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Operational Matrix Method for Solving Variable Order Fractional Integro-differential Equations

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Abstract: In this paper, operational matrix method based upon the Bernstein polynomials is proposed to solve the variable order fractional integro-differential equations in the Caputo derivative sense. We derive the Bernstein polynomials operational matrix of fractional order integration and introduce the product operational matrix of Bernstein polynomials. A truncated the Bernstein polynomials series together with the polynomials operational matrix are utilized to reduce the variable order fractional integro-differential equations to a system of algebraic equations. Only a small number of Bernstein polynomials are needed to obtain a satisfactory result. Some examples are included to demonstrate the validity and applicability of the method.

Keywords: Bernstein polynomials, variable order fractional, operational matrix, numerical solution.

1 Introduction

In recent years, the research of fractional calculus has attracted much attention[Wei, Chen, Li and Yi (2012); Zhou, Wang, Wang and Liu (2011); Yi and Chen (2012)] and successfully introduced to a variety of engineering applications, such as chaos systems [Li and Peng (2004)], viscous-elasticity [Schiessel, Metzler and Blumen (1995)], control system design [Chen and Moore (2002)] and anomalous diffusion processes [Bao (2003)]. Generally speaking, the notion of fractional calculus is to enlarge integer order to fractional order in numerical representations. To characterize anomalous diffusion phenomena, constant order fractional diffusion equations are introduced and have received tremendous success. However, it has been found that the constant order fractional diffusion equations are not capable of characterizing some complex diffusion processes. To solve this problem, the variable order fractional derivative and variable order fractional diffusion equation models have been suggested for use [Sun, Chen, Sheng, Chen (2010); Chen, Sun, Zhang (2010)].

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The work of variable order operator can be traced to Samko et al. by introducing the variable order integration and Riemann-Liouville derivative in 1993 [Samko and Ross (1993); Samko (1995)]. It has been recognized as a powerful modeling approach in the fields of viscoelasticity viscous fluid, viscoelastic deformation anomalous diffusion, etc.

During the last decade, the variable order fractional differential equations have been solved by means of the numerical methods such as the explicit difference scheme, the implicit difference scheme, the Crank-Nicholson difference scheme and so on [Lin, Liu and Anh (2009); Chen, Liu, Anh (2010); Chen, Liu, Anh (2011); Zhuang, Liu and Anh (2009); Soon, Coimbra and Kobayashi (2005); Coimbra (2003)]. On the other hand, Bhatti and Bracken [Bhatti and Bracken (2007)] solved the differential equations by using the Galerkin method based on Bernstein polynomials basis. Yousefi and Behroozifar [Yousefi and Behroozifar (2010)] presented an operational matrix method based on Bernstein polynomials for the differential equations. Mandal and Bhattacharya [Mandal, and Bhattacharya (2007)] applied Bernstein polynomials to solve the numerical solution of some classes of integral equations. Chen and Yi et al. [Chen, Yi, Chen, Yu (2012)] used Bernstein polynomials method to solve a class of fractional convection- diffusion equation with variable coefficients. In this study, our purpose is to propose a method based on Bernstein polynomials to solve the variable order fractional integro-differential equations. The variable order fractional derivative is considered in the Caputo sense.

2 Definitions and properties of variable order operator

There are several definitions of the variable order differential operator [Coimbra (2003)]. Here we adopt the definition of the variable order differential operator suggested by Coimbra

$$D_t^{\alpha(t)} y(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_{0^+}^{t} \frac{y'(\tau)}{(t-\tau)^{\alpha(t)}} d\tau + \frac{(f(0^+) - f(0^-))t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}, \ 0 < \alpha(t) < 1 \tag{1}$$

if $\alpha(t)$ is a constant, it can be reduced to the constant order Caputo definition. We assume the property of function $y(t)$ at $t = 0$ is good enough, then we can state the following Caputo type definition

$$D_t^{\alpha(t)} y(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_{0}^{t} \frac{y'(\tau)}{(t-\tau)^{\alpha(t)}} d\tau, \quad 0 < \alpha(t) < 1 \tag{2}$$

The definition of variable order integration proposed by Samko is presented as below

$$I_t^{\alpha(t)}y(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1}y(\tau)d\tau, \quad Re(\alpha(t)) > 0 \tag{3}$$

Then we present following properties for the operator $I_t^{\alpha(t)}(\cdot)$ which will be used in this paper.

Property 1: $I_t^{\alpha(t)}(t^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha(t)+\beta+1)}t^{\alpha(t)+\beta}$.

Proof. By using the definition of the operator $I_t^{\alpha(t)}(\cdot)$, we can get

$$I_t^{\alpha(t)}(t^\beta) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1}\tau^\beta d\tau,$$

Let $\tau = \varepsilon t$, according to the definition of beta function, we have

$$\begin{aligned} I_t^{\alpha(t)}(t^\beta) &= \frac{t^{\alpha(t)+\beta}}{\Gamma(\alpha(t))}\int_0^1 (1-\varepsilon)^{\alpha(t)-1}\varepsilon^\beta d\varepsilon = \frac{t^{\alpha(t)+\beta}}{\Gamma(\alpha(t))}B(\beta+1,\alpha(t)) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha(t)+\beta+1)}t^{\alpha(t)+\beta} \end{aligned},$$

where B is the beta function which is defined as follows

$$B(m,n) = \int_0^1 \tau^{m-1}(1-\tau)^{n-1}d\tau, \quad Re(m) > 0,\ Re(n) > 0.$$

Property 2: $I_t^{\alpha(t)}\left(D_t^{\alpha(t)}y(t)\right) = y(t) - y(0)$.

