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The Far-field Green's Integral in Stokes Flow from the Boundary Integral Formulation

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Abstract: In boundary integral methods for Stokes flow, the far-field Green's integral is usually taken to be zero without proof. However, this is not obviously the case, the reason being that Stokes flow is a near-field approximation and breaks down in the far-field. Here, we show that it is zero as expected by matching it to a far-field Green's integral in Oseen flow. Hence, there are similarities to the matched asymptotic procedure matching a near-field Stokes flow to a far-field Oseen flow, except in this case a different and new procedure is required to deal with the Green's integrals. In particular, the velocity is represented in the near-field by an integral distribution of stokeslets, and in the far-field by an integral distribution of oseenlets, and the two integral distributions are matched together by equating the stokeslets with the oseenlets in the matching region. A boundary integral representation is then obtained which holds throughout the whole flow region, enabling the velocity in the boundary integral scheme to be determined everywhere in the flow region.

Keywords: Stokes flow, Green's integral, Boundary Integral, matching preparation.

1 Introduction

The Boundary Element technique for Stokes flow is in widespread use for a large variety of low-Reynolds number flow applications, originating from initial boundary integral formulations, in particular by Pozrikidis (1992) and more recently by Sellier (2012). Benefits of the method include the compact format of the numerical scheme for fast calculations, straightforward reductions for flow past special bodies such as slender and thin, and the way the method implicitly deals with the exterior domain. However, in these formulations it is assumed that the far-field Green's integral is zero. This is not trivial to show, and in fact these integrals instead become singular. This difficulty is well-known in the context of calculating forces

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and moments and overcome by using matched asymptotics, in particular to an Oseen flow far-field [Kaplun and Lagerstrom (1957); Proudman and Pearson (1957); Imai (1951)]. However, this difficulty has not been addressed within the Boundary Element technique for Stokes flow, and it is necessary to show that the far-field Green's integrals can be legitimately taken to be zero if the method is to be put on a sound mathematical footing, and it is the aim of this paper to do this. This requires establishing a relation between the far-field Green's integral in Stokes flow for the inner region and the near-field Green's integral in Oseen flow for the outer region. This entails finding the order of the inertial term that is omitted in Stokes flow, and also finding the order of the error between the Green's function for Stokes flow and Oseen flow on the matched boundary. As it is known that the far-field Green's integral in Oseen flow is zero from the work of Fishwick and Chadwick (2006), then we can deduce that it is reasonable to omit the Stokes far-field Green's integral in numerical Boundary Element schemes, and give the order of the error.

2 Formulation

The problem is formulated by establishing two flow regions, and inner flow where Stokes flow is valid, and an outer flow where Oseen flow is valid. The two flows match on a common boundary, see Fig. 1.

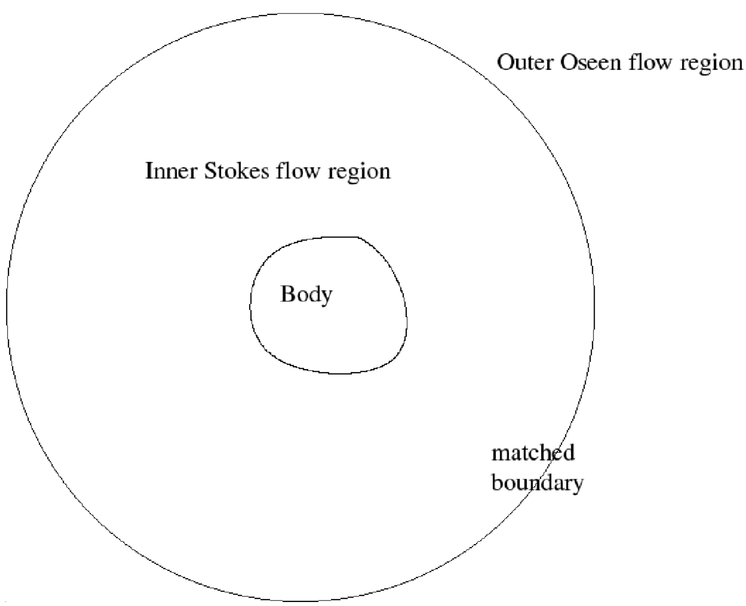

Figure 1: Flow regions with matching boundary

The problem therefore assumes that for a co-ordinate system fixed on the body, then a uniform flow field of size U prevails far from the body. Letting this lie in the x_1 direction gives an outer Oseen flow region governed by the Oseen equation

