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Numerical Algorithm to Solve Fractional Integro-differential Equations Based on Operational Matrix of Generalized Block Pulse Functions

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Abstract: In this paper, we propose a numerical algorithm for solving linear and nonlinear fractional integro-differential equations based on our constructed fractional order generalized block pulse functions operational matrix of integration. The linear and nonlinear fractional integro-differential equations are transformed into a system of algebraic equations by the matrix and these algebraic equations are solved through known computational methods. Further some numerical examples are shown to illustrate the accuracy and reliability of the proposed approach. Moreover, comparing the methodology with the known technique shows that our approach is more efficient and more convenient.

Keywords: Block pulse functions, operational matrix, fractional integro-differential equation, error analysis; numerical solution.

1 Introduction

In recent years, fractional calculus has attracted many researchers successfully in different disciplines of science and engineering. One of the main advantages of the fractional calculus is that the fractional derivatives provide a superior approach for the description of memory and hereditary properties of various materials and processes [Chen (2007); Bagley and Calico (1991); Rossikhin and Shitikova (1997)]. Differential equations involving fractional order derivatives are used to model a variety of systems, such as the field of viscoelasticity, heat conduction, electrode-electrolyte polarization, electromagnetic waves, diffusion equations and so on[Hilfer (2000); Laroche and Knittel (2005); Calderon, Vinagre and Feliu (2006); Bagley and Torvik (1984)]. Since its tremendous applications in several disciplines, a considerable attention has been given to the exact and the numerical solutions of fractional differential equations and fractional integral equations. Even

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numerical approximation of fractional differentiation of rough functions is not easy as it is an ill-posed problem.

Other than modeling aspects of these differential equations, the solution techniques and their reliability are rather more important. In order to obtain the goal of highly accurate and reliable solutions, several methods have been proposed to solve the fractional order differential and fractional order integral equations [Wang, Meng, Ma, and Wu (2013); Wei, Chen, Li and Yi (2012)]. The most commonly used methods are Variational Iteration Method [Odibat (2010)], Adomian Decomposition Method [EI-Sayed (1998); EI-Kalla (2011)], Generalized Differential Transform Method [Momani and Odibat (2007); Odibat and Momani(2008)], and Wavelet Method [Zhou, Wang, Wang and Liu (2011); Yi and Chen (2012)].

In this paper, the main objective of the present paper is to introduce the block pulse functions operational matrix method to solve the linear and nonlinear fractional integro-differential equations. The method is based on reducing the equation to a system of algebraic equations by expanding the solution as block pulse with unknown coefficients. The main characteristic of an operational method is to convert a integro-differential equation into an algebraic one. It not only simplifies the problem but also speeds up the computation.

2 Definitions of fractional derivatives and integrals

In this section, we introduce some necessary definitions and preliminaries of the fractional calculus theory which will be used in this article [Podlubny (1999)].

Definition 1. The Riemann-Liouville fractional integral operator $J^{\alpha}$of order $\alpha$is given by

$$J^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-T)^{\alpha-1}y(T)dT, \quad \alpha > 0 \tag{1}$$

$$J^0 y(t) = y(t) \tag{2}$$

Its properties as following:
(i) $J^{\alpha}J^{\beta} = J^{\alpha+\beta}$, (ii) $J^{\alpha}J^{\beta} = J^{\beta}J^{\alpha}$, (iii) $J^{\alpha}J^{\beta}y(t) = J^{\beta}J^{\alpha}y(t)$.

Definition 2. The Caputo definition of fractional differential operator is given by

$$D_*^{\alpha}y(t) = \begin{cases} \frac{d^m y(t)}{dt^m}, & \alpha = m \in N; \\ \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{y^{(m)}(T)}{(t-T)^{\alpha-m+1}}dT, & 0 \le m-1 < \alpha < m. \end{cases} \tag{3}$$

The Caputo fractional derivatives of order α is also defined as $D_*^{\alpha}y(t)$=$J^{m-\alpha}D^m y(t)$, where D^m is the usual integer differential operator of order m. The relation between the Riemann- Liouville operator and Caputo operator is given by the following expressions:

$$D_*^{\alpha} J^{\alpha} y(t) = y(t) \tag{4}$$

$$J^{\alpha} D_*^{\alpha} y(t) = y(t) - \sum_{k=0}^{m-1} y^{(k)}(0^+) \frac{t^k}{k!}, \qquad t > 0 \tag{5}$$

3 Brief review of block pulse functions

A set of block pulse functions $B_n(x)$ containing n component functions in the interval $[0,T)$ is given by [Li and Sun (2011)]

$$B_n(x) = [b_0(x), b_1(x), \ldots, b_{n-1}(x)]^T \tag{6}$$

The ith component $b_i(x)$ of the block pulse functions vector $B_n(x)$ is defined as

$$b_i(x) = \begin{cases} 1 & iT/m \le t < (i+1)T/m \\ 0 & otherwise \end{cases} \tag{7}$$

where $i = 0, 1, 2, \ldots, n-1$.