Proof. By using the property of Caputo type definition, we can obtain

$$I_t^{\alpha(t)}\left(D_t^{\alpha(t)}y(t)\right) = I_t^{\alpha(t)}\left(I_t^{1-\alpha(t)}y'(t)\right) = \int_0^t y'(\tau)d\tau = y(t) - y(0).$$

3 Bernstein polynomials and their some properties

3.1 The definition of Bernstein polynomials basis [Maleknejad, Basirat, and Hashe-mizadeh]

The Bernstein basis polynomials of degree $n$are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i(1-t)^{n-i},\ t \in [0,1] \tag{4}$$

By using the binomial expansion of $(1-t)^{n-i}$

$$(1-t)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \begin{pmatrix} n-i \\ k \end{pmatrix} t^k \tag{5}$$

They can be written as

$$B_{i,n}(t) = \sum_{k=0}^{n-i} (-1)^k \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ k \end{pmatrix} t^{i+k}, \ \ t \in [0,1] \tag{6}$$

Also, the Bernstein basis polynomials of degree $n$in $[0,R]$are given as follows

$$B_{k,n}(t) = \begin{pmatrix} n \\ k \end{pmatrix} \frac{t^k (R-t)^{n-k}}{R^n} \tag{7}$$

By substituting the binomial expansion

$$(R-t)^{n-k} = \sum_{i=0}^{n-k} (-1)^i \begin{pmatrix} n-k \\ i \end{pmatrix} R^{n-k-i} t^i.$$

Then we have the formula

$$B_{k,n}(t) = \sum_{i=0}^{n-k} (-1)^i \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} n-k \\ i \end{pmatrix} \frac{t^{k+i}}{R^{k+i}}, \quad t \in [0,R] \tag{8}$$

The Bernstein basis polynomials given by Eq.(6) can be expressed in the matrix form

$$\boldsymbol{\Phi}(t) = [B_{0,n}(t), B_{1,n}(t), \cdots, B_{n,n}(t)]^T = \boldsymbol{A}_n \boldsymbol{\Delta}_n(t) \tag{9}$$

where

$$\boldsymbol{A}_n = \begin{bmatrix} (-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{0}\binom{n-0}{1} & \cdots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ \ddots & & & \vdots \\ 0 & (-1)^0 \binom{n}{i} & \cdots & (-1)^{n-i}\binom{n}{i}\binom{n-i}{n-i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (-1)^0 \binom{n}{n} \end{bmatrix} \tag{10}$$

$$\boldsymbol{\Delta}_n(t) = [1, t, \cdots, t^n]^T \tag{11}$$

3.2 Function approximation

A function $y(t) \in L^2(0,1)$ can be expressed in terms of the Bernstein basis. In practice, only the first $(n+1)$ term of Bernstein polynomials are considered. Hence

$$y(t) \cong \sum_{i=0}^{n} c_i B_{i,n}(t) = \boldsymbol{c}^T \boldsymbol{\Phi}(t) \tag{12}$$

where $\boldsymbol{c} = [c_0, c_1, \cdots, c_n]^T$, c_i $(i = 0,1,2,\cdots,n)$ are called Bernstein coefficients. We can obtain them by

$$\boldsymbol{c} = \boldsymbol{Q}^{-1}(f, \boldsymbol{\Phi}(t)) \tag{13}$$

where

$$\boldsymbol{Q} = \int_0^1 \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^T(t)dt = \int_0^1 (\boldsymbol{A}_n\boldsymbol{\Delta}_n(t))(\boldsymbol{A}_n\boldsymbol{\Delta}_n(t))^T dt = \boldsymbol{A}_n\boldsymbol{H}\boldsymbol{A}_n^T \tag{14}$$

and $\boldsymbol{H} = \begin{bmatrix} 1 & 1/2 & \cdots & 1/n+1 \\ 1/2 & 1/3 & \cdots & 1/n+2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/n+1 & 1/n+2 & \cdots & 1/2n+1 \end{bmatrix}$.

We can also approximate the function $u(x,t)$as following:

$$u(x,t) \cong \sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij} B_{i,n}(x) B_{j,n}(t) = \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{\Phi}(t) \tag{15}$$

where $\boldsymbol{U} = [u_{ij}]_{(n+1)\times(n+1)}$.

3.3 Convergence analysis

Suppose that the function $f : [x_0, 1] \to R$ is $m+1$ times continuously differentiable, $f \in C^{m+1}[0,1]$, and $Y = Span\{B_{0,n}, B_{1,n}, B_{2,n} \cdots, B_{n,n}\}$. If $c^T\Phi(x)$ is the best approximation of f out of Y, then the mean error bound is presented as follows:

$$\left\| f - \boldsymbol{c}^T\boldsymbol{\Phi} \right\|_2 \leq \frac{MS^{m+1}}{(m+1)!} \tag{16}$$

where $M = \max\limits_{x\in[0,1]} \left| f^{(m+1)}(x) \right|$, $S = \max\{1-x_0, x_0\}$.