$$\rho U \frac{\partial u_i}{\partial x_1} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - f_i \tag{1}$$

where u_i is the fluid velocity, p is the pressure, ρ is the fluid density and μ the viscosity. For the sake of forming the Green's integral, it is also assumed that an external force $-f_i$ is applied, so equivalently the fluid exerts a force f_i at that point. All vectors are represented in Cartesian co-ordinates (x_1, x_2, x_3)in the form v_i where the vector is given by (v_1, v_2, v_3). Also the Einstein summation convection applies, so $v_i w_i = v_1 w_1 + v_2 w_2 + v_3 w_3$. Near the body, the flow is governed by an inner Stokes flow region such that,

$$0 = -\frac{\partial p^S}{\partial x_i} + \mu \frac{\partial^2 u_i^S}{\partial x_j \partial x_j} - f_i \quad , \tag{2}$$

where $\mathbf{u}^S$ is the Stokes velocity and p^S is the Stokes pressure, and there is a common boundary on which the inner and outer region can be matched. So on the matched boundary the inertia term $\rho U \frac{\partial u_i}{\partial x_1}$ is negligible compared to the other terms. Applying dimensionless analysis to Eq.1, it is seen this occurs when $Re\frac{L}{l}$ is small, where $Re = \frac{\rho U l}{\mu}$ is the Reynolds number, $L$is the length dimension of the matched region and l is a typical length dimension of the body. This means that the inertia term is of order $Re\frac{L}{l}$ smaller than the others, so this is the order of the approximation when it is omitted. Also, it is noted that sufficiently far from the body L can be sufficiently large that the inertia term cannot be neglected, and Oseen flow must be considered.

3 Green's functions

Here, we shall give the Green's functions for Oseen flow called oseenlets in both 2- and 3-dimensions. Then, in the limit of small Reynolds number it is shown that these can be approximated to the stokeslets, and the order of this approximation is given.

3.1 Oseenlets

The oseenlets were first given by Oseen (1927) and satisfy Eq.1 with $f_i^{(j)} = \delta_{ij}\delta(x)$ for the j- oseenlet where $\delta_{ij}(= 1$ when $i = j$, and is zero otherwise) is Kronecker delta and $\delta(x)$ is the Dirac delta function such that the volume integral is 1 across the origin point, but zero otherwise.

In 2-dimensions the drag oseenlet is

$$u_i^{(1)} = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_i}(\ln r + e^{kx_1}K_0(kr)) - 2ke^{kx_1}K_0(kr)\delta_{i1}\right],\ \ p^{(1)} = -\frac{1}{2\pi}\frac{\partial}{\partial x_1}(\ln r) \tag{3}$$

where r is the 2-dimensional radius, K_0 is the modified Bessel function of zero order, and $k = \frac{\rho U}{2\mu}$. The lift oseenlet in 2-dimensions is

$$u_i^{(2)} = \frac{1}{2\pi\rho U}\varepsilon_{ij3}\frac{\partial}{\partial x_j}(\ln r + e^{kx_1}K_0(kr)),\ \ p^{(2)} = -\frac{1}{2\pi}\frac{\partial}{\partial x_2}(\ln r) \tag{4}$$

where $\varepsilon_{ijk} = 1$ when $(i,j,k) = (1,2,3)$, $(2,3,1)$ and $(3,1,2)$,
$\varepsilon_{ijk} = -1$ when $(i,j,k) = (1,3,2)$, $(2,1,3)$ and $(3,2,1)$, and zero otherwise.
In 3-dimensions the oseenlets are given by

$$u_i^{(m)} = \frac{1}{4\pi\rho U}\left[\left(\frac{\partial\phi^{(m)}}{\partial x_i} - \frac{\partial\chi^{(m)}}{\partial x_i}\right) + 2k\delta_{im}\chi^{(1)}\right],\ \ p^{(m)} = -\rho U\frac{\partial\varphi^{(m)}}{\partial x_1} = \frac{1}{4\pi}\frac{\partial}{\partial x_k}\frac{1}{R}, \tag{5}$$

where R is the 3-dimensional radius, $\phi^{(m)} = \frac{\partial}{\partial x_m}\ln(R - x_1)$, and $\chi^{(m)} = e^{-k(R-x_1)}\frac{\partial}{\partial x_m}$ $\ln(R - x_1)$.

