, Alpha, Be, F, G, 3, 4, 0.2] to generate the numeric solutions $\hat{u}$ and $\hat{v}$, which are of orders 4 and 6, respectively. In Fig. 9, the numeric solutions $\hat{u}$ and $\hat{v}$ are plotted, which exhibit an exponential-like pattern.

4 Conclusions

We have developed a new modified recursion scheme for the ADM to systematically solve systems of higher-order inhomogeneous nonlinear ordinary differential equations with variable system coefficients. Our new approach provides a significant computational advantage for computing the solution's Taylor expansion series without using differentiation operators. We have presented the analysis of several cases beginning from the first-order and the second-order inhomogeneous nonlinear differential equations, inclusively, seeking a possible pattern so as to develop the new modified recursion scheme for the general pth-order differential case. Finally, we have generalized our results to the system of coupled higher-order inhomogeneous nonlinear ordinary differential equations with variable system coefficients. We have designed the MATHEMATICA function PSSOL to implement the algorithms for the general case in Subsection 2.4, which also does apply to all of our previous canonical cases.

The overall efficiency of our new approach is further enhanced by the efficient algorithms and fast subroutines crafted by Duan (2010a,b, 2011a) for easy and convenient generation of the Adomian polynomials quickly and to high orders for the solution of nonlinear differential equations.

Convergence acceleration, such as the Padé approximants, for our result is readily achieved within MATHEMATICA. Moreover, we have presented new numeric algorithms based on multistage decompositions for the general system case. We have also designed the MATHEMATICA function NSOL for computing these numeric solutions. Several numeric examples display the ease of use, practicality and efficiency of our new modification of the ADM.

A key concept is that the Adomian decomposition series is a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function, which permits solution by recursion. Our new modified recursion scheme yields an easily computable, readily verifiable and rapidly convergent sequence of analytic approximate solutions. Thus the ADM subsumes the classic power series method while extending the class of amenable nonlinearities to include any analytic nonlinearity and not just polynomial nonlinearities.

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References

Abbaoui, K.; Cherruault, Y. (1994): Convergence of Adomian's method applied to nonlinear equations. *Math. Comput. Modelling*, vol. 20, pp. 60–73.

Abbaoui, K.; Cherruault, Y.; Seng, V. (1995): Practical formulae for the calculus of multivariable Adomian polynomials. *Math. Comput. Modelling*, vol. 22, pp. 89–93.

Abdelrazec, A.; Pelinovsky, D. (2011): Convergence of the Adomian decomposition method for initial-value problems. *Numer. Methods Partial Differential Equations*, vol. 27, pp. 749–766.

Adomian, G. (1983): *Stochastic Systems*. Academic, New York.

Adomian, G. (1986): *Nonlinear Stochastic Operator Equations*. Academic, Orlando, FL.

Adomian, G. (1989): *Nonlinear Stochastic Systems Theory and Applications to Physics*. Kluwer Academic, Dordrecht.

Adomian, G. (1994): *Solving Frontier Problems of Physics: The Decomposition Method*. Kluwer Academic, Dordrecht.

Adomian, G.; Rach, R. (1983): Inversion of nonlinear stochastic operators. *J. Math. Anal. Appl.*, vol. 91, pp. 39–46.

Adomian, G.; Rach, R. (1991): Transformation of series. *Appl. Math. Lett.*, vol. 4, pp. 69–71.

Adomian, G.; Rach, R. (1992a): Nonlinear transformation of series–Part II. *Comput. Math. Appl.*, vol. 23, pp. 79–83.

Adomian, G.; Rach, R. (1992b): Inhomogeneous nonlinear partial differential equations with variable coefficients. *Appl. Math. Lett.*, vol. 5, pp. 11–12.

Adomian, G.; Rach, R. (1992c): Modified decomposition solution of nonlinear partial differential equations. *Appl. Math. Lett.*, vol. 5, pp. 29–30.

Adomian, G.; Rach, R. (1993a): Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition. *J. Math. Anal. Appl.*, vol. 174, pp. 118–137.