Proof. Considering the Taylor polynomials, we have

$$f_1(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} \cdots + f^{(m)}(x_0)\frac{(x-x_0)^m}{m!} \tag{17}$$

which we know

$$|f(x)-f_1(x)| \le \left|f^{(m+1)}(\varepsilon)\right| \frac{(x-x_0)^{m+1}}{(m+1)!} \tag{18}$$

where$\varepsilon \in (0,1)$. Since $c^T\Phi(x)$is the best approximation of$f$, then we get

$$\begin{aligned}
\left\|f-\boldsymbol{c}^T\boldsymbol{\Phi}\right\|_2^2 \le \|f-f_1\|_2^2 &= \int_0^1 (f(x)-f_1(x))^2 dx \\
&\le \int_0^1 \left(\left|f^{(m+1)}(\varepsilon)\right| \frac{(x-x_0)^{m+1}}{(m+1)!}\right)^2 dx \\
&\le \frac{M^2}{[(m+1)!]^2}\int_0^1 (x-x_0)^{2m+2}dx \\
&\le \frac{M^2 S^{2m+2}}{[(m+1)!]^2}
\end{aligned},$$

Therefore

$$\left\|f-\boldsymbol{c}^T\boldsymbol{\Phi}\right\|_2 \le \frac{MS^{m+1}}{(m+1)!}.$$

4 Operational matrix of the fractional integration

Now, we derive the Bernstein polynomials operational matrix of fractional order integration. Let

$$I_t^{\alpha(t)}(\boldsymbol{\Phi}(t)) \cong \boldsymbol{P}^{\alpha(t)}\boldsymbol{\Phi}(t) \tag{19}$$

where matrix $P^{\alpha(t)}$ is called Bernstein polynomials operational matrix of fractional order integration.

For this purpose, we use Eq.(9) and the property 1, as following

$$I_t^{\alpha(t)}(\boldsymbol{\Phi}(t)) = I_t^{\alpha(t)}(\boldsymbol{A}_n\boldsymbol{\Delta}_n(t)) = \boldsymbol{A}_n I_t^{\alpha(t)}(\boldsymbol{\Delta}_n(t)) = \boldsymbol{A}_n\boldsymbol{\Psi}(t)\boldsymbol{\Delta}_n(t) \tag{20}$$

where $\boldsymbol{\Psi}(t) = \begin{bmatrix} \frac{\Gamma(1)t^{\alpha(t)}}{\Gamma(\alpha(t)+1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{\alpha(t)}}{\Gamma(\alpha(t)+2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{\alpha(t)}}{\Gamma(\alpha(t)+3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)t^{\alpha(t)}}{\Gamma(\alpha(t)+n+1)} \end{bmatrix}.$

Eq.(20) can be written as

$$I_t^{\alpha(t)}(\boldsymbol{\Phi}(t)) = \boldsymbol{A}_n\boldsymbol{\Psi}(t)\boldsymbol{A}_n^{-1}\boldsymbol{A}_n\boldsymbol{\Delta}_n(t) = \boldsymbol{A}_n\boldsymbol{\Psi}(t)\boldsymbol{A}_n^{-1}\boldsymbol{\Phi}(t) \tag{21}$$

From Eq.(19) and Eq.(21), the Bernstein polynomials operational matrix of fractional order integration $\boldsymbol{P}^{\alpha(t)}$is given by

$$\boldsymbol{P}^{\alpha(t)} = \boldsymbol{A}_n\boldsymbol{\Psi}(t)\boldsymbol{A}_n^{-1} \tag{22}$$

In particular, for $n = 2$ the Bernstein polynomials operational matrix of fractional order integration $\boldsymbol{P}^{\alpha(t)}$ is given by

$$\boldsymbol{P}^{\alpha(t)} = \begin{bmatrix} \frac{\Gamma(1)t^{\alpha(t)}}{\Gamma(1+\alpha(t))} & \frac{\Gamma(1)t^{\alpha(t)}}{\Gamma(1+\alpha(t))} - \frac{\Gamma(2)t^{\alpha(t)}}{\Gamma(2+\alpha(t))} & \frac{\Gamma(1)t^{\alpha(t)}}{\Gamma(1+\alpha(t))} - \frac{2\Gamma(2)t^{\alpha(t)}}{\Gamma(2+\alpha(t))} + \frac{\Gamma(3)t^{\alpha(t)}}{\Gamma(3+\alpha(t))} \\ 0 & \frac{\Gamma(2)t^{\alpha(t)}}{\Gamma(2+\alpha(t))} & \frac{2\Gamma(2)t^{\alpha(t)}}{\Gamma(2+\alpha(t))} - \frac{2\Gamma(3)t^{\alpha(t)}}{\Gamma(3+\alpha(t))} \\ 0 & 0 & \frac{\Gamma(3)t^{\alpha(t)}}{\Gamma(3+\alpha(t))} \end{bmatrix}.$$

It should be noted that the operational matrix $\boldsymbol{P}^{\alpha(t)}$ is upper triangular matrix. This phenomena makes calculations fast, and it also reduces the memory space.

