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Boundary Element Analysis of Shear Deformable Shallow Shells Under Harmonic Excitation

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Abstract: In this work, the harmonic analysis of shallow shells using the Boundary Element Method, is presented. The proposed boundary element formulation is based on a direct time-domain integration using the elastostatic fundamental solutions for both in-plane elasticity and shear deformable plates. Shallow shell was modeled coupling boundary element formulation of shear deformable plate and two-dimensional plane stress elasticity. Effects of shear deformation and rotatory inertia were included in the formulation. Domain integrals related to inertial terms were treated using the Dual Reciprocity Boundary Element Method. Numerical examples are presented to demonstrate the efficiency and accuracy of the proposed formulation.

Keywords: Boundary element method, Shear deformable shallow shells, Thick shells, Harmonic analysis, Dual reciprocity boundary element method, Harmonic excitation.

1 Introduction

Dynamic plate bending problems appear on civil, mechanical, aerospatial and naval applications. The complexity involved in the dynamic response of plates turns these problems in a challenging one from mathematical point of view. In general, numerical methods represent the only way to obtain approximate solutions for dynamic analysis. However, the use of traditional methods based on domain discretization requires refined meshes involving a high number of degrees of freedom, which requires a significant computational effort [Hall (2013); Gao and Davies (2002)].

Nowadays, the Boundary Element Method (BEM) has emerged as an accurate and efficient numerical method for plate and shear deformable shell static analysis [Zhang and Atluri (1986); Wrobel and Aliabadi (2002); Rashed (2000); Dirgantara and Aliabadi (1999); Wen, Aliabadi, and Young (2000a) and Wen and Aliabadi (2000)]. [Providakis and Beskos (1999)] presents a review of all the work on plate

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dynamics up to 1998. [Beskos (2003)] deals with the dynamic analysis of various structures and soil-structural systems by the direct conventional boundary element method (BEM) in both the frequency and time domains. Dynamic analysis of thin shallow shells using BEM is presented in and Providakis and Beskos (1991). Time-domain dynamic analysis of shear deformable plates using elastodynamic fundamental solutions or Laplace or Fourier transformations of these fundamental solutions were used in [Duddeck (2010); Wen, Aliabadi, and Young (2000b); Wen and Aliabadi (2006) and Wen, Adetoro, and Xu (2008)]. In those works, harmonic BEM analysis using direct time integration and elastostatic fundamental solutions was not performed. In [Nardini and Brebbia (1982)] a time-domain direct formulations for dynamic analysis of plates (plane stress) is presented. [Useche and Albuquerque (2012); Useche, Albuquerque, and Sollero (2012)] presents a time-domain direct formulations based on elastostatic fundamental solutions for dynamic analysis of shear deformable plates. However, to the best of the author knowledge, the Boundary Element Method has not been used for the harmonic analysis of shear deformable elastic shallow shells.

This work presents the harmonic analysis of shear deformable double-curved shallow shells using a boundary element formulation. This formulation is based on direct time integration and elastostatic fundamental solutions. Effects of shear deformation and rotatory inertia are included in the formulation. Shells were modeled by coupling the boundary element formulation for shear deformable plates based on the Reissner plate theory and two-dimensional plane stress elasticity, as presented in Dirgantara and Aliabadi (1999); Wen and Aliabadi (2000)]. The Dual Reciprocity Boundary Element Method for the treatment of domain integrals involving inertial mass, was used. Numerical examples are presented and results were compared with those obtained using both analytical and finite element solutions.

2 Shallow shell dynamic equations

Consider a shallow shell of uniform thickness h and mass density ρ, occupying the area Ω, in the x_1x_2 plane, bounded by the contour $\Gamma = \Gamma_w \bigcup \Gamma_q$ with $\Gamma = \Gamma_w \bigcap \Gamma_q \equiv 0$, as presented in figure 1. The dynamic bending response for the shallow shell was modeled coupling the classical Reissner plate theory and the two-dimensional plane stress elasticity as presented in [Wen, Aliabadi, and Young (2000a)].