3.2 Stokeslets

The stokeslets can be obtained from the oseenlets by considering $kr \to 0$ for 2-dimensional flow, and $kR \to 0$ for 3-dimensional flow. In 2-dimensional flow, noting that $e^{kx_1} = 1 + kx_1 + O(k^2r^2)$, and $K_0(kr) = -\ln r + O(r^2 \ln r)$ where $O(...)$ means 'of the order of', then Eq.3 becomes

$$u_i^{(1)} = \frac{1}{4\pi\mu}\left[\delta_{i1}\ln r - \frac{x_1x_i}{r^2}\right](1 + O(kr)),\ \ p^{(1)} = -\frac{1}{2\pi}\frac{x_1}{r^2} \quad , \tag{6}$$

and Eq.4 becomes

$$u_i^{(2)} = \frac{1}{4\pi\mu}\left[\delta_{i2}\ln r - \frac{x_2x_i}{r^2}\right](1 + O(kr)) + C_i,\ \ p^{(2)} = -\frac{1}{2\pi}\frac{x_2}{r^2} \tag{7}$$

where $C_i = \frac{\delta_{i2}}{4\pi\mu}$. Combining Eq.6 and Eq.7 gives

$$u_i^{(m)} = \frac{1}{4\pi\mu}\left[\delta_{im}\ln r - \frac{x_mx_i}{r^2}\right](1 + O(kr)) + C_i^{(m)},\ \ p^{(m)} = -\frac{1}{2\pi}\frac{x_m}{r^2} \quad , \tag{8}$$

where $C_i^{(m)} = \frac{\delta_{i2}\delta_{m2}}{4\pi\mu}$. So up to the order kr and a constant, the 2-dimensional stokeslets

$$u_i^{(m)} \approx u_i^{(m)S} = \frac{1}{4\pi\mu}\left[\delta_{im}\ln r - \frac{x_m x_i}{r^2}\right] \quad , \quad p^{(m)} = p^{(m)S} = -\frac{1}{2\pi}\frac{x_m}{r^2} \quad , \tag{9}$$

are obtained from Eq.8.

For 3-dimensional flow, we consider $kR \to 0$ in Eq.5. This gives $\frac{\partial}{\partial x_i}\left(\phi^{(m)} - \chi^{(m)}\right) = k\left(\frac{\delta_{im}}{R} - \frac{x_i x_m}{R^3}\right)(1+O(kR))$ and $\chi^{(1)} = \left(-\frac{1}{R} + k - \frac{kx_1}{R}\right)(1+O(kR))$, and so

$$u_i^{(m)} = -\frac{1}{8\pi\mu}\left[\frac{\delta_{im}}{R} + \frac{x_i x_m}{R^3}\right](1+O(kR)). \tag{10}$$

So up to the order kR and a constant, the 3-dimensional stokeslets

$$u_i^{(m)} \approx u_i^{(m)S} = -\frac{1}{8\pi\mu}\left[\frac{\delta_{im}}{R} + \frac{x_i x_m}{R^3}\right], \quad . \quad p^{(m)} = p^{(m)S} = -\frac{1}{2\pi}\frac{x_m}{r^2} \tag{11}$$

are obtained.

4 Green's integral formulation

4.1 Outer region

The Green's integral formulation for Oseen flow can be obtained by following Oseen (1927) by considering the integral

$$\int_{\Sigma} \begin{array}{l} \left(-\rho U \frac{\partial u_i^{(m)}(\mathbf{z})}{\partial y_1} - \frac{\partial p^{(m)}(\mathbf{z})}{\partial y_i} - \mu \frac{\partial^2 u_i^{(m)}(\mathbf{z})}{\partial y_j \partial y_j} + f_i^{(m)}(\mathbf{z})\right) u_i(\mathbf{y}) \\ + \left(-\rho U \frac{\partial u_i(\mathbf{y})}{\partial y_1} - \frac{\partial p(\mathbf{y})}{\partial y_i} + \mu \frac{\partial^2 u_i(\mathbf{y})}{\partial y_j \partial y_j} - f_i(\mathbf{y})\right) u_i^{(m)}(\mathbf{z}) \end{array} d\Sigma = 0$$