Any functions $y(x)$ of Lebesgue measure may be expanded into an n-term block pulse functions series in $x \in [0,T)$ as

$$y(x) \approx \sum_{i=0}^{n-1} c_i b_i(x) = C^T B_n(x) \tag{8}$$

The constant coefficients $c_i$in Eq.(8) are given by

$$c_i = (1/h) \int_{ih}^{(i+1)h} y(x) dx \tag{9}$$

where $h = T/n$ is the duration of each component block pulse functions along time scale.

One of the important properties of block pulse functions is the disjointness of them, which can directly be obtain from the definition of block pulse functions. Indeed

$$b_i(x) b_j(x) = \begin{cases} b_i(x), & i = j \\ 0, & i \neq j \end{cases} \tag{10}$$

where $i, j = 0, 1, \ldots, n-1$.

The orthogonality of block pulse functions is derived immediately form

$$\int_0^1 b_i(x) b_j(x) dx = h\delta_{ij} \tag{11}$$

where δ_{ij} is the Kronecker delta.

4 Error Analysis

In this section, we analyze the error when a differentiable function $y(x)$is represented in a series of block pulse functions over the interval $I = [0,1)$. We need the following theorem.

Theorem 4.1 Suppose $y(x)$ is continuous in I, is differentiable in $(0,1)$, and there is a number M such that $|y'(x)| \leq M$, for every $x \in I$. Then

$$|y(b) - y(a)| \leq M\,|b-a|\,,$$

for all $a, b \in I$.

Proof. See [Vladimir (2006)].

Now, we assume that $y(x)$ is a differentiable function on I such that $|y'(x)| \leq M$. We define the error between $y(x)$ and its block pulse functions expansion over every subinterval I_i as follows:

$$e_i(x) = c_i - y(x), \quad x \in I_i \tag{12}$$

where $I_i = \left[\frac{i}{n}, \frac{i+1}{n}\right)$.
It can be shown that

$$\|e_i\|^2 = \int_{i/n}^{(i+1)/n} e_i^2(x)dx = \int_{i/n}^{(i+1)/n} (c_i - y(x))^2 dx = \frac{1}{n}(c_i - y(\eta))^2, \quad \eta \in I_i \tag{13}$$

where we used mean value theorem for integral. Using Eq.(9) and the mean value theorem, we have

$$c_i = n\int_{i/n}^{(i+1)/n} y(x)dx = n\frac{1}{n}y(\varsigma) = y(\varsigma), \quad \varsigma \in I_i \tag{14}$$

Substituting Eq.(14) into Eq.(13) and using Theorem 4.1, we have

$$\|e_i\|^2 = \frac{1}{n}(y(\varsigma) - y(\eta))^2 \leq \frac{M^2}{n}\,|\varsigma - \eta|^2 \leq \frac{M^2}{n^3} \tag{15}$$

This leads to

$$\begin{aligned} \|e(x)\|^2 &= \int_0^1 e^2(x)dx = \int_0^1 \left(\sum_{i=0}^{n-1} e_i(x)\right)^2 dx \\ &= \int_0^1 \left(\sum_{i=0}^{n-1} e_i^2(x)\right) dx + 2\sum_{i\leq j}\int_0^1 e_i(x)e_j(x)dx \end{aligned} \tag{16}$$

Since for $i \neq j,\ \ I_i \cap I_j = \emptyset$, then

$$\|e(x)\|^2 = \sum_{i=0}^{n-1} \left(\int_0^1 e_i^2(x)dx \right) = \sum_{i=0}^{n-1} \|e_i\|^2 \tag{17}$$

Substituting Eq.(15) into Eq.(17), we get

$$\|e(x)\|^2 \leq \frac{M^2}{n^2} \tag{18}$$

hence, $\|e(x)\| = O\left(\frac{1}{n}\right)$, where $e(x) = y_n(x) - y(x)$ and $y_n(x) = \sum\limits_{i=0}^{n-1} c_i b_i(x)$.

5 Block pulse operational matrix for fractional calculus

In this part, we may simply introduce the operational matrix of fractional integration of block pulse functions, more detailed introduction can be found in the Ref. [Li and Sun (2011)].

