

Universal Reliability Method for Structural Models with Both Random and Fuzzy Variables

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Abstract: The conventional probabilistic reliability model for structures is based on the “probability assumption” and “binary-state assumption”. These assumptions are often offset the reality of practical engineering and lead to a wrong conclusion. In fact, besides randomness, fuzziness which is different from randomness in nature is also a prevalent uncertainty factor and plays an important role in structural reliability assessment. In this paper, a novel structural reliability model with random variables and fuzzy variables is established by using the fuzzy set theory, possibility theory and probability measure for fuzzy events, based on the “mixed probability and possibility assumption” and “fuzzy state assumption”. The presented universal structural reliability model can be regarded as the unification of probability reliability theory and possibility reliability theory. The universal reliability model can degenerate into probabilistic reliability model or possibility reliability model spontaneously for pure random basic variables or fuzzy basic variables. The Monte-Carlo simulation combing with optimization method is applied to calculate the failure probability of the structures. Numerical examples revealed the feasibility of the proposed reliability model for structures.

Keywords: structural reliability, random variables, fuzzy variables, membership function, fuzzy failure domain.

1 Introduction

The conventional reliability theory is based on probability theory. The two fundamental assumptions [Cai, Wen and Zhang (1991)] of the probabilistic reliability theory are:

A1(Probability assumption): The system behavior is fully characterized in the

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context of probability measures. The uncertainties are usually modeled as random variables.

A2(Binary-state assumption): The system alternates between two states: safety or failure. That is the system must be perfect functioning or in complete failure.

In the conventional probabilistic reliability theory for structures, as the probabilistic assumption and binary-state assumption, the basic uncertain parameters are expressed as random variables and a limit state function is applied to determine safety or failure of the structures. But this may be not the reality in practical engineering. More often than not, uncertainties emerged in the structural reliability problems are not only randomness but also fuzziness which also plays an important role in reliability assessment. For examples, the “safety” and “failure” for structures are often vague concepts which can’t be determined crisply by the limit state function; in addition, some of the basic uncertain parameters might not be random but fuzzy in nature, e.g. “large deflection” for a beam. Thus new reliability models for structures exhibiting both fuzziness and randomness have been developed. Considering the basic parameters as random variables, the failure criterion is fuzzy, probabilistic reliability methods with fuzzy safety state [Huang (2012); Wang, Li, Huang and Liu (2013)] have been discussed, which involve more generalities than the conventional probabilistic reliability model. In some cases, the loads and the corresponding responses of the structures have not only randomness but also fuzziness, several authors [Liu, Qiao and Wang (1997); Möller, Graf and Beer (2003); Holický(2006); Wang, Huang, Li, Pang and Xiao (2012)] investigated the structural reliability models by considering the basic parameters as fuzzy random variables based on fuzzy random theory [Shapiro (2009); Liu and Liu (2003)]. In addition, one of the most familiar cases is that some of the basic parameters are stochastic and expressed as random variables while the others are fuzzy and expressed as fuzzy variables. Because the definitions of probability and possibility are fundamentally deferent, it is a challenge problem to deal with reliability problems with both fuzzy and random variables. On the whole, two strategies have been developed for structural reliability problems with both random variables and fuzzy variables in literatures. One is that, relying on equivalent transformations, such as entropy based transformation and transformation by scaling of fuzzy membership function, the mixed random and fuzzy variables are transformed into pure random variables or pure fuzzy variables, then the hybrid reliability problems are tackled by using probabilistic method or possibility method [Haldar and Reddy (1992); Chakraborty and Sam (2006); Liu, Choi, Youn and Gorsich (2006)]. The other is that, using α -cuts of the fuzzy variables or fuzzy sets, the hybrid reliability problems are tackled in the manner of probabilistic method and a probabilistic reliability index associated with membership function is generated [Adduri and Penmetsa (2008;2009); Kala

(2007)]. Whereas the existing methods seem to be suitable to perform reliability evaluation under mixed random and fuzzy variables, there are problems with them. The equivalent transformation methods always lead to losing some of the information [Chakraborty and Sam (2006)]; moreover, since fuzziness and randomness are different in nature, transformations between them are irrational. While the α -cuts methods generate a probabilistic reliability index, a fuzzy number, associated with membership function which is intricate and inconvenient for practical application. This paper investigates the structural reliability model with mixed random and fuzzy variables in a novel point of view.

In this paper, different with the existing reliability methods for mixed random variables and fuzzy variables, the reliability of structure is evaluated not by the pure probability or possibility measure but the probability measure for fuzzy events [Zadeh (1968)] without transforming the basic uncertain variables. In order to use the probability measure for fuzzy events to evaluate the reliability, the concept of fuzzy failure criterion in the random variables subspace is proposed. Then the membership functions of the fuzzy fail domain and fuzzy safety domain are deduced rigorously from the possibility distribution functions of the fuzzy variables and failure criterion (both crisp and fuzzy failure criteria are considered) in basic variables space based on possibility theory and fuzzy set theory.

The organization of this paper is as follows. In Section 2, a brief description on conventional probabilistic structural reliability model and probabilistic structural reliability method with fuzzy state are introduced. In Section 3, the fuzzy failure domain and safety domain defined on random variables subspace are studied and the pos-probability reliability model for structures with crisp failure criterion defined in the basic uncertain variables space is proposed; then the relationships between the proposed model and probabilistic reliability model or fuzzy reliability model are discussed. In Section 4, considering the failure criterion in basic variable space is fuzzy, the proposed pos-probability reliability model is developed with more generalities. In Section 5, three examples are followed to demonstrate the proposed model, and Monte-Carlo simulation combining with nonlinear programming (NLP) method is applied to calculate the failure probability of the structures. A conclusion is arrived in Section 6.

The novelties of the paper are that: (1) the conception of fuzzy failure in random variables subspace for structural reliability problems with mixed random and fuzzy variables is proposed; (2) using possibility theory and fuzzy set theory, the membership functions of the fuzzy failure domain and fuzzy safety domain in random variables subspace are deduced from the joint possibility distribution of the fuzzy variables and the failure criterion in basic variables space; (3) the proposed reliability model is compatible with the probabilistic reliability model, POSBIST relia-

bility model (reliability model based on possibility assumption and binary-state assumption) [Cai, Wen and Zhang (1995); Guo, Lv and Feng (2002)] and POSFUST reliability model (reliability model based on possibility assumption and fuzzy state assumption) [Cai, Wen and Zhang (1991)] can degenerate into them spontaneously, which can be regarded as a universal structural reliability model with random and fuzzy variables.

2 Probabilistic reliability model for structures with fuzzy failure

In the conventional probabilistic reliability method, failure of the structures can be determined by a limit state function crisply. Consider a performance function of a certain structure as $G(\mathbf{x})$, where $\mathbf{x}=(x_1, x_2, \dots, x_n)^T \in R^n$ is the basic random vector associated with its joint probability density function (JPDF) $f(\mathbf{x})$. $G(\mathbf{x})=0$ is the limit state function of the structure, which divides the basic variables space R^n into the failure domain $\Omega_{xf}=\{\mathbf{x}:G(\mathbf{x})<0, \mathbf{x} \in R^n\}$ and the safe domain $\Omega_{xs}=\{\mathbf{x}:G(\mathbf{x})\geq 0, \mathbf{x} \in R^n\}$. The failure probability of the structure P_f can be calculated as follows

$$P_f = \int_{\Omega_{xf}} f(\mathbf{x}) d\mathbf{x} \quad (1)$$

The reliability of the structure P_s can be obtained from the following equation

$$P_f + P_s \equiv 1 \quad (2)$$

There are many methods that can be used to calculate the probability reliability, such as, the well known first-order reliability method (FORM) [Chau, Han, Bai, and Jiang (2012)], Hasofer-Lind approach [Hasofer and Lind (1974); Santos, Martoli and Beck (2012)] and Monte-Carlo simulation [Radhika, Panda and Manohar (2008)] etc. In practical application, the responses of the complex structures are usually obtained by using finite element method (FEM), thus the limit state functions are implicit. Under the circumstance, the Monte-Carlo simulation can't work well either because of huge calculation costs. Stochastic finite element method (SFEM) [Kamiński and Szafran, (2012)] and response surface method (RSM) [Li, Luo, and Sun (2011); Panda and Manohar (2008)] have been proposed to tackle the problems with implicit limit state functions.