Equations of motion for an infinitesimal plate element are given by [Reddy (2004)]:

$$L^b_{ik} w_k + q^*_i = \Lambda^b_{ik} \ddot{w}_k + \Lambda^{bm}_{i\alpha} \ddot{u}_\alpha \tag{1}$$

$$L^m_{\alpha\beta} u_\beta = \Lambda^{bm}_{\alpha\beta} \ddot{w}_\beta + \Lambda^m_{\alpha\beta} \ddot{u}_\beta \tag{2}$$

Indicial notation is used throughout this work. Greek indices vary from 1 to 2 and

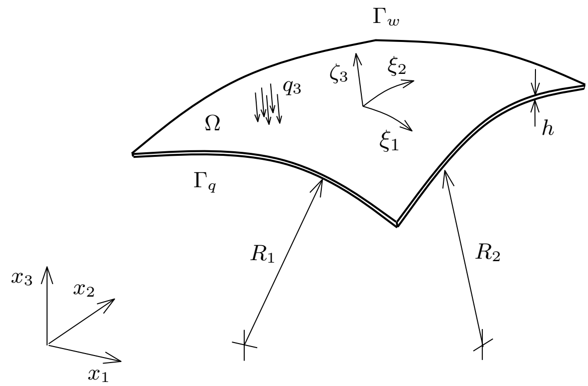

Figure 1: Shallow shell geometry

Latin indices take values from 1 to 3. Einstein's summation convention is used unless otherwise indicated. In these equations, w_α represents rotations with respect to x_1 and x_2 axes, and w_3 represents transverse deflection; $\ddot{w}_\alpha$ denotes angular accelerations with respect to x_1 and x_2 axes, respectively, $\ddot{w}_3$ represents the transverse acceleration; u_α and $\ddot{u}_\alpha$ represents membrane displacements and accelerations along x_α axis, respectively; Tensors Λ^b_{ij}, Λ^{bm}_{ij} and Λ^m_{ij} are defined as: $\Lambda^b_{\alpha\beta} = I_2\delta_{\alpha\beta}$ and $\Lambda^b_{33} = I_0$; $\Lambda^{bm}_{\alpha\beta} = I_1\delta_{\alpha\beta}$; $\Lambda^m_{\alpha\beta} = I_0\delta_{\alpha\beta}$; $\delta_{\alpha\beta}$ is the Kronecker's delta and I_i is the mass inertias. The differential operator L_{ik} in equation (1), is given by [Dirgantara and Aliabadi (1999)]:

$$L^b_{\alpha\beta} = \frac{D(1+\nu)}{2}\left[(\nabla^2 - \lambda^2)\delta_{\alpha\beta} + \frac{(1+\nu)}{(1-\nu)}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\right] \tag{3}$$

$$L^b_{\alpha 3} = -L^b_{3\alpha} = -\frac{(1-\nu)D}{2}\lambda^2\frac{\partial}{\partial x_\alpha} \tag{4}$$

$$L^b_{33} = -\frac{(1-\nu)D}{2}\lambda^2\nabla^2 \tag{5}$$

where $D = Eh^3/12(1-\nu^2)$ is the bending stiffness of the plate, E is the Young modulus, ν is the Poisson's ratio and $\lambda^2 = 10/h$. In equation (1) q_i is the equivalent body force, which is given by (q^*_α=0):

$$q^*_3 = q_3 - B(\kappa_{11} + \nu\kappa_{22})\frac{\partial u_1}{\partial x_1} - B(\kappa_{11} + \nu\kappa_{22})\frac{\partial u_2}{\partial x_2}$$
$$-B(\kappa^2_{11} + \nu\kappa^2_{22} + 2\nu\kappa_{11}\kappa_{22})w_3 \tag{6}$$

where q_3 is a distributed transverse load and $\kappa_{\alpha\beta}$ represents the curvature tensor of the shell surface. Similarly, in equation (2) $L^m_{\alpha\beta}$ is the Navier differential operator for two-dimensional plane stress problems:

$$L^m_{\alpha\beta} = B\nabla^2\delta_{\alpha\beta} + \frac{B(1+\nu)}{2}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}(1-2\delta_{\alpha\beta}) \tag{7}$$

where $B = Eh/(1-\nu^2)$ is the in-plane stiffness of the plate.