If J^α is fractional integration operator of block pulse functions, we can get:

$$J^\alpha \left(B_n(x)\right) \approx P^\alpha B_n(x) \tag{19}$$

where

$$P^\alpha = h^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{n-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{n-2} \\ 0 & 0 & 1 & \cdots & \xi_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}, \quad k = 1, 2, \ldots, n-1.$$

P^α is called the block pulse operational matrix of fractional integration.

6 The algorithm for finding numerical solution of fractional integro - differential equations

6.1 Linear multi-order fractional integro-differential equations

Consider the linear multi-order fractional integro-differential equations

$$\sum_{i=1}^{r} a_i(x) D_*^{\alpha_i} y(x) = D_*^{\alpha} y(x) + \lambda_1 \int_0^x k_1(x,t) y(t)dt + \lambda_2 \int_0^1 k_2(x,t) y(t)dt + f(x) \tag{20}$$

subject to initial conditions

$$y^{(s)}(0) = 0, \quad s = 0,1,\dots,\lceil\alpha\rceil - 1 \tag{21}$$

where $\alpha > \alpha_1 > \alpha_2 > \cdots \alpha_r$, D_*^{α}denotes the Caputo fractional order derivative of order α, $a_i(x)$ is known function, $\lceil\alpha\rceil$ is the ceiling function, $f(x)$ is input term and $y(x)$ is the output response. $k_1(x,t), k_2(x,t)$ are given functions. λ_1, λ_2 are real constants.

Now we approximate $D_*^{\alpha}y(x)$, $k_1(x,t), k_2(x,t)$ and $f(x)$ in terms of block pulse functions as follows

$$D_*^{\alpha}y(x) \approx C^T B_n(x),\ k_1(x,t) \approx B_n^T(x)K_1 B_n(t), k_2(x,t) \approx B_n^T(x)K_2 B_n(t) \tag{22}$$

and

$$f(x) \approx F^T B_n(x) \tag{23}$$

where $K_1 = [k_{ij}^1]_{n\times n}$, $K_2 = [k_{ij}^2]_{n\times n}$ and $F = [f_0, f_1, \dots f_{n-1}]^T$. A similar approximation scheme is follow for variable coefficient $a_i(x)$ as well.

Let

$$a_i(x) \approx \left(A_n^i\right)^T B_n(x) \tag{24}$$

where A_n^i is known $n \times 1$ column vector.

Now using Eq.(19) and (22) together with above approximation of $a_i(x)$, we obtain

$$D_*^{\alpha_i}y(x) = J^{\alpha-\alpha_i}\left(D_*^{\alpha}y(x)\right) \approx J^{\alpha-\alpha_i}\left(C^T B_n(x)\right) = C^T P^{\alpha-\alpha_i} B_n(x) \tag{25}$$

and

$$y(x) \approx C^T P^{\alpha} B_n(x) \tag{26}$$

Let $E = [e_0, e_1, \dots, e_{n-1}] = C^T P^{\alpha}$, then

$$\begin{aligned}
\int_0^x k_1(x,t)y(t)dt &= \int_0^x B_n^T(x)K_1 B_n(t) B_n^T(t)[C^T P^{\alpha}]^T dt \\
&= B_n^T(x)K_1 \int_0^x B_n(t)B_n^T(t)E^T dt \\
&= B_n^T(x)K_1 \int_0^x diag(E)B_n(t)dt \\
&= B_n^T(x)K_1 diag(E)P^1 B_n(x) = \tilde{Q}^T B_n(x)
\end{aligned} \tag{27}$$

where $\tilde{Q}$ is a n-vector with elements equal to the diagonal entries of the following matrix

$$Q = K_1 diag(E) P^1 \tag{28}$$

and

$$\begin{aligned}
\int_0^1 k_2(x,t)y(t)dt &= \int_0^1 B_n^T(x)K_2B_n(t)B_n^T(t)E^T dt \\
&= B_n^T(x)K_2\int_0^1 B_n(t)B_n^T(t)dtE^T \\
&= \frac{1}{n}B_n^T(x)K_2E^T \\
&= \frac{1}{n}EK_2^TB_n(x)
\end{aligned} \tag{29}$$

Substituting the above equations into Eq.(20), we have

$$\sum_{i=1}^{r}\left(A_n^i\right)^T B_n(x)B_n^T(x)[P^{\alpha-\alpha_i}]^T C{=}C^T B_n(x)+\lambda_1\tilde{Q}^T B_n(x)+\frac{\lambda_2}{n}EK_2^T B_n(x)+F^T B_n(x) \tag{30}$$