In practical engineering, the "failure" and "safety" of the structures are often fuzzy events. Thus the failure domain and safe domain of the structure should be expressed as Fuzzy (F) sets [Zadeh (1965)] with more generality. The probability measure for fuzzy events proposed by professor Zadeh (1968) can be used to evaluate the reliability of the structure, which is defined by the following Lebesgue-

Stieltjes integral

$$P(\tilde{A}) = \int_{R^n} \mu_{\tilde{A}}(\mathbf{x}) dP \quad (3)$$

Where \tilde{A} is a F subset in n -dimensional Euclidean space R^n associated with its membership function (MF) $\mu_{\tilde{A}}(\mathbf{x}), \mathbf{x} \in R^n$.

Then, the failure probability and safety probability of the structure can be calculated by the following Riemann integrals

$$P_{\text{FF}} = \int_{R^n} \mu_f(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = E[\mu_f(\mathbf{x})] \quad (4)$$

$$P_{\text{sF}} = \int_{R^n} \mu_s(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = E[\mu_s(\mathbf{x})] \quad (5)$$

Where $\mu_f(\mathbf{x})$ and $\mu_s(\mathbf{x})$ are MFs of the fuzzy failure domain (FFD) F_{xf} and fuzzy safety domain (FSD) F_{xs} . F_{xf} and F_{xs} are the F subsets of the universe of discourse R^n in which the random vector \mathbf{x} is defined.

3 Pos-probability reliability model with mixed random and fuzzy variables

In this section, the structural reliability method with mixed random and fuzzy variables for crisp failure criterion in basic variables space is studied.

3.1 The generalized assumptions for structural reliability problems with mixed variables

Similar with the probabilistic reliability model, we study the reliability model with mixed variables based on the following two assumptions:

A1' (Mixed probability and possibility assumption): The basic uncertain vector of the structure is $\mathbf{z}=(\mathbf{x}^T, \mathbf{y}^T)^T$, where \mathbf{x} is a n -dimensional random vector $\mathbf{x}=(x_1, x_2, \dots, x_n)^T$ characterized by its JPDPDF $f(\mathbf{x})$, \mathbf{y} is a m -dimensional fuzzy vector $\mathbf{y}=(y_1, y_2, \dots, y_m)^T$ characterized by its joint possibility distribution function (JPoDF) $\pi_y(\mathbf{y})$. Denote the n -dimensional Euclidean space \mathbf{x} defined in by Ω_x , the m -dimensional Euclidean space \mathbf{y} defined in by Ω_y . Thus the state variables \mathbf{z} belongs to a $m+n$ -dimensional Euclidean space denoted by $\Omega_z=\Omega_x \oplus \Omega_y$.

A2(Binary-state assumption): The limit state function defined in the basic variables space Ω_z deduced from the physics model is denoted by $G(\mathbf{z})=0$, where the failure surface $G(\mathbf{z})=0$ divides the basic variables space Ω_z into two parts: the failure domain $\Omega_{zf}=\{\mathbf{z}; G(\mathbf{z}) < 0, \mathbf{z} \in \Omega_z\}$ and the safety domain $\Omega_{zs}=\{\mathbf{z}; G(\mathbf{z}) \geq 0, \mathbf{z} \in \Omega_z\}$. That is if $\mathbf{z}=\mathbf{z} \in \Omega_{zf}$, the structure is failed; if $\mathbf{z}=\mathbf{z} \in \Omega_{zs}$, the structure is safe.

Since the topic discussed in this paper is the “reliability” of the structures, there should exist definite failure state $\mathbf{z}^*=(\mathbf{x}^{*T}, \mathbf{y}^{*T})^T \in \Omega_z$ and definite safety state $\mathbf{z}^{**}=(\mathbf{x}^{**T}, \mathbf{y}^{**T})^T \in \Omega_z$, that is, $\pi_y(\mathbf{y}^*)=1$ and $G(\mathbf{z}^*) < 0$ for $\mathbf{z}=\mathbf{z}^*$; $\pi_y(\mathbf{y}^{**})=1$ and $G(\mathbf{z}^{**}) \geq 0$ for $\mathbf{z}=\mathbf{z}^{**}$. As shown in Fig.1.

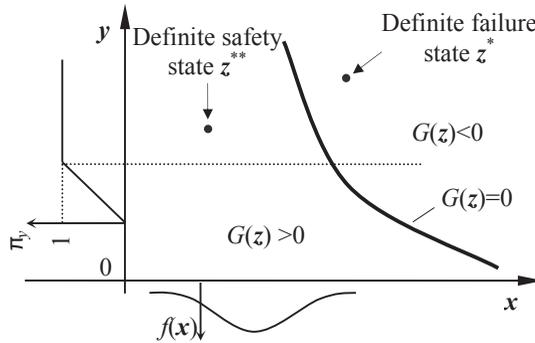


Figure 1: The limit state function of the structure in Ω_z

3.2 Fuzzy failure of the structure in random variables subspace Ω_x

As far as the generalized reliability problem in section 3.1 is concerned, we define the performance variable of the structure as

$$M = G(\mathbf{z}) = G(\mathbf{x}, \mathbf{y}) \tag{6}$$

Where $G(\mathbf{z})$ is the performance function of the structure. Because \mathbf{x} is a random vector and \mathbf{y} is a fuzzy vector, thus M can be regarded as a fuzzy random variable [Liu and Liu (2003)] which is a random variable taking fuzzy variable value.

We discuss the failure criterion of the structure both in the universe of discourse Ω_z and Ω_x as follows:

1. The failure criterion of the structure, $G(\mathbf{z})=0$, is defined in Ω_z . In the universe of discourse Ω_z , the state of a certain realization of $\mathbf{z}=\mathbf{z}_0$ can be determined crisply by the value of $M = G(\mathbf{z}_0)$. That is, if $G(\mathbf{z}_0) < 0$, the structure is completely failed; $G(\mathbf{z}_0) \geq 0$, the structure is completely safe. In other words, the failure criterion in Ω_z is crisp, and expressed as the limited state function or crisp failure domain Ω_{zf} and safety domain Ω_{zs} .
2. In the universe of discourse Ω_x , the random variables subspace, another failure criterion can be deduced from the failure criterion in Ω_z . But it can't

simply be expressed as the limit state function or crisp failure and safety domains. While \mathbf{x} , the random component of \mathbf{z} , takes the realization $\mathbf{x}=\mathbf{x}_0$, since M can be regarded as a fuzzy random variable, $M|_{\mathbf{x}=\mathbf{x}_0} = G(\mathbf{x}_0, \mathbf{y})$ turns out to be a fuzzy variable. Thus $M|_{\mathbf{x}=\mathbf{x}_0} \geq 0$ or $M|_{\mathbf{x}=\mathbf{x}_0} < 0$ are fuzzy events, the structure state $\mathbf{x}=\mathbf{x}_0$ in universe of discourse Ω_x is ambiguous or fuzzy in the terms of mathematics. Namely, the failure and safety of the structure is a fuzzy event in the universe of discourse Ω_x . Thus the fuzzy failure criterion in Ω_x can be expressed as FFD and FSD denoted by F_{xf} and F_{xs} .

In summary, as far as the structural reliability problems with mixed random and fuzzy variables are concerned, the crisp failure criterion defined in the basic uncertain variables space could induce a fuzzy failure criterion defined in the random variables subspace. The fuzzy failure criterion can be expressed as F sets (FFD and FSD).

A standard procedure is developed to determine the MFs of F_{xf} and F_{xs} based on the fuzzy set theory and possibility theory in the following section.