where $\mathbf{y}$ is a vector position of the exterior domain integrated space Σ (which is an area integral in 2-dimensional space and a volume integral in 3-dimensional space, and $\mathbf{z} = \mathbf{x} - \mathbf{y}$, so the differential equation for the Green's functions satisfies the conjugate Oseen equation since $\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$. Rearranging this then gives

$$\int_{\Sigma} \left[\begin{array}{l} -\rho U \frac{\partial}{\partial y_1}\left(u_i^{(m)}(\mathbf{z}) u_i(\mathbf{y})\right) - \frac{\partial}{\partial y_i}\left(p^{(m)}(\mathbf{z}) u_i(\mathbf{y}) + p(\mathbf{y}) u_i^{(m)}(\mathbf{z})\right) \\ -\mu \frac{\partial}{\partial y_j}\left(\frac{\partial u_i^{(m)}(\mathbf{z})}{\partial y_j} u_i(\mathbf{y})\right) + \mu \frac{\partial}{\partial y_j}\left(\frac{\partial u_i(\mathbf{y})}{\partial y_j} u_i^{(m)}(\mathbf{z})\right) \end{array} \right] d\Sigma$$
$$= \int_{\Sigma} \left(-f_i^{(m)}(\mathbf{z}) u_i(\mathbf{y}) + f_i(\mathbf{y}) u_i^{(m)}(\mathbf{z})\right) d\Sigma$$

since $\frac{\partial u_i(\mathbf{y})}{\partial y_i} = \frac{\partial u_i^{(m)}}{\partial y_i} = 0$ from the continuity equation, and upon expanding the two terms with the factor $\mu \frac{\partial u_i^{(m)}}{\partial y_j} \frac{\partial u_i}{\partial y_j}$ cancel. Applying the divergence theorem then gives the Oseen Green's integral representation for the velocity

$$u_m(\mathbf{x}) = -\int_{\partial\Sigma} \left[\begin{array}{l} \rho U u_i^{(m)}(\mathbf{z}) u_i(\mathbf{y}) n_1 + \left(p^{(m)}(\mathbf{z}) u_i(\mathbf{y}) + p(\mathbf{y}) u_i^{(m)}(\mathbf{z}) \right) n_i \\ +\mu \left(\frac{\partial u_i^{(m)}(\mathbf{z})}{\partial y_j} u_i(\mathbf{y}) - \frac{\partial u_i(\mathbf{y})}{\partial y_j} u_i^{(m)}(\mathbf{z}) \right) n_j \end{array} \right] d\partial\Sigma \quad (12)$$

since from Fishwick and Chadwick (2006) the far-field integral bounding the exterior domain Σ in the Oseen representation is zero, see Fig. 2, where $\partial\Sigma$ bounds Σ, so is a curve in 2-dimensions and a surface in 3-dimensions, and

$$\int_{\Sigma} f_i^{(m)}(\mathbf{z}) u_i(\mathbf{y}) d\Sigma = \int_{\Sigma} \delta_{ij} \delta(\mathbf{z}) u_i(\mathbf{y}) d\Sigma = u_m(\mathbf{x}).$$

It is also noted that the normal n_i pointing out exterior to the non-zero internal $\partial\Sigma$ is a normal that points into Σ, see Fig. 2.