Define $[P^{\alpha-\alpha_i}]^T C = [v_0^i, v_1^i, \dots, v_{n-1}^i]^T = [V_n^i]^T$, then Eq.(30) becomes

$$\sum_{i=1}^{r}\left(A_n^i\right)^T diag(V_n^i)B_n(x) = C^T B_n(x)+\lambda_1\tilde{Q}^T B_n(x)+\frac{\lambda_2}{n}EK_2^T B_n(x)+F^T B_n(x) \tag{31}$$

Dispersing Eq.(31), we get

$$\sum_{i=1}^{r}\left(A_n^i\right)^T diag(V_n^i) = C^T+\lambda_1\tilde{Q}^T+\frac{\lambda_2}{n}EK_2^T+F^T \tag{32}$$

which is a linear system of algebraic equations. By solving this system we can obtain the approximation of Eq.(26).

6.2 Nonlinear multi-order fractional integro-differential equations

In this section we deal with nonlinear multi-order fractional integro-differential equation of the form

$$\sum_{i=1}^{r}a_i(x)D_*^{\alpha_i}y(x) = D_*^{\alpha}y(x)+\lambda_1\int_0^x k_1(x,t)[y(t)]^p dt+\lambda_2\int_0^1 k_2(x,t)[y(t)]^q dt+f(x) \tag{33}$$

subject to initial conditions $y^{(s)}(0) = 0, \quad s = 0, 1, \ldots, \lceil \alpha \rceil - 1.$

where $p, q \in N$, and the other parameters and variables are the same as the section 6.1. While dealing with such a situation, the same procedure (as in linear case) of expansion of fractional order derivatives via block pulse functions is adopted with exception at the term containing $[y(t)]^p$, $[y(t)]^q$.

From Eq.(26), we have $y(x) \approx EB_n(x)$ and hence

$$[y(t)]^p \approx [EB_n(t)]^p = [e_0^p, e_1^p, \ldots, e_{n-1}^p]B_n(t) = E_p B_n(t) \tag{34}$$

and

$$[y(t)]^q \approx [EB_n(t)]^q = [e_0^q, e_1^q, \ldots, e_{n-1}^q]B_n(t) = E_q B_n(t) \tag{35}$$

Following the procedure of section 5.1and using the Eq.(34) and Eq.(35), the Eq.(33) is transformed into a nonlinear system of algebraic equations

$$\sum_{i=1}^{r} \left(A_n^i\right)^T diag(V_n^i) = C^T + \lambda_1 \tilde{W}^T + \frac{\lambda_2}{n} E_q K_2^T + F^T \tag{36}$$

where $\tilde{W}$ is a n-vector with elements equal to the diagonal entries of the following matrix

$$W = K_1 diag(E_p) P^1 \tag{37}$$

Solving the system of equations given by Eq.(37), the approximate numerical solution $y(x)$ is obtained. The Eq.(27) can be solved by iterative numerical technique such as Newton's method. Also the Matlab function "fsolve" is available to deal with such a nonlinear system of algebraic equations.

7 Numerical examples

In order to illustrate the effectiveness of the proposed method, we consider numerical examples of linear and nonlinear nature.

Example 7.1. Consider the following linear equation:

$$x^2 D^{1.5} y(x) + x D^{0.5} y(x) = D^{1.7} y(x) + \int_0^x (x-t) y(t) dt + \int_0^1 (x+t) y(t) dt + f(x) \tag{38}$$

with this condition $y'(0) = y(0) = 0$ and

$$\begin{aligned} f(x) = &\left(\frac{\Gamma(3)}{\Gamma(1.5)} + \frac{\Gamma(3)}{\Gamma(2.5)}\right) x^{2.5} + \left(\frac{\Gamma(4)}{\Gamma(2.5)} + \frac{\Gamma(4)}{\Gamma(3.5)}\right) x^{3.5} - \frac{\Gamma(3)}{\Gamma(1.3)} x^{0.3} \\ &- \frac{\Gamma(4)}{\Gamma(2.3)} x^{1.3} - \frac{x^4}{12} - \frac{x^5}{20} - \frac{7x}{12} - \frac{9}{20} \end{aligned},$$

The exact solution of this problem is $y(x) = x^2 + x^3$. **Table 1** shows the approximate solutions and exact solutions for different n. **Fig. 1** shows the absolute errors between the numerical solutions and exact solution.