3.3 The FFD and FSD in random variables subspace Ω_x

According to possibility theory [Zadeh (1978)], a possibility measure $\text{Pos}_y(\cdot)$ and a necessity measure $\text{Nec}_y(\cdot)$ defined on the σ -algebra $p(\Omega_y)$, $p(\cdot)$ denotes power set, can be induced by the JPoDF $\pi_y(\mathbf{y})$. That is

$$\text{Pos}_y(A) = \sup_{\mathbf{y} \in A} \pi_y(\mathbf{y}) \tag{7}$$

$$\text{Nec}_y(A) = 1 - \text{Pos}_y(A^c) = 1 - \sup_{\mathbf{y} \in A^c} \pi_y(\mathbf{y}) \tag{8}$$

Where A is a subset of Ω_y , A^c is the complimentary set of A . If the component $y_i, i=1,2,\dots,m$, are independent fuzzy variables characterized by their possibility distribution functions (PoDFs) $\pi_i(y_i)$, the upper two equations can be written as follows:

$$\text{Pos}_y(A) = \sup_{\mathbf{y}=(y_1,y_2,\dots,y_m)^T \in A} \pi_1(y_1) \wedge \pi_2(y_2) \wedge \dots \wedge \pi_m(y_m) \tag{9}$$

$$\text{Nec}_y(A) = 1 - \sup_{\mathbf{y}=(y_1,y_2,\dots,y_m)^T \in A^c} \pi_1(y_1) \wedge \pi_2(y_2) \wedge \dots \wedge \pi_m(y_m) \tag{10}$$

Where “ \wedge ” is Zadeh operator, which denotes min.

Thus the triplet $(\Omega_y, p(\Omega_y), \text{Pos}_y)$ is referred to as a possibility space. $\text{Pos}_y(A)$ is interpreted as possibility of the event A in universe of discourse Ω_y ; and $\text{Nec}_y(A)$ is interpreted as necessity of the event A .

Then let random variables subspace Ω_x , in which the random component $\mathbf{x}=(x_1, x_2, \dots, x_n)^T$ of \mathbf{z} is defined, be the universe of discourse. A possibility measure $\text{Pos}_x(\cdot)$ defined on σ -algebra $p(\Omega_x)$ can be induced by the possibility measure $\text{Pos}_y(\cdot)$ and the limit state function defined in Ω_z . $\text{Pos}_x(\cdot)$ is defined as follows

$$\begin{aligned} \text{Pos}_x(B) &= \text{Pos}_y\left(\bigcup_{\mathbf{x} \in B} \{\mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\}\right) \\ &= \sup_{\mathbf{x} \in B} \text{Pos}_y\{\mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\} \end{aligned} \tag{11}$$

Where B is a subset of Ω_x .

It can be proved that $\text{Pos}_x(\cdot)$ obeys the possibility axioms:

Axiom 1: $\text{Pos}_x(\phi) = 0$.

Where ϕ denotes the empty set.

Proof:

$$\text{Pos}_x(\phi) = \sup_{\mathbf{x} \in \phi} \text{Pos}_y\{\mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0\} = \text{Pos}_y(\phi) = 0 \tag{12}$$

Axiom 2: $\text{Pos}_x(\Omega_x) = 1$.

Proof: Using Eqs. (7) and (11) we can obtain

$$\begin{aligned} \text{Pos}_x(\Omega_x) &= \sup_{\mathbf{x} \in \Omega_x} \text{Pos}_y\{\mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\} \\ &= \sup_{\mathbf{x} \in \Omega_x} \left\{ \sup_{\mathbf{y} \in \{G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\}} \mu_y(\mathbf{y}) \right\} \end{aligned} \tag{13}$$

Because there exists state $\mathbf{z}=\mathbf{z}^* \in \Omega_z$, as discussed in section 3.1, satisfying $\pi_y(\mathbf{y}^*)=1$ and $G(\mathbf{z}^*) < 0$. Thus we can conclude that

$$\text{Pos}_x(\{\mathbf{z} = \mathbf{z}^*\}) = 1 \tag{14}$$

And according to the properties of possibility measure, we can easily obtain $\text{Pos}_x(\Omega_x) \leq 1$. Combining with Eqs.(13) **Axiom 2** results.

Axiom 3: For any arbitrary collection of sets $\{C_j\} \subset p(\Omega_x)$

$$\text{Pos}_x\left(\bigcup_j C_j\right) = \sup_j \text{Pos}_x(C_j) \tag{15}$$

Proof: From Eqs.(11),

$$\begin{aligned}
 \text{Pos}_x \left(\bigcup_j C_j \right) &= \text{Pos}_y \left(\bigcup_{\mathbf{x} \in \bigcup_j C_j} \{ \mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y \} \right) \\
 &= \text{Pos}_y \left(\bigcup_j \bigcup_{\mathbf{x} \in C_j} \{ \mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y \} \right) \\
 &= \sup_j \text{Pos}_y \bigcup_{\mathbf{x} \in C_j} \{ \mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y \} \\
 &= \sup_j \text{Pos}_x \{ C_j \}
 \end{aligned} \tag{16}$$

In summary, $\text{Pos}_x(\cdot)$ is a possibility measure defined on $p(\Omega_x)$, and the triplet $(\Omega_x, p(\Omega_x), \text{Pos}_x)$ is referred to as a possibility space.

According to possibility theory and fuzzy set theory, the possibility measure $\text{Pos}_x(\cdot)$ can induce a F set defined in the universe of discourse Ω_x characterized by its MF $\mu_x(\mathbf{x})$

$$\mu_x(\mathbf{x}) = \text{Pos}_x \{ \omega \in \Omega_x : \omega = \mathbf{x} \} \tag{17}$$

From Eqs.(11), $\mu_x(\mathbf{x})$ can be written as

$$\mu_x(\mathbf{x}) = \text{Pos}_y \{ \mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y \} = \text{Pos}_y (\Omega_{zf}) \tag{18}$$

Eqs.(18) reveals the significance of $\mu_x(\mathbf{x})$: in the universe of discourse Ω_x , the failure possibility of the structure for any arbitrary state \mathbf{x} is $\mu_x(\mathbf{x})$. Thus it is convenient to define $\mu_x(\mathbf{x})$ as the MF of the FFD F_{xf} in universe of discourse Ω_x discussed in section 3.2. That is

$$\mu_{xf}(\mathbf{x}) = \mu_x(\mathbf{x}) = \text{Pos}_y (\Omega_{zf}) \tag{19}$$

Where $\mu_{xf}(\mathbf{x})$ is the MF of the FFD F_{xf} .

According to possibility theory, $1-\mu_x(\mathbf{x})$ should be also a membership function of some F set in Ω_x . From Eqs.(7) and (11) we can conclude

$$\begin{aligned}
 1 - \mu_x(\mathbf{x}) &= 1 - \text{Pos}_y \{ \mathbf{y} : G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y \} \\
 &= \text{Nec}_y \{ \mathbf{y} : G(\mathbf{x}, \mathbf{y}) \geq 0, \mathbf{y} \in \Omega_y \}
 \end{aligned} \tag{20}$$

Eqs.(20) reveals the significance of $1-\mu_x(\mathbf{x})$: in the universe of discourse Ω_x , the structure safety necessity for any arbitrary state \mathbf{x} is $1-\mu_x(\mathbf{x})$. Again, it is convenient to define $1-\mu_x(\mathbf{x})$ as the MF of the FSD F_{xs} in universe of discourse Ω_x . That is

$$\mu_{xs}(\mathbf{x}) = 1 - \mu_{xf}(\mathbf{x}) = \text{Nec}_y \{ \mathbf{y} : G(\mathbf{x}, \mathbf{y}) \geq 0, \mathbf{y} \in \Omega_y \} = \text{Nec}_y (\Omega_{zs}) \tag{21}$$

Finally, from Eqs.(7), (19) and (21), the membership functions $\mu_{xf}(\mathbf{x})$ and $\mu_{xs}(\mathbf{x})$ of F_{xf} and F_{xs} can be obtained by the following equations

$$\mu_{xf}(\mathbf{x}) = \sup_{\mathbf{y} \in \{\mathbf{y}: G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\}} \pi_y(\mathbf{y}) \tag{22}$$

$$\mu_{xs}(\mathbf{x}) = 1 - \sup_{\mathbf{y} \in \{\mathbf{y}: G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\}} \pi_y(\mathbf{y}) \tag{23}$$