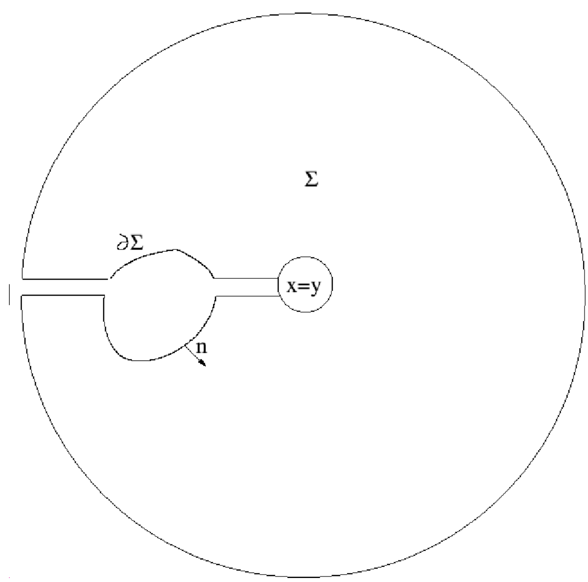

Figure 2: The Green's integral

4.2 Inner region

The same approach can be applied to give the Green's integral representation for the inner Stokes flow as

$$u_m^S(\mathbf{x}) = -\int\limits_{\partial\Sigma} \left[\begin{array}{l} \left(p^{(m)S}(\mathbf{z})u_i^S(\mathbf{y}) + p^S(\mathbf{y})u_i^{(m)S}(\mathbf{z})\right) n_i + \\ \mu \left(\frac{\partial u_i^{(m)S}(\mathbf{z})}{\partial y_j} u_i^S(\mathbf{y}) - \frac{\partial u_i^S(\mathbf{y})}{\partial y_j} u_i^{(m)S}(\mathbf{z}) \right) n_j \end{array} \right] d\partial\Sigma \qquad (13)$$

5 Matching

Eq.12 is matched with Eq.13 on the matching boundary between the inner and outer regions. This matching introduces an approximation error into the formulation, which is given next.

The inertia term in the integrand of Eq.12 is of order $Re\frac{L}{l}$ smaller than the other terms which gives an approximation $\left(1 + O(Re\frac{L}{l})\right)$.

In 2-dimensions, the constant term $C_i^{(m)}$ gives the leading order approximation to the velocity oseenlet $\left(1 + O(\frac{1}{\ln kr})\right) = \left(1 + O(\frac{1}{\ln Re\frac{L}{l}})\right)$ on the matching boundary where $r = O(L)$. Therefore, Eq.12 is given as

$$u_m(\mathbf{x}) = -\int\limits_{\partial\Sigma} \left[\begin{array}{l} \left(p^{(m)S}(\mathbf{z})u_i^S(\mathbf{y}) + p^S(\mathbf{y})u_i^{(m)S}(\mathbf{z})\right) n_i \\ +\mu \left(\frac{\partial u_i^{(m)S}(\mathbf{z})}{\partial y_j} u_i^S(\mathbf{y}) - \frac{\partial u_i^S(\mathbf{y})}{\partial y_j} u_i^{(m)S}(\mathbf{z}) \right) n_j \end{array} \right] d\partial\Sigma \left(1 + O(\frac{1}{\ln Re\frac{L}{l}})\right). \qquad (14)$$

In 3-dimensional flow, this is

$$u_m(\mathbf{x}) = -\int\limits_{\partial\Sigma} \left[\begin{array}{l} \left(p^{(m)S}(\mathbf{z})u_i^S(\mathbf{y}) + p^S(\mathbf{y})u_i^{(m)S}(\mathbf{z})\right) n_i \\ +\mu \left(\frac{\partial u_i^{(m)S}(\mathbf{z})}{\partial y_j} u_i^S(\mathbf{y}) - \frac{\partial u_i^S(\mathbf{y})}{\partial y_j} u_i^{(m)S}(\mathbf{z}) \right) n_j \end{array} \right] d\partial\Sigma \left(1 + O(Re\frac{L}{l})\right). \qquad (15)$$

6 Summary

The order of the approximation of the matching is given by Eq.14 for 2-dimensional flow and Eq.15 for 3-dimensional flow. To reduce this error as much as possible, the matching boundary can be taken to be of order $L = l$ where l is the body dimension. The error in the approximations then becomes $O\left(\frac{1}{\ln Re}\right)$ for 2-dimensional flow and $O(Re)$ in 3-dimensional flow. This also means that a model where Oseen flow is assumed throughout the domain will be a good approximation for low Reynolds number flow, even though the Oseen flow linearization assumption breaks down on the body surface, because close to the body the Oseen flow reduces to Stokes flow to the approximations given. This gives an explanation why Oseen flow models give good approximations to low Reynolds number flow even when this is assumed close to the body.

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