Table 1: The approximate solution and exact solution for different n.

x	$n = 8$	$n = 16$	$n = 32$	$n = 64$	Exact solution
0	0.000724	0.000112	0.000017	0.000002	0.00000
1/8	0.014822	0.015551	0.016598	0.017166	0.017578
2/8	0.074531	0.076098	0.077320	0.077880	0.078125
3/8	0.192880	0.193159	0.193600	0.193699	0.193359
4/8	0.381498	0.378505	0.377205	0.376375	0.375000
5/8	0.652950	0.644293	0.640080	0.637743	0.634765
6/8	1.020901	1.003194	0.994423	0.989762	0.984375
7/8	1.481340	1.468939	1.452946	1.444649	1.435546

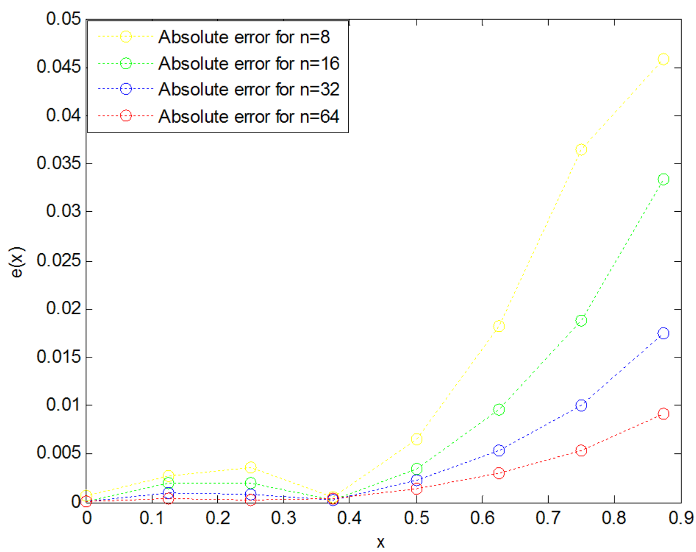

Figure 1: The absolute errors for different n.

From the **Table 1** and **Fig. 1**, we can see clearly that the absolute errors become more and more small when $m$increases.

Example 7.2. Consider this equation:

$$(x^2-1)D^{1.6}y(x)+(x^2+1)D^{1.2}y(x)+x^2D^{0.75}y(x)= \\ D^{2.3}y(x)+\frac{1}{4}\int_0^x (x-t)y(t)dt+\frac{1}{2}\int_0^1 xty(t)dt+f(x) \tag{39}$$

where

$$f(x)=\frac{\Gamma(4.5)}{\Gamma(2.9)}(x^{3.9}-x^{1.9})+\frac{\Gamma(4.5)}{\Gamma(3.3)}(x^{4.3}+x^{2.3})+\frac{\Gamma(4.5)}{\Gamma(3.75)}x^{4.75}-\frac{\Gamma(4.5)}{\Gamma(2.2)}x^{1.2}-\frac{x^{5.5}}{99}-\frac{x}{11}.$$

Such that $y''(0)=y'(0)=y(0)=0$, the exact solution is $y(x)=x^{\frac{7}{2}}$.
The numerical results for $n=8$, 16, 32, 64 are shown in **Figs.2-5**. From the **Figs.2-5**, we can find easily that the numerical solutions are in good agreement with the exact solutions. The absolute errors for different values of $m$are shown in **Table 2**. Through **Table 2**, we can also see that the errors are smaller and smaller when m increases.

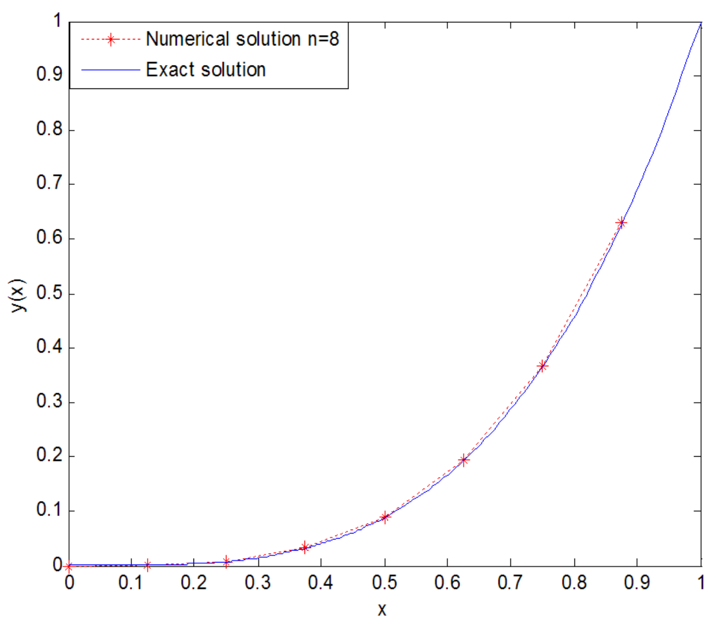

Figure 2: Comparison of Num. sol. and Exa. Sol. of $n=8$.