Especially, if the component $y_i, i=1,2,\dots,m$, are independent fuzzy variables characterized by the PoDFs $\pi_i(y_i)$, the upper two equations can be written as follows

$$\mu_{xf}(\mathbf{x}) = \sup_{\mathbf{y} \in \{\mathbf{y}: G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\}} \pi_1(y_1) \wedge \pi_2(y_2) \wedge \dots \wedge \pi_m(y_m) \tag{24}$$

$$\mu_{xs}(\mathbf{x}) = 1 - \sup_{\mathbf{y} \in \{\mathbf{y}: G(\mathbf{x}, \mathbf{y}) < 0, \mathbf{y} \in \Omega_y\}} \pi_1(y_1) \wedge \pi_2(y_2) \wedge \dots \wedge \pi_m(y_m) \tag{25}$$

In summary, as far as the generalized assumptions with mixed variables mentioned in section 3.1 are concerned, the crisp failure criterion in basic variables space $\Omega_z, G(z)=0$, can be transformed into a fuzzy failure criterion in the random variables subspace Ω_x , expressed as the FFD F_{xf} and FSD F_{xs} characterized by their MFs $\mu_{xf}(\mathbf{x})$ and $\mu_{xs}(\mathbf{x})$ as shown in Eqs.(22-25). For any arbitrary state $\mathbf{x}=\mathbf{x}_0$, $\mu_{xf}(\mathbf{x}=\mathbf{x}_0)=1$ indicates: $\mu_{xs}(\mathbf{x}=\mathbf{x}_0)=0$, and the structure is definitely failed at $\mathbf{x}=\mathbf{x}_0$. $\mu_{xf}(\mathbf{x}=\mathbf{x}_0)=0$ indicates: $\mu_{xs}(\mathbf{x}=\mathbf{x}_0)=1$, and the structure is definitely safe at $\mathbf{x}=\mathbf{x}_0$. While $0 < \mu_{xf}(\mathbf{x}=\mathbf{x}_0) < 1$ indicates: $0 < \mu_{xs}(\mathbf{x}=\mathbf{x}_0) < 1$, the structure is fuzzy safe and fuzzy failed at $\mathbf{x}=\mathbf{x}_0$ and compatible with the concept “failure” at degree $\mu_{xf}(\mathbf{x}=\mathbf{x}_0)$, with the concept “safety” at degree $\mu_{xs}(\mathbf{x}=\mathbf{x}_0)$.

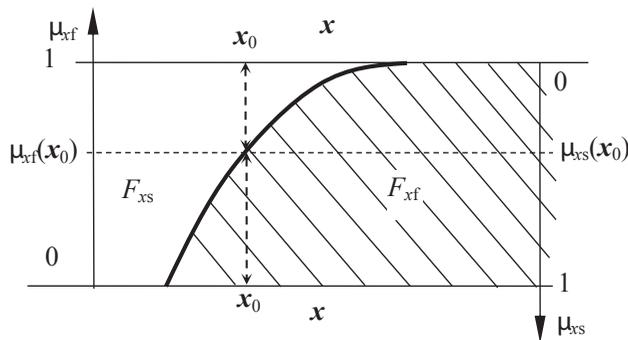


Figure 2: FFD and FSD in the universe of discourse Ω_x

Eqs.(21) indicates that: the FFD F_{xf} and FSD F_{xs} are complement F sets for each other, as shown in Fig.2. Mathematically

$$F_{xs}^c = F_{xf} \quad \text{and} \quad F_{xf}^c = F_{xs} \tag{26}$$

This is compatible with the contrariety of the concepts “failure” and “safety”.

3.4 The reliability assessment of the structure

Based on the analysis in section 3.3, using Eqs.(4-5) we can calculate the failure probability and safe probability of the structure as follows

$$P_{fF} = \int_{\Omega_x} \mu_{xf}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \tag{27}$$

$$P_{sF} = \int_{\Omega_x} \mu_{xs}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \tag{28}$$

Where $\mu_{xf}(\mathbf{x})$ and $\mu_{xs}(\mathbf{x})$ are MFs of the FFD F_{xf} and FSD F_{xs} and can be obtained from Eqs.(22-25).

Since the FFD F_{xf} and FSD F_{xs} are complement F sets for each other, there must be

$$\mu_{xs}(\mathbf{x}) + \mu_{xf}(\mathbf{x}) \equiv 1 \tag{29}$$

Combining Eqs.(27-29), we can obtain

$$P_{sF} + P_{fF} \equiv 1 \tag{30}$$

As shown in Eqs.(29-30), we can simply calculate the MF $\mu_{xf}(\mathbf{x})$ for FFD F_{xf} and the failure probability P_{fF} , while omitting the calculation for the MF $\mu_{xs}(\mathbf{x})$ of FSD F_{xs} and the safe probability P_{sF} in practical application.

3.5 Further discussions

Considering two special cases, we take further study on the pos-probability reliability model proposed in previous sections.

CASE 1: Consider the fuzzy component \mathbf{y} takes some definite real value $\mathbf{y}=\mathbf{y}_0$ without fuzziness, component \mathbf{x} is still a random vector associated with its JPDPF $f(\mathbf{x})$.

Because \mathbf{y} takes definite real value, the performance variable of the structure $M = G(\mathbf{x},\mathbf{y}_0)$ is simply a random variable. Thus, the original reliability problem with mixed variables is transformed into a reliability problem with pure random variables. We apply the pos-probability reliability model proposed in this paper to

settle the problem. The real vector $\mathbf{y}=\mathbf{y}_0$ can be regarded as a fuzzy vector with its JPoDF as follows

$$\pi_{\mathbf{y}}(\mathbf{y}) = \begin{cases} 1 & \mathbf{y} = \mathbf{y}_0 \\ 0 & \mathbf{y} \neq \mathbf{y}_0 \end{cases} \quad (31)$$

Using Eqs.(22-23), the MFs of the FFD and FSD in $\Omega_{\mathbf{x}}$ result in

$$\mu_{x_f}(\mathbf{x}) = \begin{cases} 1 & G(\mathbf{x}, \mathbf{y}_0) < 0 \\ 0 & G(\mathbf{x}, \mathbf{y}_0) \geq 0 \end{cases} \quad (32)$$

$$\mu_{x_s}(\mathbf{x}) = \begin{cases} 1 & G(\mathbf{x}, \mathbf{y}_0) \geq 0 \\ 0 & G(\mathbf{x}, \mathbf{y}_0) < 0 \end{cases} \quad (33)$$

Thus the FFD F_{x_f} and FSD F_{x_s} degenerate into crisp failure domain $\Omega_{x_f}=\{\mathbf{x}: G(\mathbf{x})<0, \mathbf{x}\in\Omega_{\mathbf{x}}\}$ and crisp safe domain $\Omega_{x_s}=\{\mathbf{x}:G(\mathbf{x})\geq 0, \mathbf{x}\in\Omega_{\mathbf{x}}\}$.

Using Eqs.(27-28), we can show that

$$P_{fF} = \int_{G(\mathbf{x}, \mathbf{y}_0) < 0} f(\mathbf{x}) d\mathbf{x} \quad (34)$$

$$P_{sF} = \int_{G(\mathbf{x}, \mathbf{y}_0) \geq 0} f(\mathbf{x}) d\mathbf{x} \quad (35)$$

Comparing with Eqs.(1-2), the reliability measurement P_{fF} and P_{sF} degenerate into P_f and P_s , and the pos-probability reliability model degenerates into the traditional probabilistic reliability model.

CASE 2: Consider the random component \mathbf{x} takes some determined real value $\mathbf{x}=\mathbf{x}_0$ without randomness, component \mathbf{y} is still a fuzzy vector characterized by its JPoDF $\pi_{\mathbf{y}}(\mathbf{y})$.