5 The operational matrix of integration

We frequently encounter the integration of the vector $\Phi(t)$defined in Eq.(9) by

$$\int_0^t \boldsymbol{\Phi}(x)dx \cong \boldsymbol{P}\boldsymbol{\Phi}(t) \tag{23}$$

where $\boldsymbol{P}$ is the $(n+1)\times(n+1)$operational matrix of integration and is given as

$$\int_0^t \boldsymbol{\Phi}(x)dx = \int_0^t \boldsymbol{A}_n\boldsymbol{\Delta}_n(x)dx = \boldsymbol{A}_n\int_0^t \boldsymbol{\Delta}_n(x)dx = \boldsymbol{B}\boldsymbol{T} \tag{24}$$

where $\boldsymbol{B}$ is an $(n+1)\times(n+1)$ matrix:

$$\boldsymbol{B} = \boldsymbol{A}_n \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1/2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/(n+1) \end{bmatrix} \text{ and } \boldsymbol{T} = [t, t^2, \ldots, t^{n+1}]^T.$$

Now, we approximate the elements of vector $\boldsymbol{X}$ in terms of $\boldsymbol{\Phi}(t)$. By Eq.(9), we have $\boldsymbol{\Delta}_n(t) = \boldsymbol{A}_n^{-1}\boldsymbol{\Phi}(t)$, then for $k = 0, 1, \ldots, n$,

$$t^k = \boldsymbol{A}_{[k+1]}^{-1}\boldsymbol{\Phi}(t) \tag{25}$$

where $\boldsymbol{A}^{-1}_{[k+1]}$ is the $k+1th$ row of $\boldsymbol{A}_n^{-1}$ for $k=0,1,\dots,n$. We just need to approximate $t^{n+1}=\boldsymbol{c}^T_{n+1}\boldsymbol{\Phi}(t)$, by using Eq. (12) and Eq. (13), we have

$$\begin{aligned}\boldsymbol{c}_{n+1}&=\boldsymbol{Q}^{-1}\int_0^1 t^{n+1}\boldsymbol{\Phi}(t)dt\\&=\boldsymbol{Q}^{-1}\left[\begin{array}{cccc}\int_0^1 t^{n+1}B_{0,n}(t)dt & \int_0^1 t^{n+1}B_{0,n}(t)dt & \cdots & \int_0^1 t^{n+1}B_{0,n}(t)dt\end{array}\right]^T\\&=\frac{\boldsymbol{Q}^{-1}}{2n+2}\left[\begin{array}{cccc}\dfrac{\binom{n}{0}}{\binom{2n+1}{n+1}} & \dfrac{\binom{n}{1}}{\binom{2n+1}{n+2}} & \cdots & \dfrac{\binom{n}{n}}{\binom{2n+1}{2n+1}}\end{array}\right]^T\end{aligned}$$

Let $\boldsymbol{E}=[\boldsymbol{A}^{-1}_{[2]},\boldsymbol{A}^{-1}_{[3]},\dots\boldsymbol{A}^{-1}_{[n+1]},\boldsymbol{c}^T_{n+1}]^T$, then, $\boldsymbol{T}\cong\boldsymbol{E\Phi}(t)$. Therefore we have the operational matrix of integration $\boldsymbol{P}=\boldsymbol{BE}$.

6 The product operational matrix

It is always necessary to compute the product of $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Phi}(t)^T$, which is called the product matrix the Bernstein polynomials. Let

$$\boldsymbol{\Pi}(t)=\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(t)^T \tag{26}$$

then multiplying the matrix $\boldsymbol{\Pi}(t)$ with the vector c which is defined in Eq.(12) we obtain

$$\boldsymbol{c}^T\boldsymbol{\Pi}(t)=\boldsymbol{\Phi}(t)^T\hat{\boldsymbol{C}} \tag{27}$$

where $\hat{\boldsymbol{C}}$ is an $(n+1)\times(n+1)$ matrix and is called the coefficient matrix. Then we get

$$\begin{aligned}\boldsymbol{c}^T\boldsymbol{\Pi}(t)&=\boldsymbol{c}^T\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(t)^T=\boldsymbol{c}^T\boldsymbol{\Phi}(t)(\boldsymbol{\Delta}_n^T(t)\boldsymbol{A}_n^T)\\&=[\boldsymbol{c}^T\boldsymbol{\Phi}(t),t(\boldsymbol{c}^T\boldsymbol{\Phi}(t)),\dots,t^n(\boldsymbol{c}^T\boldsymbol{\Phi}(t))]\boldsymbol{A}_n^T\\&=\left[\sum_{i=0}^n c_iB_{i,n}(t),\sum_{i=0}^n c_itB_{i,n}(t),\dots,\sum_{i=0}^n c_it^nB_{i,n}(t)\right]\boldsymbol{A}_n^T\end{aligned} \tag{28}$$

Now, we approximate all functions $t^kB_{i,n}(t)$ in terms of $\boldsymbol{\Phi}(t)$. Let

$$e_{k,i}=[e_0^{k,i},e_1^{k,i},\dots,e_n^{k,i}]^T \tag{29}$$

By Eq.(12), we have

$$t^kB_{i,n}(t)\cong\boldsymbol{e}^T_{k,i}\boldsymbol{\Phi}(t),\quad i,k=0,1,2,\dots,n \tag{30}$$