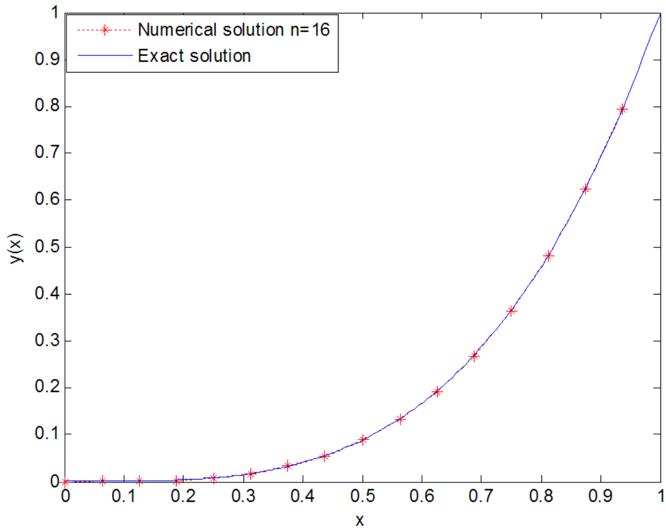

Figure 3: Comparison of Num. sol. and Exa. Sol. of $n = 16$.

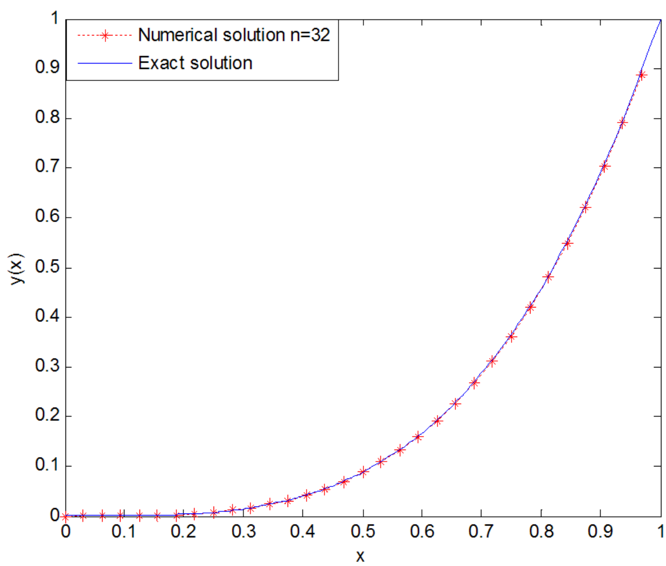

Figure 4: Comparison of Num. sol. and Exa. Sol. of $n = 32$.

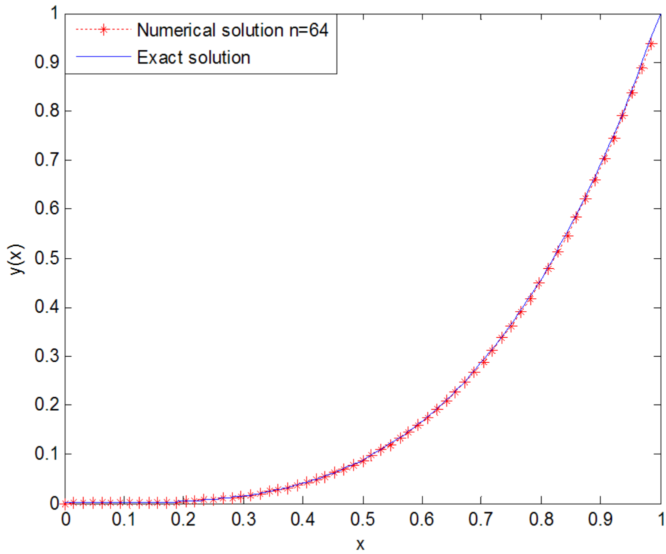

Figure 5: Comparison of Num. sol. and Exa. Sol. of $n = 64$.