Because \mathbf{x}_0 is a real vector, the performance variable $M = G(\mathbf{x}_0, \mathbf{y})$ is simply a fuzzy variable. The original mixed variables reliability problem degenerates into a fuzzy reliability problem. Again the pos-probability reliability model is applied. The real vector \mathbf{x}_0 can be regarded as a random vector with its JPDF as follows

$$f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0) \quad (36)$$

Where $\delta(\mathbf{x}-\mathbf{x}_0)$ is the δ -function at point $\mathbf{x}=\mathbf{x}_0$, satisfying the following equations

$$\int_{\Omega_{\mathbf{x}}} \delta(\mathbf{x} - \mathbf{x}_0) d\mathbf{x} = 1 \quad (37)$$

$$\delta(\mathbf{x} - \mathbf{x}_0) = \begin{cases} +\infty & \mathbf{x} = \mathbf{x}_0 \\ 0 & \mathbf{x} \neq \mathbf{x}_0 \end{cases} \quad (38)$$

Combining Eqs.(22-23), (27-28) and (37-38), we can show that

$$P_{fF} = \mu_{xf}(\mathbf{x} = \mathbf{x}_0) = \sup_{\mathbf{y} \in \Omega_y, G(\mathbf{x}_0, \mathbf{y}) < 0} \pi_y(\mathbf{y}) \quad (39)$$

$$P_{sF} = \mu_{xs}(\mathbf{x} = \mathbf{x}_0) = 1 - \sup_{\mathbf{y} \in \Omega_y, G(\mathbf{x}_0, \mathbf{y}) \geq 0} \pi_y(\mathbf{y}) \quad (40)$$

Thus, the reliability measurement P_{fF} and P_{sF} degenerate into possibility of failure and possibility of safety for the structure. And the pos-probability reliability model degenerates into the POSBIST reliability model [Cai, Wen and Zhang (1995); Guo, Lv and Feng (2002)].

In summary, the conventional probabilistic reliability model and POSBIST reliability model for structures can be regarded as the special cases of the proposed pos-probability reliability model. In other words, the proposed model unifies the probabilistic and POSBIST reliability models and can be regarded as a universal method.

4 Universal reliability model with fuzzy and random variables for fuzzy failure criterion

In the binary-state assumption (A2) shown in section 3.1, the failure criterion in basic variables space Ω_z is crisp and characterized by the limit state function $G(\mathbf{z})=0$. But often the reality in practical application is that, the failure criterion in Ω_z might be fuzzy because of a variety of causes. In this section, with more generalities, we develop the pos-probability reliability model by replacing the binary-state assumption with the following fuzzy state assumption (A2').

A2'(fuzzy state assumption): The failure criterion defined in Ω_z is fuzzy and expressed as FFD F_{zf} and FSD F_{zs} characterized by their MFs $\mu_{zf}(\mathbf{x}, \mathbf{y})$ and $\mu_{zs}(\mathbf{x}, \mathbf{y})$, where

$$\mu_{zs}(\mathbf{x}, \mathbf{y}) + \mu_{zf}(\mathbf{x}, \mathbf{y}) \equiv 1 \quad (41)$$

While assumption A1' is reserved.

As shown in Eqs.(22-23), the MFs $\mu_{xf}(\mathbf{x})$ and $\mu_{xs}(\mathbf{x})$ rely on the JPoDF $\pi_y(\mathbf{y})$ of fuzzy component vector \mathbf{y} and the failure domain Ω_{zf} and safe domain Ω_{zs} in Ω_z . The failure domain $\Omega_{zf} = \{\mathbf{z}: G(\mathbf{z}) < 0, \mathbf{z} \in \Omega_z\}$ can be written in the form of characteristic function as follows

$$\Omega_{zf} \Rightarrow \varphi_{zf}(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & G(\mathbf{x}, \mathbf{y}) < 0 \\ 0 & G(\mathbf{x}, \mathbf{y}) \geq 0 \end{cases} \quad (42)$$

Where $\varphi_{zf}(\mathbf{x}, \mathbf{y})$ is the characteristic function of the crisp set Ω_{zf} .

Thus Eqs.(22) can be written as

$$\mu_{xf}(\mathbf{x}) = \sup_{\Omega_y} [\pi_y(\mathbf{y}) \wedge \varphi_{zf}(\mathbf{x}, \mathbf{y})] \tag{43}$$

If the failure domain in Ω_z is a F set, we can replace the characteristic function $\varphi(\mathbf{x}, \mathbf{y})$ with the MF $\mu_{zf}(\mathbf{x}, \mathbf{y})$ conveniently, and obtain the MF of FFD F_{xf} in Ω_x .

$$\mu_{xf}(\mathbf{x}) = \sup_{\Omega_y} [\pi_y(\mathbf{y}) \wedge \mu_{zf}(\mathbf{x}, \mathbf{y})] \tag{44}$$

As a matter of fact, the upper equation can be also obtained by generalizing Eqs.(19) to F sets. That is

$$\mu_{xf}(\mathbf{x}) = \text{Pos}_y(F_{zf}) \Rightarrow \mu_{xf}(\mathbf{x}) = \sup_{\Omega_y} [\pi_y(\mathbf{y}) \wedge \mu_{zf}(\mathbf{x}, \mathbf{y})] \tag{45}$$

The FFD and FSD in Ω_z should be the complement F sets for each other. That is

$$F_{zS}^c = F_{zf} \quad \text{and} \quad F_{zf}^c = F_{zS} \tag{46}$$

Similarly, we can obtain the MF $\mu_{xs}(\mathbf{x})$ of FSD F_{xs} in Ω_x by generalizing the Eqs.(21) to F sets. That is

$$\mu_{xs}(\mathbf{x}) = \text{Nec}_y(F_{zs}) \Rightarrow \mu_{xs}(\mathbf{x}) = 1 - \sup_{\Omega_y} [\pi_y(\mathbf{y}) \wedge \mu_{zf}(\mathbf{x}, \mathbf{y})] = 1 - \mu_{xf}(\mathbf{x}) \tag{47}$$

Thus Eqs.(29) is reserved.

Finally, the failure probability and safe probability can be calculated by using Eqs.(27-28). Combining Eqs. (27-28), (44) and (47), Eqs.(30) is reserved too.

Similar with the discussions in section 3.5, the proposed pos-probability reliability model with fuzzy criterion can degenerate into the probabilistic model with crisp or fuzzy states, POSBIST reliability model and POSFUST reliability model, which can be regarded as a universal structural reliability model with random and fuzzy variables.

5 Examples

5.1 A simple example

Considering the typical stress-strength reliability problem, the stress S is a uniform random variable associated with the range $[-1, 1]$ and its probability density function

(PDF) $f(S)$, the strength R is fuzzy variable which follows a triangular distribution. Its PoDF $\pi_R(R)$ is shown in Fig.3 and the following equation.

$$\pi_R(R) = \begin{cases} 0 & R < 0 \text{ or } R > 2 \\ R & 0 \leq R < 1 \\ 2 - R & 1 \leq R \leq 2 \end{cases} \quad (48)$$

The Euclidean space R^1 in which S defined is denoted by Ω_S , while the Euclidean space R^2 vector $(R,S)^T$ defined in is denoted by Ω_z .

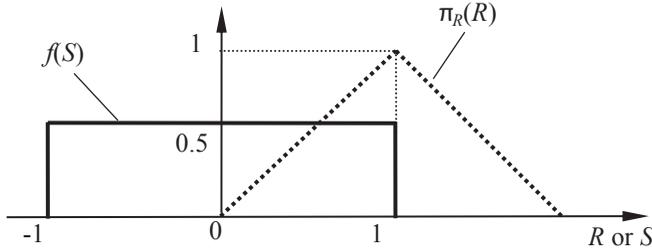


Figure 3: The PDF of S and PoDF of R

5.1.1 Reliability problems

We calculate the reliability of the structure with the following two cases:

Case 1: The failure criterion in Ω_z is crisp and characterized by the limit function as follows

$$R - S = 0 \quad (49)$$

Certainly, $R - S < 0$ denotes failure.