By using Eq.(13) for $i,k = 0,1,2,\dots,n$, we obtain

$$
\begin{aligned}
\boldsymbol{e}_{k,i} &= \boldsymbol{Q}^{-1}\int_0^1 t^k B_{i,n}(t)\boldsymbol{\Phi}(t)dt \\
&= \boldsymbol{Q}^{-1}\left[\ \int_0^1 t^k B_{i,n}(t)B_{0,n}(t)dt \int_0^1 t^k B_{i,n}(t)B_{1,n}(t)dt \cdots \int_0^1 t^k B_{i,n}(t)B_{n,n}(t)dt\ \right]^T \\
&= \frac{\boldsymbol{Q}^{-1}\binom{n}{i}}{2n+k+1}\left[\ \frac{\binom{n}{0}}{\binom{2n+k}{i+k}} \quad \frac{\binom{n}{1}}{\binom{2n+k}{i+k+1}} \quad \cdots \quad \frac{\binom{n}{n}}{\binom{2n+k}{i+k+n}}\ \right]^T
\end{aligned}
\tag{31}
$$

Thus we get finally

$$
\begin{aligned}
\sum_{i=0}^{n} c_i t^k B_{i,n}(t) \cong \sum_{i=0}^{n} c_i\left(\sum_{j=0}^{n} e_j^{k,i} B_{j,n}(t)\right) &= \sum_{j=0}^{n} B_{j,n}(t)\left(\sum_{i=0}^{n} c_i e_j^{k,i}\right) \\
&= \boldsymbol{\Phi}(t)^T\left[\ \sum_{i=0}^{n} c_i e_0^{k,i} \quad \sum_{i=0}^{n} c_i e_1^{k,i} \quad \cdots \quad \sum_{i=0}^{n} c_i e_n^{k,i}\ \right]^T \\
&= \boldsymbol{\Phi}(t)^T[\boldsymbol{e}_{k,0},\boldsymbol{e}_{k,1},\dots,\boldsymbol{e}_{k,n}]\boldsymbol{c} \\
&= \boldsymbol{\Phi}(t)^T\boldsymbol{E}_{k+1}\boldsymbol{c}
\end{aligned}
\tag{32}
$$

where $\boldsymbol{E}_{k+1}$ is an $(n+1)\times(n+1)$ matrix. Then we define $\tilde{\boldsymbol{E}}_{k+1} = \boldsymbol{E}_{k+1}\boldsymbol{c}$. If we choose an $(n+1)\times(n+1)$ matrix $\tilde{\boldsymbol{C}} = [\tilde{\boldsymbol{E}}_1,\tilde{\boldsymbol{E}}_2,\dots,\tilde{\boldsymbol{E}}_{n+1}]$, then by Eq. (29) and Eq. (33) we have

$$
\begin{aligned}
\boldsymbol{c}^T\boldsymbol{\Pi}(t) &= \left[\sum_{i=0}^{n} c_i B_{i,n}(t), \sum_{i=0}^{n} c_i t B_{i,n}(t), \dots, \sum_{i=0}^{n} c_i t^n B_{i,n}(t)\right]\boldsymbol{A}_n^T \\
&\cong \boldsymbol{\Phi}(t)^T\tilde{\boldsymbol{C}}\boldsymbol{A}_n^T
\end{aligned}
\tag{33}
$$

and therefore we have the coefficient matrix, as

$$
\hat{\boldsymbol{C}} = \tilde{\boldsymbol{C}}\boldsymbol{A}_n^T \tag{34}
$$

7 Numerical solution of variable order fractional integro-differential equations

Consider the following equations

$$
D_t^{\alpha(t)}y(t) = \lambda_1\int_0^t k_1(t,s)y(s)ds + \lambda_2\int_0^1 k_2(t,s)y(s)ds + f(t) \tag{35}
$$

with initial condition

$$y(0) = b \tag{36}$$

where $0 < \alpha(t) < 1$ and $D_t^{\alpha(t)}$ denotes the variable order fractional derivative defined by Coimabra. $k_1(t,s), k_2(t,s), f(t)$ are the known functions. λ_1, λ_2, b are the real constants.
By previous section, the function $D_t^{\alpha(t)} y(t)$ of the Eq.(35) can be approximated as:

$$D_t^{\alpha(t)} y(t) \cong \boldsymbol{c}^T \boldsymbol{\Phi}(t) \tag{37}$$

Using the property 2, we have

$$\begin{aligned} y(t) &= I_t^{\alpha(t)} \left(D_t^{\alpha(t)} y(t) \right) + y(0) \cong \boldsymbol{c}^T \boldsymbol{P}^{\alpha(t)} \boldsymbol{\Phi}(t) + \boldsymbol{u}^T \boldsymbol{\Phi}(t) \\ &= [\boldsymbol{c}^T \boldsymbol{P}^{\alpha(t)} + \boldsymbol{u}^T] \boldsymbol{\Phi}(t) = \boldsymbol{v}^T \boldsymbol{\Phi}(t) \end{aligned} \tag{38}$$

where $\boldsymbol{v}^T = \boldsymbol{c}^T \boldsymbol{P}^{\alpha(t)} + \boldsymbol{u}^T$ and $y(0) \cong \boldsymbol{u}^T \boldsymbol{\Phi}(t)$.
The functions approximating $k_1(t,s)$ and $k_2(t,s)$ by Bernstein polynomials can be given as