Adomian, G.; Rach, R. (1993b): A new algorithm for matching boundary conditions in decomposition solutions. *Appl. Math. Comput.*, vol. 58, pp. 61–68.

Adomian, G.; Rach, R. (1993c): Solution of nonlinear partial differential equations in one, two, three, and four dimensions. *World Sci. Ser. Appl. Anal.*, vol. 2, pp. 1–13.

Adomian, G.; Rach, R. (1994): Modified decomposition solution of linear and nonlinear boundary-value problems. *Nonlinear Anal.*, vol. 23, pp. 615–619.

Adomian, G.; Rach, R. (1996): Modified Adomian polynomials. *Math. Comput. Modelling*, vol. 24, pp. 39–46.

Adomian, G.; Rach, R.; Meyers, R. (1991): Numerical algorithms and decomposition. *Comput. Math. Appl.*, vol. 22, pp. 57–61.

Adomian, G.; Rach, R.; Shawagfeh, N. T. (1995): On the analytic solution of the Lane-Emden equation. *Found. Phys. Lett.*, vol. 8, pp. 161–181.

Adomian, G.; Rach, R. C.; Meyers, R. E. (1997): Numerical integration, analytic continuation, and decomposition. *Appl. Math. Comput.*, vol. 88, pp. 95–116.

Al-Sawalha, M. M.; Noorani, M. S. M.; Hashim, I. (2008): Numerical experiments on the hyperchaotic Chen system by the Adomian decomposition method. *Int. J. Comput. Methods*, vol. 5, pp. 403–412.

Arenas, A. J.; González-Parra, G.; Jódar, L.; Villanueva, R. J. (2009): Piecewise finite series solution of nonlinear initial value differential problem. *Appl. Math. Comput.*, vol. 212, pp. 209–215.

Azreg-Aïnou, M. (2009): A developed new algorithm for evaluating Adomian polynomials. *CMES-Comput. Model. Eng. Sci.*, vol. 42, pp. 1–18.

Bellman, R. E.; Adomian, G. (1985): *Partial Differential Equations: New Methods for their Treatment and Solution*. D. Reidel, Dordrecht.

Cherruault, Y. (1989): Convergence of Adomian's method. *Kybernetes*, vol. 18, pp. 31–38.

Cherruault, Y.; Adomian, G. (1993): Decomposition methods: A new proof of convergence. *Math. Comput. Modelling*, vol. 18, pp. 103–106.

Dai, H. H.; Schnoor, M.; Atluri, S. N. (2012): A simple collocation scheme for obtaining the periodic solutions of the Duffing equation, and its equivalence to the high dimensional harmonic balance method: subharmonic oscillations. *CMES-Comput. Model. Eng. Sci.*, vol. 84, pp. 459–497.

Duan, J. S. (2010a): Recurrence triangle for Adomian polynomials. *Appl. Math. Comput.*, vol. 216, pp. 1235–1241.

Duan, J. S. (2010b): An efficient algorithm for the multivariable Adomian polynomials. *Appl. Math. Comput.*, vol. 217, pp. 2456–2467.

Duan, J. S. (2011a): Convenient analytic recurrence algorithms for the Adomian polynomials. *Appl. Math. Comput.*, vol. 217, pp. 6337–6348.

Duan, J. S. (2011b): New recurrence algorithms for the nonclassic Adomian polynomials. *Comput. Math. Appl.*, vol. 62, pp. 2961–2977.

Duan, J. S. (2011c): New ideas for decomposing nonlinearities in differential equations. *Appl. Math. Comput.*, vol. 218, pp. 1774–1784.

Duan, J. S.; Chaolu, T.; Rach, R. (2012): Solutions of the initial value problem for nonlinear fractional ordinary differential equations by the Rach-Adomian-Meyers modified decomposition method. *Appl. Math. Comput.*, vol. 218, pp. 8370–8392.

Duan, J. S.; Chaolu, T.; Rach, R.; Lu, L. (2013): The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations. *Comput. Math. Appl.*, vol. 66, pp. 728–736.

Duan, J. S.; Guo, A. P. (2010): Reduced polynomials and their generation in Adomian decomposition methods. *CMES-Comput. Model. Eng. Sci.*, vol. 60, pp. 139–150.