Table 2: The absolute errors for different values of n.

x	$n = 8$	$n = 16$	$n = 32$	$n = 64$
0	0	0	0	0
1/8	1.247555e-004	3.866262e-005	5.639161e-006	5.063613e-006
2/8	4.688266e-004	8.920155e-005	3.528902e-005	7.476278e-006
3/8	9.063457e-004	6.093125e-005	2.137330e-005	3.044334e-006
4/8	1.371242e-003	1.363454e-004	6.368124e-005	8.108133e-006
5/8	1.808436e-003	6.109401e-004	1.440001e-004	1.743114e-005
6/8	2.161233e-003	1.507216e-003	2.808616e-004	3.306478e-005
7/8	2.359972e-003	2.036846e-003	5.018022e-004	5.806481e-005

Example 7.3. Consider the following nonlinear equation:

$$\sin x \cdot D^{1.5}y(x) + \cos x \cdot D^{0.8}y(x) =$$
$$D^{2.2}y(x) + \frac{1}{3}\int_0^x (x+t)[y(t)]^2 dt + \frac{1}{4}\int_0^1 (x-t)[y(t)]^3 dt + f(x) \tag{40}$$

such that $y''(0) = y'(0) = y(0) = 0$, the exact solution of the equation is $y(x) = x^3$

and

$$f(x) = \frac{\Gamma(4)}{\Gamma(2.5)}x^{1.5}\sin x + \frac{\Gamma(4)}{\Gamma(3.2)}x^{2.2}\cos x - \frac{\Gamma(4)}{\Gamma(1.8)}x^{0.8} - \frac{5x^8}{56} - \frac{x}{40} + \frac{1}{44}.$$

Fig.6-9 show the numerical solutions and exact solution for $n = 16,\ 32,\ 64, 128$.

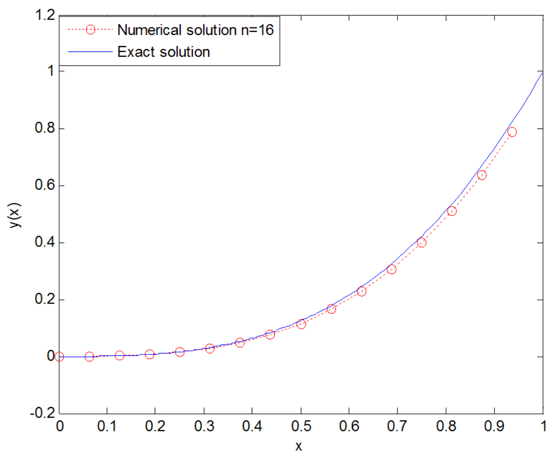

Figure 6: Comparison of Num. sol. and Exa. Sol. of $n = 16$for Example 3.

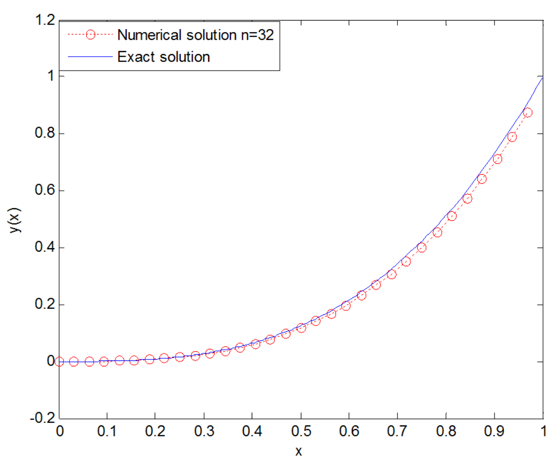

Figure 7: Comparison of Num. sol. and Exa. Sol. of $n = 32$for Example 3.

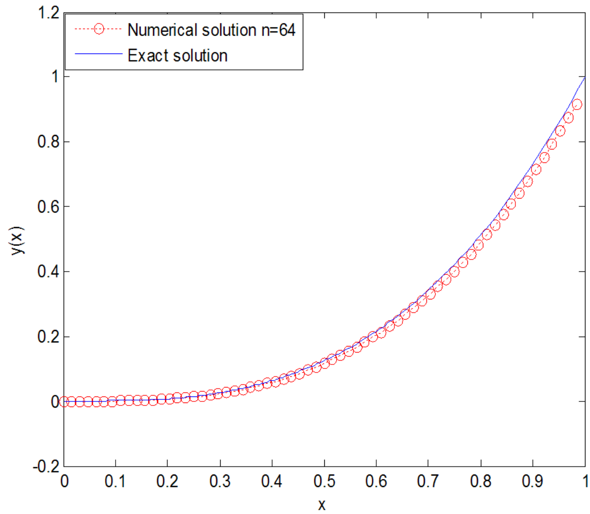

Figure 8: Comparison of Num. sol. and Exa. Sol. of $n = 64$for Example 3.