Case 2: The failure criterion in Ω_z is fuzzy and the MF of the FFD is

$$\mu_{zf}(R,S) = \begin{cases} 0 & S < R \\ S - R & 0 \leq S - R \leq 1 \\ 1 & S - R > 1 \end{cases} \quad (50)$$

5.1.2 Solutions

Case 1: Since R is fuzzy variable, the failure criterion of the structure in Ω_S is fuzzy. Using Eqs.(22), the MF of the FFD in Ω_S is

$$\begin{aligned} \mu_{Sf}^{(\text{Case 1})}(S) &= \sup_{R-S < 0} \pi_R(R) = \sup_{R-S < 0} \max \{ \min \{ R, 2 - R \}, 0 \} \\ &= \min \{ \max \{ S, 0 \}, 1 \} \end{aligned} \quad (51)$$

Using Eqs.(27), the failure probability can be obtain

$$\begin{aligned}
 P_{fF}^{(\text{Case 1})} &= \int_{-\infty}^{+\infty} \mu_{sf}(S) f(S) dS \\
 &= \int_{-\infty}^{+\infty} 0.5 * \min \{ \max \{ S, 0 \}, 1 \} dS \\
 &= \int_0^1 0.5S dS = 0.25
 \end{aligned}
 \tag{52}$$

Case 2: Using Eqs.(4), we can obtain the MF $\mu_{sf}(S)$ of the FFD in Ω_S

$$\begin{aligned}
 \mu_{sf}^{(\text{Case 2})}(S) &= \sup_{(R,S)^T \in R^2} \pi_R(R) \wedge \mu_{zf}(R,S) \\
 &= \sup_{(R,S)^T \in R^2} \max \{ \min \{ R, 2 - R \}, 0 \} \wedge \min \{ \max \{ S - R, 0 \}, 1 \} \\
 &= \min \{ \max \{ 0.5S, 0 \}, 1 \}
 \end{aligned}
 \tag{53}$$

Using Eqs.(27), the failure probability can be obtained

$$\begin{aligned}
 P_{fF}^{(\text{Case 2})} &= \int_{-\infty}^{+\infty} \mu_{sf}(S) f(S) dS \\
 &= \int_{-\infty}^{+\infty} \min \{ \max \{ 0.5S, 0 \}, 1 \} dS \\
 &= \int_0^1 0.25S dS = 0.125
 \end{aligned}
 \tag{54}$$

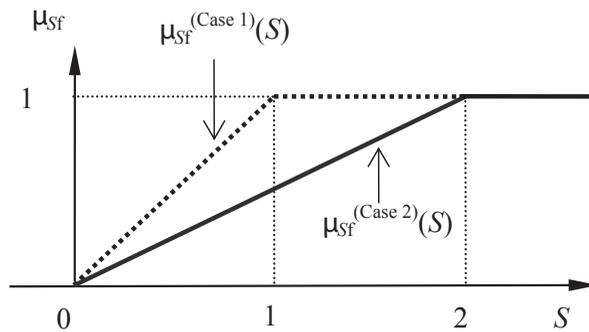


Figure 4: The FFDs in Ω_S of the two cases

The FFDs in Ω_S of the two cases are shown in Fig.4.

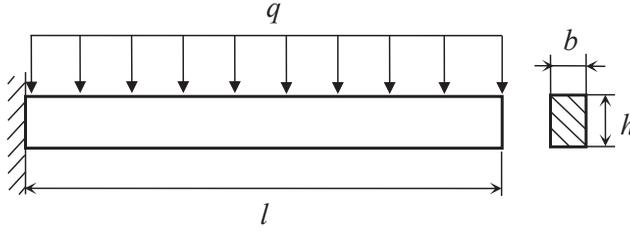


Figure 5: Cantilever under uniformly distributed load

5.2 Cantilever under uniformly distributed load

A cantilever under uniformly distributed load is shown in Fig.5. The length of the cantilever is $l=2000\text{mm}$. The width of the section b/mm , the depth of the section h/mm , the Young's modulus E/MPa and the uniformly distributed load q are assumed to follow the normal random distribution. The distribution parameters of the basic random variables are shown in Tab.1. Large deflection leads to the failure of the cantilever, where large deflection can be regarded as a fuzzy variable denoted by Δ^*/mm . Δ^* follows a rising half-trapezoidal distribution, its PoDF is as follows

$$\pi_{\Delta^*}(\Delta^*) = \begin{cases} 0 & \Delta^* < 25 \\ 0.2(\Delta^* - 25) + 1 & 25 \leq \Delta^* \leq 30 \\ 1 & \Delta^* > 30 \end{cases} \quad (55)$$

Table 1: Parameters of the normal random variables

	b/mm	h/mm	$q/(\text{N}/\text{mm})$	E/Mpa
Mean value	120	240	260	2.0×10^5
Standard deviation	5.2	9	10.2	6.1×10^3

5.2.1 The failure criterion

Large deflection leads to the failure of the cantilever. The failure criterion can be expressed as the limit state function as follows

$$\Delta^* - \Delta(\mathbf{x}) = 0 \quad (56)$$

Where Δ is the deflection of the cantilever, and $\mathbf{x}=(b,h,q,E)^T$ is the basic random vector of the structure. $\Delta^*-\Delta < 0$ denotes failure.

According to the engineering mechanics principles, we can obtain the deflection of the cantilever

$$\Delta(\mathbf{x}) = \frac{3}{2} \frac{ql^4}{Ebh^3} \tag{57}$$

Thus the limit state function can be written as

$$\Delta^* - \frac{3}{2} \frac{ql^4}{Ebh^3} = 0 \tag{58}$$

5.2.2 FFD in the random variables subspace

According to Eqs.(24) and (58), the MF of FFD in the random variables subspace in which $\mathbf{x}=(b,h,q,E)^T$ defined is

$$\mu_{x_f}(\mathbf{x}) = \sup_{\Delta^* - \Delta(\mathbf{x}) < 0} \pi_{\Delta^*}(\Delta^*) \tag{59}$$

That is

$$\mu_{x_f}(\mathbf{x}) = \begin{cases} 0 & \Delta(\mathbf{x}) < 25 \\ 0.2[\Delta(\mathbf{x}) - 30] + 1 & 25 \leq \Delta(\mathbf{x}) \leq 30 \\ 1 & \Delta(\mathbf{x}) > 30 \end{cases} \tag{60}$$

5.2.3 Failure probability of the cantilever

Using Eqs.(27) we can obtain the failure probability of the structure P_{fF}

$$\begin{aligned} P_{fF} &= \int_{25}^{30} \{0.2[\Delta(\mathbf{x}) - 30] + 1\} f_{\Delta}(\Delta) d\Delta(\mathbf{x}) + \int_{30}^{+\infty} f_{\Delta}(\Delta) d\Delta(\mathbf{x}) \\ &= \int_{\mathbf{x}=(b,h,E,q)^T \in R^4} \min \{ \max \{ \{0.2[\Delta(\mathbf{x}) - 30] + 1\}, 0 \}, 1 \} f_b(b) f_h(h) f_E(E) f_q(q) d\mathbf{x} \end{aligned} \tag{61}$$

Where $f_{\Delta}(\Delta)$ is the PDF of random variable Δ , $f_b(b), f_h(h), f_E(E)$ and $f_q(q)$ are PDFs of the normal random variables b, h, E and q .

According to Eqs.(61), the failure probability P_{fF} can be calculated by using the Monte-Carlo simulation, with number of simulations $N=10^6$, $P_{fF} \approx 0.0536$.

The relationship between the failure probability P_{fF} and the distribution parameters of q is shown in Tab.2 and Fig.6.

From Tab.2 and Fig.6 we can conclude that:

Table 2: The relationship of P_{fF} and the distribution parameters of q

Standard deviation of q	Mean value of $q/(N/mm)$							
	245	250	255	260	265	270	275	280
8.7	9.3×10^{-4}	0.00227	0.00440	0.00782	0.0124	0.0196	0.0299	0.0448
10.2	0.0457	0.0478	0.0500	0.0536	0.0598	0.0678	0.0776	0.0916
11.7	0.0926	0.0944	0.0969	0.1011	0.1067	0.1154	0.1270	0.1429
13.2	0.1444	0.1466	0.1493	0.1540	0.1605	0.1688	0.1804	0.1965

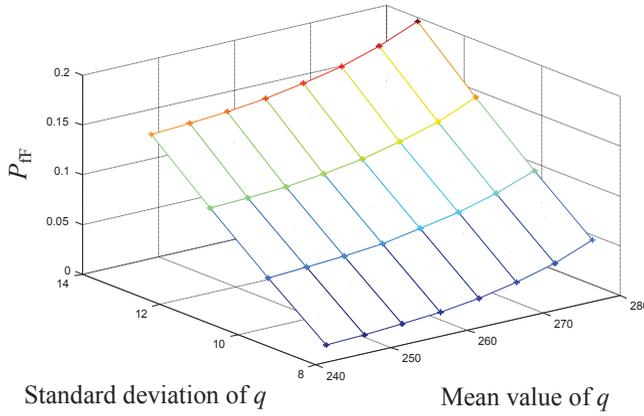


Figure 6: Relationship of P_{fF} and distribution parameters of q

1. The failure probability P_{fF} increases with increasing mean value or standard deviation of q , which shows the rationality of the pos-probability reliability model in a certain extent.
2. For fixed mean value of q , the relationship of P_{fF} and deviation of q approximates linear; but for fixed standard deviation of q P_{fF} increases more quickly for larger mean value of q .