Duan, J. S.; Rach, R. (2011a): A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations. *Appl. Math. Comput.*, vol. 218, pp. 4090–4118.

Duan, J. S.; Rach, R. (2011b): New higher-order numerical one-step methods based on the Adomian and the modified decomposition methods. *Appl. Math. Comput.*, vol. 218, pp. 2810–2828.

Duan, J. S.; Rach, R. (2012): Higher-order numeric Wazwaz-El-Sayed modified Adomian decomposition algorithms. *Comput. Math. Appl.*, vol. 63, pp. 1557–1568.

Duan, J. S.; Rach, R.; Baleanu, D.; Wazwaz, A. M. (2012): A review of the Adomian decomposition method and its applications to fractional differential equations. *Commun. Frac. Calc.*, vol. 3, pp. 73–99.

Duan, J. S.; Rach, R.; Lin, S. M. (2013): Analytic approximation of the blow-up time for nonlinear differential equations by the ADM-Padé technique. *Math. Methods Appl. Sci.*, vol. 36, pp. 1790–1804.

Duan, J. S.; Rach, R.; Wang, Z. (2013): On the effective region of convergence of the decomposition series solution. *J. Algorithms Comput. Tech.*, vol. 7, pp. 227–247.

Duan, J. S.; Rach, R.; Wazwaz, A. M. (2013): Solution of the model of beam-type micro- and nano-scale electrostatic actuators by a new modified Adomian decomposition method for nonlinear boundary value problems. *Int. J. Nonlinear Mech.*, vol. 49, pp. 159–169.

Duan, J. S.; Rach, R.; Wazwaz, A. M.; Chaolu, T.; Wang, Z. (2013): A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions. *Appl. Math. Model.*, vol. 37, pp. 8687–8708.

Duan, J. S.; Wang, Z.; Fu, S. Z.; Chaolu, T. (2013): Parametrized temperature distribution and efficiency of convective straight fins with temperature-dependent thermal conductivity by a new modified decomposition method. *Int. J. Heat Mass Transfer*, vol. 59, pp. 137–143.

Gabet, L. (1994): The theoretical foundation of the Adomian method. *Comput. Math. Appl.*, vol. 27, pp. 41–52.

Ghosh, S.; Roy, A.; Roy, D. (2007): An adaptation of Adomian decomposition for numeric-analytic integration of strongly nonlinear and chaotic oscillators. *Comput. Methods Appl. Mech. Engrg.*, vol. 196, pp. 1133–1153.

Hinch, E. J. (1991): *Perturbation Methods*. Cambridge University Press, Cambridge.

Lai, H. Y.; Chen, C. K.; Hsu, J. C. (2008): Free vibration of non-uniform Euler-Bernoulli beams by the Adomian modified decomposition method. *CMES–Comput. Model. Eng. Sci.*, vol. 34, pp. 87–116.

Rach, R. (1984): A convenient computational form for the Adomian polynomials. *J. Math. Anal. Appl.*, vol. 102, pp. 415–419.

Rach, R. (1987): On the Adomian (decomposition) method and comparisons with Picard's method. *J. Math. Anal. Appl.*, vol. 128, pp. 480–483.

Rach, R. (2008): A new definition of the Adomian polynomials. *Kybernetes*, vol. 37, pp. 910–955.

Rach, R. (2012): A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011. *Kybernetes*, vol. 41, pp. 1087–1148.

Rach, R.; Adomian, G. (1990): Multiple decompositions for computational convenience. *Appl. Math. Lett.*, vol. 3, pp. 97–99.

Rach, R.; Adomian, G.; Meyers, R. E. (1992): A modified decomposition. *Comput. Math. Appl.*, vol. 23, pp. 17–23.

Rach, R.; Duan, J. S. (2011): Near-field and far-field approximations by the Adomian and asymptotic decomposition methods. *Appl. Math. Comput.*, vol. 217, pp. 5910–5922.

Rach, R.; Duan, J. S.; Wazwaz, A. M. (2013): Solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. *J. Math. Chem.,* doi: 10.1007/s10910-013-0260-6.