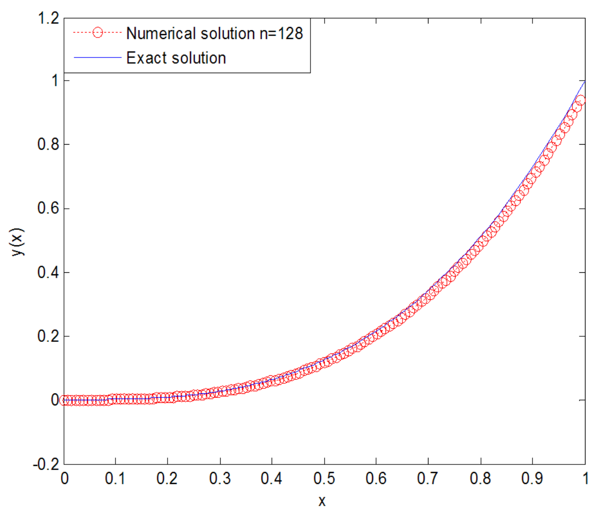

Figure 9: Comparison of Num. sol. and Exa. Sol. of $n = 128$for Example 3.

We can see that the numerical solutions are more and more close to the exact solution with the value of $n$becomes large by taking a closer look at **Fig. 6-9**.

Example 7.4. Consider this equation:

$$(x^2+x)D^{\alpha}y(x) = D^{\alpha+1}y(x) + \int_0^x (e^t+1)[y(t)]^2dt + \int_0^1 xt[y(t)]^2dt + f(x) \qquad (41)$$

where $f(x) = (x^2+x)(e^x-1) - e^x - \frac{(e^x-x-1)^3}{3} - x(\frac{e^2}{4} - 2e + \frac{11}{3})$. With initial conditions $y'(0) = y(0) = 0$. The exact solution of the problem for $\alpha = 1$ is $y(x) = e^x - x - 1$.

The comparison of numerical results for $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 1$ and the exact solution for $\alpha = 1$ are shown in **Fig. 10**. From the **Fig. 10**, we can see clearly that the numerical solutions are in very good agreement with the exact solution when $\alpha = 1$. It is evident from the **Fig. 10** that, as α close to 1, the numerical solution by the block pulse functions, converge to the exact solution.

On one hand, because of the construction of block pulse functions is very simple, compared with the Ref. [Chen, Sun, Li, Fu (2013)], taking advantage of above method can avoid constructing the operational matrix of block pulse functions. One the other hand, since the block pulse operational matrix for the fractional integration is upper triangular matrix, compared with the second kind Chebyshev wavelets operational matrix for the fractional integration in Ref. [Chen, Sun, Li, Fu (2013)], it greatly reduces the memory space.

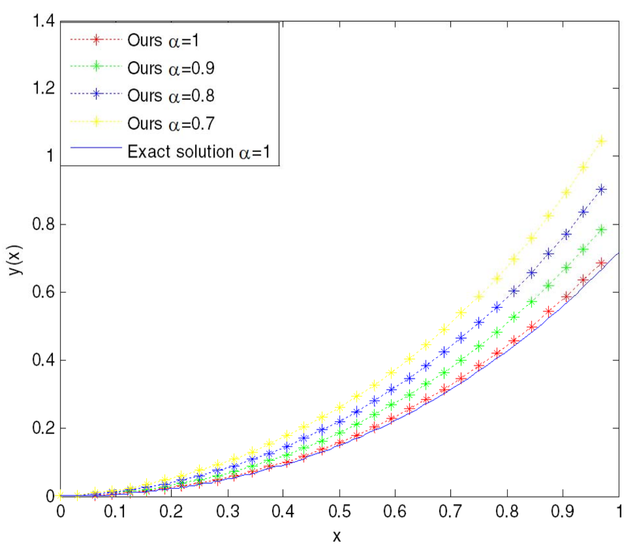

Figure 10: Numerical solution and exact solution of $\alpha = 1$.

8 Conclusion

In the present manuscript, the application and scope of the generalized block pulse functions have been extended to fractional order linear and nonlinear integro-differential equations successfully. We construct fractional orders generalized block pulse functions operational matrix of integration and use this to solve the fractional linear and nonlinear integro-differential equations numerically. By solving the linear and nonlinear system, numerical solutions are obtained. The convergence analysis of block pulse functions is proposed. The numerical results show that the approximation is in very good coincidence with the exact solution.

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