$$k_1(t,s) = \boldsymbol{\Phi}^T(t) \boldsymbol{K}_1 \boldsymbol{\Phi}(s), \quad k_2(t,s) = \boldsymbol{\Phi}^T(t) \boldsymbol{K}_2 \boldsymbol{\Phi}(s) \tag{39}$$

where $\boldsymbol{K}_1$ and are defined in Eq.(15). Using Eq.(39), Eq.(38), Eq.(27) and Eq.(23) we can write the Volterra part of Eq.(35) as

$$\begin{aligned} \int_0^t k_1(t,s) y(s) ds &\cong \int_0^t \boldsymbol{\Phi}^T(t) \boldsymbol{K}_1 \boldsymbol{\Phi}(s) \boldsymbol{\Phi}(s)^T \boldsymbol{v} ds \\ &= \boldsymbol{\Phi}^T(t) \boldsymbol{K}_1 \int_0^t \boldsymbol{\Pi}(s)^T \boldsymbol{v} ds \\ &= \boldsymbol{\Phi}^T(t) \boldsymbol{K}_1 \hat{\boldsymbol{V}}^T \int_0^t \boldsymbol{\Phi}(s) ds \\ &= \boldsymbol{\Phi}^T(t) \boldsymbol{K}_1 \hat{\boldsymbol{V}}^T \boldsymbol{P} \boldsymbol{\Phi}(t) \end{aligned} \tag{40}$$

and making use of Eq.(39) and Eq.(38) we have the Fredholm part of Eq.(35) as follows:

$$\begin{aligned} \int_0^1 k_2(t,s) y(s) ds &\cong \int_0^1 \boldsymbol{\Phi}^T(t) \boldsymbol{K}_2 \boldsymbol{\Phi}(s) \boldsymbol{\Phi}(s)^T \boldsymbol{v} ds \\ &= \boldsymbol{\Phi}^T(t) \boldsymbol{K}_2 \int_0^1 \boldsymbol{\Phi}(s) \boldsymbol{\Phi}(s)^T ds \boldsymbol{v} \\ &= \boldsymbol{\Phi}^T(t) \boldsymbol{K}_2 \boldsymbol{Q} \boldsymbol{v} \end{aligned} \tag{41}$$

Also the right hand side of Eq.(35) can be written as

$$f(t) \cong \boldsymbol{f}^T \boldsymbol{\Phi}(t) \tag{42}$$

where $f = [f_0, f_1, \cdots, f_n]^T$.

Substituting Eq.(37), Eq.(40), Eq(41) and Eq.(42) into Eq.(35), we get

$$\boldsymbol{c}^T \boldsymbol{\Phi}(t) = \lambda_1 \boldsymbol{\Phi}^T(t) \boldsymbol{K}_1 \hat{\boldsymbol{V}}^T \boldsymbol{P}\boldsymbol{\Phi}(t) + \lambda_2 \boldsymbol{\Phi}^T(t) \boldsymbol{K}_2 \boldsymbol{Q}\boldsymbol{v} + \boldsymbol{f}^T \boldsymbol{\Phi}(t) \tag{43}$$

Dispersing Eq.(43) at the points $t_i = \frac{i}{n+1}$, $i = 0, 1, 2, \cdots, n$, we obtain a linear system of algebraic equations. By solving this system we may obtain the approximate solution of Eq.(35) according to Eq.(38).

8 Error analysis

In this section, we will present the error analysis of the method for the variable order fractional integro-differential equations. Using Eq.(43), for $t \in [0, 1)$, the error function $R_n(t)$ is defined as

$$R_n(t) = \left| \boldsymbol{c}^T \boldsymbol{\Phi}(t) - \lambda_1 \boldsymbol{\Phi}^T(t) \boldsymbol{K}_1 \hat{\boldsymbol{V}}^T \boldsymbol{P}\boldsymbol{\Phi}(t) - \lambda_2 \boldsymbol{\Phi}^T(t) \boldsymbol{K}_2 \boldsymbol{Q}\boldsymbol{v} - \boldsymbol{f}^T \boldsymbol{\Phi}(t) \right| \tag{44}$$

Since the truncated Bernstein polynomials series is an approximate solution of Eq.(35), we should have $R_n(t) \cong 0$. In the error function, the larger the value of n, the more accurate the approximation solution of equation. The optimum value of n is determined by the prescribed accuracy.

Set $t = t_i$, where t_i, $i = 0, 1, 2, \cdots, n$ are the discrete points from $[0, 1)$. Then our aim is to have $R_n(t_i) \leq 10^{-r_i}$, where r_i is any positive integer. Let $Max\{r_i\} = r$, we can increase n as long as the following inequality holds at each point t_i

$$R_n(t_i) \leq 10^{-r} \tag{45}$$

in other words, by increasing n the error function $R_n(t_i)$ approaches zero. If $R_n(t) \to 0$ when n is sufficiently large enough, then the error decreases.

9 Numerical examples

In this section, we will use the Bernstein polynomials operational matrix of fractional order integration to analyze the variable order fractional integro-differential equations.