Assuming the PoDF of Δ^* vary with the parameter t , $t > 0$, as shown in Eqs.(62) and Fig.7. We analyze the reliability of the structure.

$$\pi_{\Delta^*}(\Delta^*) = \begin{cases} 0 & \Delta^* < 30 - t \\ \frac{1}{t}(\Delta^* - 30) + 1 & 30 - t \leq \Delta^* \leq 30 \\ 1 & \Delta^* > 30 \end{cases} \quad (62)$$

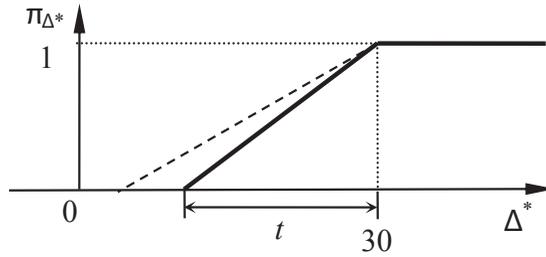


Figure 7: The PoDF of Δ^* with parameter t

The failure probability can be obtained as follows

$$P_{fF} = \int_{\mathbf{x}=(b,h,E,q)^T \in R^4} \min \left\{ \max \left\{ \left\{ \frac{1}{t} [\Delta(\mathbf{x}) - 30] + 1 \right\}, 0 \right\}, 1 \right\} f_b(b) f_h(h) f_E(E) f_q(q) \, d\mathbf{x} \tag{63}$$

The failure probability P_{fF} is calculated for vary t by using Monte-Carlo simulation. And the relationship between P_{fF} and t is shown in Tab.3 and Fig.8.

Table 3: P_{fF} for varying t

t	4	5	6	7	8	9	10
P_{fF}	0.0020	0.0048	0.0117	0.0234	0.0437	0.0774	0.1334

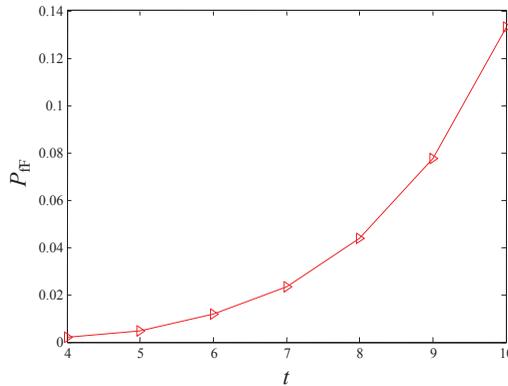


Figure 8: The failure probability versus t

As shown in Tab.3 and Fig.8, the failure probability increases with increasing t . Δ^* can be regarded as the generalized strength of the structure. From Eqs.(56) and Fig.7 we can conclude that larger t indicates “smaller” Δ^* and of course leads to decreasing reliability. Thus the pos-probability reliability model proposed in this paper reasonably shows the influence of PoDF of Δ^* on the reliability of the structure.

5.3 Reliability for ring-stiffened cylindrical shell

The ring-stiffened cylindrical shell is a prevalent structural form in the pressure structure design, e.g. the pressure hull of the submarine. High-strength steel is often adopted in the ring-stiffened cylindrical shell structure. Thus the structure might fail if buckling under a high static pressure.

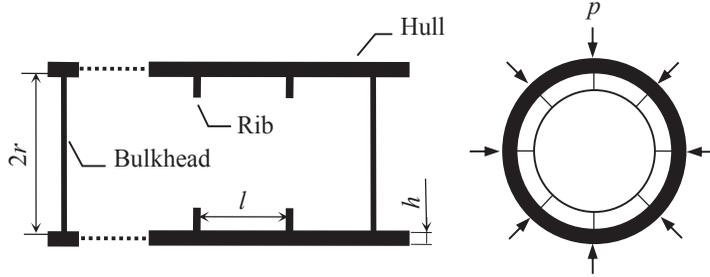


Figure 9: Ring-stiffened cylindrical shell

A certain ring-stiffened cylindrical shell is shown in Fig.9. The Poisson ratio of the material is $\nu=0.3$. The thickness h/mm of the shell and Young’s modulus E/MPa are normal random variables, their distribution parameters are shown in Tab.4. The static pressure p/MPa , rib spacing l/mm and inner radius r/mm , are fuzzy variables associated with their PoDFs as follows

$$\pi_p(p) = \begin{cases} 0 & p > 4.2 \\ 1 - 100(p - 4.1)^2 & 4.1 \leq p \leq 4.2 \\ 1 & p < 4.1 \end{cases} \quad (64)$$

$$\pi_l(l) = \begin{cases} 0 & l > 650 \\ 1 - (650 - l)^2 / 400 & 630 \leq l \leq 650 \\ 1 & l < 630 \end{cases} \quad (65)$$

$$\pi_r(r) = \begin{cases} 0 & l > 3500 \\ 1 - (r - 3450)^2 / 2500 & 21 \leq l \leq 23 \\ 1 & l < 3450 \end{cases} \quad (66)$$

Table 4: Distribution parameters for the normal random variables

	Thickness h/mm	Young's modulus E/Mpa
Mean value	23.5	2.0×10^5
Standard deviation	0.3	6.0×10^3

Analyze the reliability of the structure for buckling.

5.3.1 The limit state function

The critical pressure p_{cr} of the shell can be obtained by the following equation

$$p_{cr} = C_g C_s p_E \tag{67}$$

Where C_g is the geometrical correction factor, C_s is the residual stress correction factor, and $C_g=0.93$, $C_s=0.93$; p_E is the Euler pressure, for $\nu=0.3$ that is

$$p_E = E \left(\frac{h}{r} \right)^2 \frac{0.6}{u - 0.37} \tag{68}$$

Where u is a non-dimensional factor

$$u = \frac{0.642l}{\sqrt{rh}} \tag{69}$$

Combining Eqs.(67-69), we can obtain the limit state function as follows

$$G_{shell}(\mathbf{X}, \mathbf{Y}) = C_g C_s E \left(\frac{h}{r} \right)^2 \frac{0.6\sqrt{rh}}{0.642l - 0.37\sqrt{rh}} - p = 0 \tag{70}$$

Where $G_{shell} < 0$ denotes failure, $X=(h,E)^T$, $Y=(p,l,r)^T$, X is random vector and Y is fuzzy vector.

5.3.2 Reliability problems

We analyze the reliability of the shell for the following four cases:

Case 1: p , l and r take definite values $p=4.1MPa$, $l=630mm$, $r=3450mm$; the failure criterion is crisp as $G_{shell}=0$.

Case 2: p , l and r take definite values $p=4.2MPa$, $l=650mm$, $r=3500mm$; the failure criterion is crisp as $G_{shell}=0$.

Case 3: p , l and r take fuzzy values as shown in Eqs.(64-66); the failure criterion is crisp as $G_{shell}=0$.