Rach, R.; Wazwaz, A. M.; Duan, J. S. (2013): A reliable modification of the Adomian decomposition method for higher-order nonlinear differential equations. *Kybernetes*, vol. 42, pp. 282–308.

Rèpaci, A. (1990): Nonlinear dynamical systems: On the accuracy of Adomian's decomposition method. *Appl. Math. Lett.*, vol. 3, pp. 35–39.

Sen, A. K. (1988): An application of the Adomian decomposition method to the transient behavior of a model biochemical reaction. *J. Math. Anal. Appl.*, vol. 131, pp. 232–245.

Serrano, S. E. (2010): *Hydrology for Engineers, Geologists, and Environmental Professionals: An Integrated Treatment of Surface, Subsurface, and Contaminant Hydrology, Second Revised Edition*. HydroScience, Ambler, PA.

Serrano, S. E. (2011): *Engineering Uncertainty and Risk Analysis: A Balanced Approach to Probability, Statistics, Stochastic Modeling, and Stochastic Differential Equations, Second Revised Edition*. HydroScience, Ambler, PA.

Sidi, A. (2003): *Practical Extrapolation Methods: Theory and Applications*. Cambridge University Press, Cambridge.

Vadasz, P.; Olek, S. (2000): Convergence and accuracy of Adomian's decomposition method for the solution of Lorenz equations. *Int. J. Heat Mass Transfer*, vol. 43, pp. 1715–1734.

Wazwaz, A. M. (1997): *A First Course in Integral Equations*. World Scientific, Singapore and River Edge, NJ.

Wazwaz, A. M. (1999): A reliable modification of Adomian decomposition method. *Appl. Math. Comput.*, vol. 102, pp. 77–86.

Wazwaz, A. M. (2001): A new algorithm for solving differential equations of Lane-Emden type. *Appl. Math. Comput.*, vol. 118, pp. 287–310.

Wazwaz, A. M. (2002a): *Partial Differential Equations: Methods and Applications*. A. A. Balkema, Lisse, The Netherlands.

Wazwaz, A. M. (2002b): A new method for solving singular initial value problems in the second order ordinary differential equations. *Appl. Math. Comput.*, vol. 128, pp. 45–57.

Wazwaz, A. M. (2009): *Partial Differential Equations and Solitary Waves Theory*. Higher Education Press, Beijing, and Springer-Verlag, Berlin.

Wazwaz, A. M. (2011): *Linear and Nonlinear Integral Equations: Methods and Applications*. Higher Education Press, Beijing, and Springer-Verlag, Berlin.

Wazwaz, A. M.; El-Sayed, S. M. (2001): A new modification of the Adomian decomposition method for linear and nonlinear operators. *Appl. Math. Comput.*, vol. 122, pp. 393–405.

Wazwaz, A. M.; Rach, R.; Duan, J. S. (2013a): Adomian decomposition method for solving the Volterra integral form of the Lane-Emden equations with initial values and boundary conditions. *Appl. Math. Comput.*, vol. 219, pp. 5004–5019.

Wazwaz, A. M.; Rach, R.; Duan, J. S. (2013b): A study on the systems of the Volterra integral forms of the Lane-Emden equations by the Adomian decomposition method. *Math. Methods Appl. Sci.,* doi: 10.1002/mma.2776.