Example 1. Consider the following variable order fractional integro-differential equation

$$D_t^{\alpha(t)} y(t) = \int_0^x (t+s) y(s) ds + \int_0^1 (t-s) y(s) ds + f(t),$$

with initial condition $y(0) = 1$, and

$$f(t) = \frac{\Gamma(2)}{\Gamma\left(\frac{5+t}{3}\right)} t^{\frac{2+t}{3}} + \frac{\Gamma(3)}{\Gamma\left(\frac{8+t}{3}\right)} t^{\frac{5+t}{3}} - \frac{11}{6}t - \frac{3}{2}t^2 - \frac{5}{6}t^3 - \frac{7}{12}t^4 + \frac{13}{12}.$$

The exact solution of this equation is $y(t) = 1 + t + t^2$. In this problem, $\alpha(t) = \frac{1-t}{3}$. **Figs. 1-3** show the numerical solutions for $n = 2$, $n = 4$, $n = 6$.

From **Figs. 1-3** we can see that the numerical solutions are very good agreement with the exact solution when $n$increases. The calculating results show that combining with the Bernstein polynomials operational matrix of fractional order integration, the method in this paper can be effectively used in numerical calculus for variable order fractional integro-differential equations.

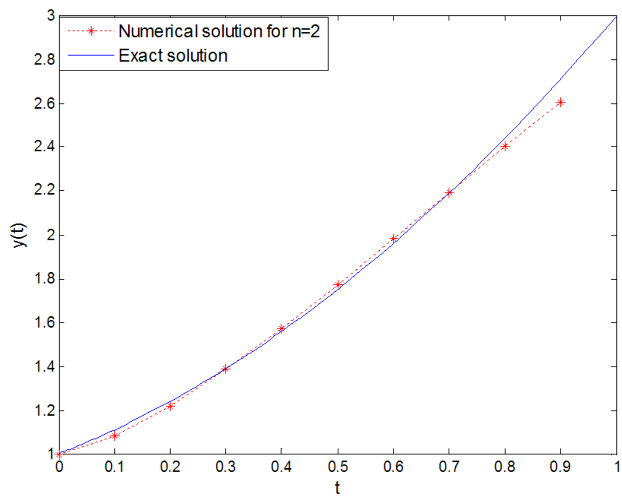

Figure 1: The comparison between approximate and exact solutions for $n = 2$.

Example 2. Consider this equation

$$D_t^{\frac{t}{2}} y(t) = \frac{1}{10}\int_0^t tsy(s)ds + \frac{1}{3}\int_0^1 (t+s)y(s)ds + f(t),$$

such that $y(0) = 0$ and $f(t) = \frac{\Gamma(7)}{\Gamma(7-\frac{t}{2})} + \frac{\Gamma(8)}{\Gamma(8-\frac{t}{2})} - \frac{t^9}{80} - \frac{t^{10}}{90} - \frac{5t}{56} - \frac{17}{216}$. We applied the Bernstein polynomials approach to solve this problem for various values of n. The exact solution is $y(t) = t^6 + t^7$. The absolute errors for $n = 4$, $n = 5$, $n = 6$, $n = 7$ are shown in **Table 1**.

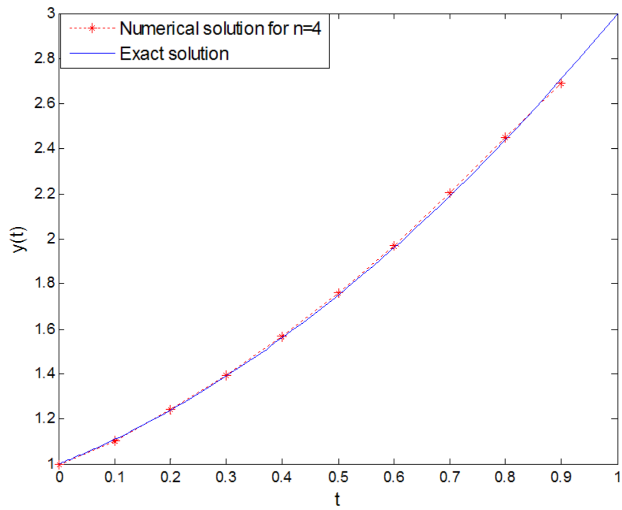

Figure 2: The comparison between approximate and exact solutions for $n = 4$.

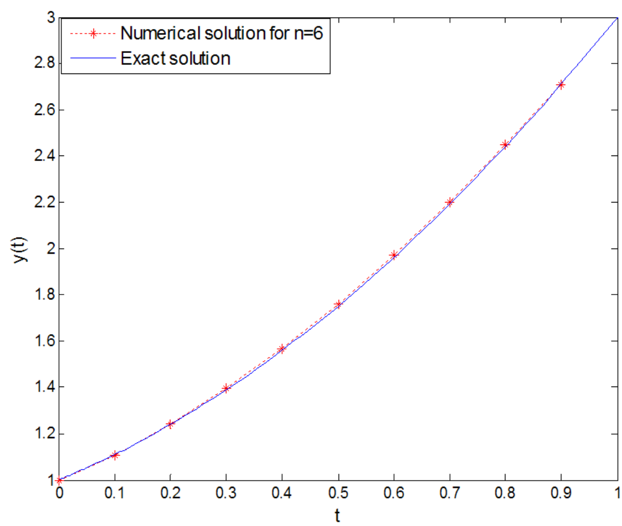

Figure 3: The comparison between approximate and exact solutions for $n = 6$.