Case 4: p , l and r take fuzzy values as shown in Eqs.(64-66); the failure criterion in basic variables space is also fuzzy and the MF of the FFD is

$$\mu_{Mf}(M) = \begin{cases} e^{-100M^2} & M \geq 0 \\ 1 & M < 0 \end{cases} \quad (71)$$

Where

$$M = C_g C_s E \left(\frac{h}{r} \right)^2 \frac{0.6\sqrt{rh}}{0.642l - 0.37\sqrt{rh}} - p \quad (72)$$

5.3.3 Calculation

Because the limit state function is nonlinear and complicated, it is difficult to obtain the analytical expressions for MF $\mu_{Xf}(\mathbf{X})$ of FFD in the random variables subspace. To overcome the technical difficulty, numerical methods are adopted, and MATLAB procedure is used in implementing the computation of the MF $\mu_{Xf}(\mathbf{X})$ and failure probability.

According to Eqs.(27), the failure probability can be obtained using the Monte-Carlo simulation

$$P_{fF} \approx \bar{P}_{fF} = \frac{1}{N} \sum_{i=1}^N \mu_{Xf}(\mathbf{X}_i^*) \quad (73)$$

Where \mathbf{X}_i^* is the i th sampling value of random vector \mathbf{X} , N is the number of simulations. According to the Monte-Carlo simulation theory, N can be determined as follows

$$N > \frac{1 - P_{fF}}{\varepsilon^2 P_{fF}} \quad (74)$$

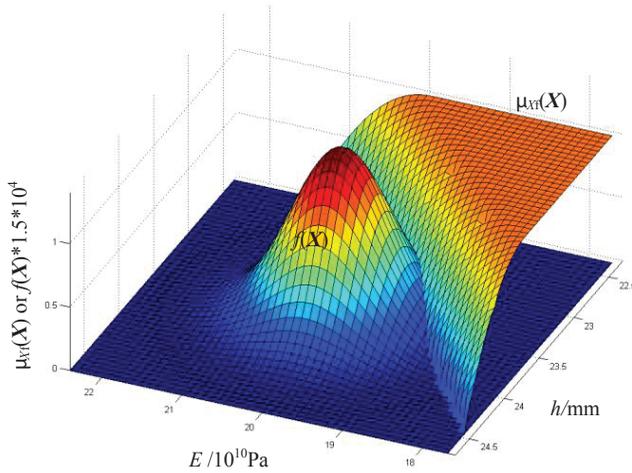
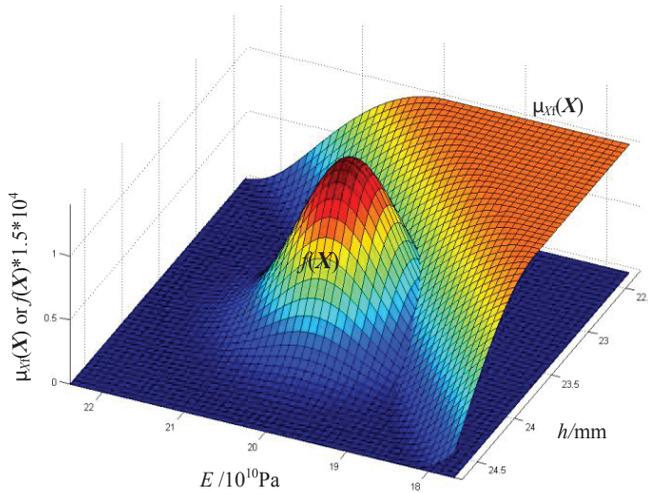
Where, ε is the coefficient of variation of \bar{P}_{fF} .

$\mu_{Xf}(\mathbf{X}_i^*)$ can be obtained by solving the following nonlinear optimization problem for **Case 3**

$$\begin{cases} \mu_{Xf}(\mathbf{X}_i^*) = \max \pi_p(p) \wedge \pi_l(l) \wedge \pi_r(r) \\ \text{s.t. } G_{shell}(\mathbf{X}_i^*, \mathbf{Y}) < 0 \end{cases} \quad (75)$$

$\mu_{Xf}(\mathbf{X}_i^*)$ can be obtained by solving the nonlinear optimization problem as shown in the following equation for **Case 4**

$$\begin{cases} \mu_{Xf}(\mathbf{X}_i^*) = \max \pi_p(p) \wedge \pi_l(l) \wedge \pi_r(r) \wedge \mu_{Mf}(M) \\ \text{s.t. } \mathbf{Y} \in R^3, \mathbf{X} = \mathbf{X}_i^* \end{cases} \quad (76)$$

Figure 10: FFD in random variables subspace for **Case 3**Figure 11: FFD in random variables subspace for **Case 4**

There are many existing methods that can be used to tackle the upper optimization problems, such as NLP method [Santos, Matioli and Beck (2012)], particle swarm optimization (PSO) [Yang and Sun (2013)] and genetic algorithm. Especially, according to the NLP theory, if $\pi_y(\mathbf{y})$ (where $\pi_y(\mathbf{y}) = \pi_p(p) \wedge \pi_l(l) \wedge \pi_r(r)$), $\mu_{Mf}(M)$

and G_{shell} are convex functions, Eqs.(75-76) degenerate into convex programming problems which are easier to solve.

Using MATLAB procedure of NLP method, $\mu_{Xf}(X)$ for $E/MPa \in [1.76 \times 10^5, 2.24 \times 10^5]$, $h/mm \in [22.3, 23.7]$ is calculated as shown in Fig.10 and 11, where $f(X)$ is the JPDP of X .

With number of simulations $N=10^5$, we can obtain the failure probability P_{fF} of the structure for the four cases shown in Tab.5.

Table 5: P_{fF} for the four cases

	Case 1	Case 2	Case 3	Case 4
P_{fF}	$< 10^{-3}$	0.083	0.026	0.115

From Tab.5 we can conclude that:

1. If the fuzziness of p, l and r are omitted, there would be significant difference with P_{fF} .
2. Denote the failure domain of the four cases by $F_{Case1}, F_{Case2}, F_{Case3}$ and F_{Case4} . It is easy to find out that $F_{Case1}, F_{Case2}, F_{Case3}$ and F_{Case4} are related as follows

$$F_{Case1} \subset F_{Case3} \subset F_{Case2} \tag{77}$$

$$F_{Case3} \subset F_{Case4} \tag{78}$$

Results in Tab.5 reveal the orderliness shown in Eqs.(77-78), which prove the rationality of the universal reliability model proposed in this paper.

Monte-Carlo simulation in section 5.3 for the proposed universal reliability model might be costly, because the optimization problem as shown in Eqs.(75 or 76) should be solved for every X point viable and the computational cost increases with increasing complexity of the limit state function and the dimensional size of fuzzy vector Y . In addition, it is invalid for problems with implicit limit state function (e.g. the responses of the structure are obtained by FEM). Thus, the calculation procedures with high efficiency and for problems with implicit limit state function should be investigated further for realistic application.

6 Conclusion

Both randomness and fuzziness play important roles in practical reliability assessment for structures. The probabilistic reliability theory and possibility reliability theory have been established separately to deal with randomness and fuzziness. The fundamental assumptions, probability assumption and binary-state assumption, of the conventional probabilistic reliability theory are generalized into the “mixed probability and possibility assumption” and “fuzzy state assumption”, then a universal structural reliability model for mixed fuzzy and random variables is proposed in this paper. The new model can deal with the reliability problem involving pure random or fuzzy variables and both of them. It unifies the probability reliability theory and possibility reliability theory.

As far as the proposed universal reliability model is concerned, the conventional computational procedures for reliability analysis, such as FORM, Hasofer-Lind method are invalid. The Monte-Carlo method can be used to calculate the failure probability for problems with explicit limit state function but not the best, because the computational cost increases with the increasing complexity of the limit state function and the dimensional size of the basic fuzzy vector y . In addition, the Monte-Carlo method can't tackle problems with implicit limit state functions. Thus, the calculation procedures with high efficiency and for problems with implicit limit state function should be investigated in further for realistic application.

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Appendix A: Acronyms

FORM	First-order reliability method
F	Fuzzy
FEM	Finite element method
FFD	Fuzzy failure domain
FSD	Fuzzy safety domain
JPDF	Joint probability density function
JPoDF	Joint possibility distribution function
MF	Membership function

NLP	Nonlinear programming
PDF	Probability density function
PoDF	Possibility distribution function
PSO	Particle swarm optimization
RSM	Response surface method
SFEM	Stochastic finite element method