Appendix A: The MATHEMATICA routine PSSOL

```
PSSOL[Init_,Alpha_,Be_,F_,G_,M_]:=Module[{p,q,s,j,ps,g,be,n,i,
j1,m,r,k,k1,nu,U,CC,tr,alpha,A,T,se,c},
p=Table[Length[Init[[k]]],{k,1,Length[Init]}];
q=Length[p];
ps=Sum[p[[k]],{k,1,q}];
For[k=1,k<=q,k++,For[n=0,n<=M,n++,
g[k,n]=1/n!*(D[G[[k]],{x,n}])/.x->0;
be[k,n]=1/n!*(D[Be[[k]],{x,n}])/.x->0 ] ];
Module[{},For[k=1,k<=q,k++,
For[s=1,s<=q,s++,
For[j=0,j<=p[[s]]-1,j++,
For[n=0,n<=M,n++, alpha[k,s,j,n]=
1/n!*(D[Alpha[[k]][[s]][[j+1]],{x,n}])/.x->0]]]]];
U=Flatten[Table[Table[Subscript[v, i,j],{j,0,M}],
  {i,1,Sum[p[[r]],{r,1,q}]}]];
CC=Flatten[Table[Table[Table[Product[r+j,{j,1,h}]*c[i,r+h],
{r,0,M}], {h,0,p[[i]]-1}],{i,1,q}]];
tr=Table[U[[j1]]->CC[[j1]],{j1,1,(M+1)*Sum[p[[r]],{r,1,q}]}];
Module[{},For[k=1,k<=q,k++,
 A[k,0]=F[[k]]/.Subscript[v, j_]->Subscript[v, j,0];
Table[T[i,j],{i,1,M},{j,1,i}];
se=Table[_,{ps}]/.List->Sequence;
For[r=1,r<=M,r++, T[r,1]=Table[Subscript[v, i,r]*D[A[k,0],
 Subscript[v, i,0]],{i,1,ps}];
For[k1=2,k1<=r,k1++,T[r,k1]=Union[Flatten[Table[
D[Map[#*Subscript[v, i,1]/(Exponent[#,Subscript[v, i,1]]+1)&,
T[r-1,k1-1]], Subscript[v, i,0]], {i,1,ps}]]]];
For[k1=2,k1<=Floor[r/2],k1++,T[r,k1]=T[r,k1]\[Union]
(T[r-k1,k1]/.Flatten[Table[Subscript[v, i,j]->
Subscript[v, i,j+1],{i,1,ps},{j,1,r-2*k1+1}]])];
A[k,r]=(Sum[Total[T[r,k1] ],{k1,1,r}]);
```

```
If[EvenQ[r], Do[T[r/2,k1]=., {k1,1,r/2}]]; ];
 Table[A[k,r],{r,0,M}]]];
For[k=1,k<=q,k++, For[r=0,r<=M,r++,
A[k,r]= A[k,r]/.tr]];
For[k=1,k<=q,k++,
For[j=0,j<=p[[k]]-1, j++, c[k,j]=Init[[k]][[j+1]]/j!]];
For[n=0,n<=M,n++, For[k=1,k<=q,k++,
c[k,p[[k]]+n]=(g[k,n]-Sum[Sum[
Sum[Product[m+j,{j,1,p[[s]]-1-nu}]*alpha[k,s,p[[s]]-1-nu,n-m]*
c[s,m+p[[s]]-1-nu],{nu,0,p[[s]]-1}],{s,1,q}]+be[k,n-m]*A[k,m],
 {m,0,n}]) / Product[n+i,{i,1,p[[k]]}]]]];
For[k=1,k<=q,k++,
v[k]=Sum[c[k,j]*x^j,{j,0,p[[k]]+M}]]; j=.;
Table[v[k],{k,1,q}] ];
```

Appendix B: The MATHEMATICA routine NSOL

```
NSOL[Init_,Alpha_,Be_,F_,G_,M_,X_,h_]:=
Module[{p,q,ps,xn,N1,k,in,al,be,f,g,j1,sol,new},
p=Table[Length[Init[[k]]],{k,1,Length[Init]}];
q=Length[p]; ps=Sum[p[[k]],{k,1,q}];
N1=Floor[X/h]; xn[0]=0;
sol={Table[Init[[k]][[1]],{k,1,Length[Init]}]};
in=Init;al=Alpha; be=Be;f=F; g=G;
For[k=1,k<=N1, k++, xn[k]=N[xn[k-1]+h];
new=Table[Table[D[PSSOL[in,al,be,f,g,M][[k]],{x,j1}],
{j1,0,p[[k]]-1}], {k,1,q}]/.{x->(xn[k]-xn[k-1])}};
sol=Append[sol,Table[new[[k]][[1]],{k,1,q}]];
in=new; al=al/.{x->h+x}; be=be/.{x->h+x};
g=g/.{x->h+x};];
Prepend[Transpose[sol],Table[xn[k],{k,0,N1}]]];
```