Table 1: The absolute errors of different $n$for **Example 2**.

t	$n=4$	$n=5$	$n=6$	$n=7$
0	0	0	0	0
0.1	1.211436e-002	2.884210e-003	2.894806e-004	2.107654e-006
0.2	9.994910e-004	5.582717e-004	9.768165e-005	7.584658e-007
0.3	4.246281e-003	4.727489e-004	2.687613e-005	5.452959e-006
0.4	3.793182e-004	5.698137e-005	1.022243e-005	5.800553e-006
0.5	4.905035e-003	1.126630e-004	3.214812e-005	3.502632e-006
0.6	3.367787e-003	2.214333e-003	1.553171e-003	1.525140e-004
0.7	3.148376e-003	2.014514e-003	1.358878e-003	5.296317e-004
0.8	9.898867e-003	7.310422e-003	5.538599e-003	1.558219e-003
0.9	1.053979e-001	4.933938e-002	4.088800e-002	4.035161e-002

From the **Table 1**, we find that the numerical solutions are more and more close to the exact solution when $n$becomes large. That is to say, the numerical solutions converge to the exact solution with increasing the value of n.

Example 3. Consider the below variable order fractional integro-differential equation

$$D_t^t y(t) = \int_0^t (t-s)y(s)ds + \int_0^1 s\sin t \cdot y(s)ds + f(t),$$

where $f(t) = \frac{\Gamma(23/4)t^{\frac{19}{4}-t}}{\Gamma(23/4-t)} + \frac{\Gamma(36/5)t^{\frac{31}{5}-t}}{\Gamma(36/5-t)} - \frac{16}{621}t^{\frac{27}{4}} - \frac{25}{1476}t^{\frac{41}{5}} - \frac{299}{1107}\sin t$, with this supplementary condition $y(0) = 0$. **Fig. 4** shows the numerical solutions for $n = 4$, $n = 5$, $n = 6$ with the exact solution $y(t) = t^{\frac{19}{4}} + t^{\frac{31}{5}}$. The absolute errors for different values n are also shown in **Table 2**.

From **Fig.4** and **Table 2**, we can see clearly that the numerical solutions are very good coincidence with the exact solution and the absolute errors become more and more small. They demonstrate the simplicity, and powerfulness of the proposed method. What's more, the method in this paper is easy to implementation.

10 Conclusion

We derive the Bernstein polynomials operational matrix of fractional order integration, and use it to solve the variable order fractional integro-differential equations. The matrix, operational matrix of integration $\boldsymbol{P}$, product matrix $\boldsymbol{\Pi}$ and coefficient matrix $\hat{\boldsymbol{C}}$ for the Bernstein polynomials have been used for transforming the initial problem into a linear algebraic system equations that can be solved easily. The

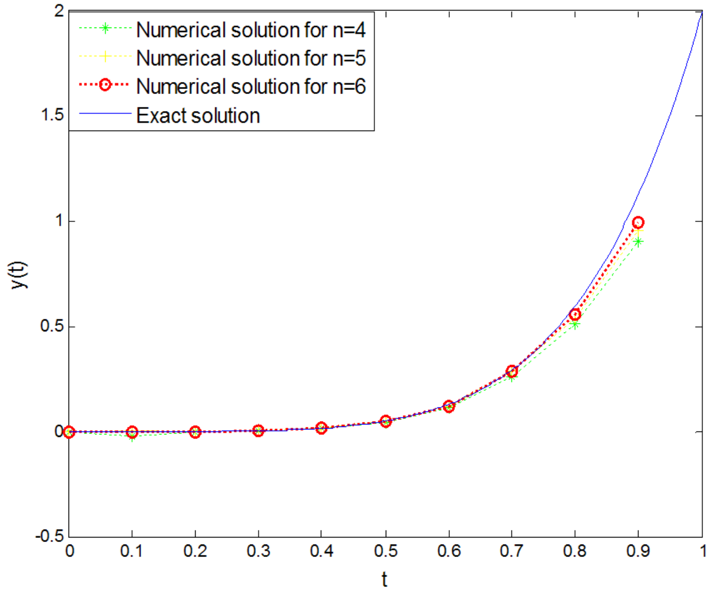

Figure 4: The comparison between approximate and exact solutions for different n.

Table 2: The absolute errors of different $n$for **Example 3**.

t	$n=4$	$n=5$	$n=6$	$n=7$
0	0	0	0	0
0.1	1.847813e-002	3.170885e-003	1.312325e-003	8.246755e-004
0.2	2.970928e-003	2.544115e-004	1.488098e-005	8.785142e-006
0.3	3.790940e-003	4.382297e-004	1.280260e-004	9.112653e-005
0.4	9.028638e-005	3.395943e-004	8.417671e-005	6.645263e-005
0.5	7.835446e-003	3.330079e-003	3.725454e-003	1.256150e-003
0.6	1.631390e-002	1.294155e-002	1.080649e-002	9.275812e-003
0.7	3.355224e-002	3.032422e-002	2.646955e-002	9.578623e-003
0.8	8.508671e-002	8.232376e-002	7.040846e-002	6.632365e-003
0.9	2.204358e-001	2.025485e-001	1.815759e-001	8.658236e-002

solution is convergent, even though the size of increment may be large. Several examples are given to demonstrate the powerfulness of the proposed method.

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