3 Boundary integral formulation for shallow shells

The derivation of the integral formulation for equations (1) is based on application of the boundary element method to the Reissner plate theory as presented in [VanderWeen (1982)], were the integral representations related to the governing equations for bending and transverse shear stress resultants are derived by using the weighted residual method, and making use of the Green's identity. Thus, by integration of equations (1), the following equations are obtained:

$$\begin{aligned}
& c_{ij}w_j(\mathbf{x}') + \int_\Gamma P_{ik}(\mathbf{x}',\mathbf{x})w_j(\mathbf{x})d\Gamma = \int_\Gamma W_{ij}(\mathbf{x}',\mathbf{x})p_j(\mathbf{x})d\Gamma \\
& - \int_\Gamma \kappa_{\alpha\beta}B\frac{1-\nu}{2}\Big[u_\alpha(\mathbf{x})n_\beta + u_\beta(\mathbf{x})n_\alpha \\
& + \frac{2\nu}{1-\nu}u_\gamma(\mathbf{x})n_\gamma\delta_{\alpha\beta}\Big]W_{i3}(\mathbf{x}',\mathbf{x})d\Gamma \\
& + \int_\Omega \kappa_{\alpha\beta}B\frac{1-\nu}{2}\Big[u_\alpha(\mathbf{X})W_{i3,\beta}(\mathbf{x}',\mathbf{x}) + u_\beta(\mathbf{x})W_{i3,\alpha}(\mathbf{x}',\mathbf{X}) \\
& + \frac{2\nu}{1-\nu}u_\gamma(\mathbf{x})W_{i3,\gamma}(\mathbf{x}',\mathbf{X})\delta_{\alpha\beta}\Big]d\Omega \\
& - \int_\Omega \kappa_{\alpha\beta}B[(1-\nu)\kappa_{\alpha\beta} + \nu\delta_{\alpha\beta}\kappa_{\gamma\gamma}]w_3(\mathbf{x})W_{i3}(\mathbf{x}',\mathbf{X})d\Omega \\
& + \int_\Omega W_{i3}(\mathbf{x}',\mathbf{X})q_3(\mathbf{x})d\Omega + \int_\Omega W_{ij}(\mathbf{x}',\mathbf{x})\Lambda^b_{ij}\ddot{w}_j(\mathbf{X})d\Omega \\
& + \int_\Omega W_{i\alpha}(\mathbf{x}',\mathbf{X})\Lambda^{bm}_{i\alpha}\ddot{u}_\alpha(\mathbf{X})d\Omega
\end{aligned} \tag{8}$$

In a similar way, the derivation of the integral formulation for equations (2) is based on application of the boundary element method to the two-dimensional elasticity equations, as presented in [Dirgantara and Aliabadi (1999)]. Thus, by integration of equations (2) the following equations are obtained:

$$c_{\theta\alpha}(\mathbf{x}')u_\alpha(\mathbf{x}') + \int_\Gamma T_{\theta\alpha}(\mathbf{x}',\mathbf{x})u_\alpha(\mathbf{x})d\Gamma$$
$$+ \int_\Omega U_{\theta\alpha,\beta}(\mathbf{x}',\mathbf{X})B[\kappa_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}\kappa_{\gamma\gamma}]w_3(\mathbf{X})d\Omega$$
$$= \int_\Gamma U_{\theta\alpha}(\mathbf{x}',\mathbf{x})t_\alpha(\mathbf{x})d\Gamma + \int_\Omega U_{\theta\alpha}(\mathbf{x}',\mathbf{X})\Lambda^m_{\theta\alpha}\ddot{u}_\alpha(\mathbf{X})d\Omega \tag{9}$$
$$+ \int_\Omega U_{\theta\alpha}(\mathbf{x}',\mathbf{X})\Lambda^{bm}_{\theta\alpha}\ddot{w}_\alpha(\mathbf{X})d\Omega$$

In these equations, **x**' and **x** represents collocation y field points, respectively; W_{ik} and P_{ik} are fundamental solutions for shear deformable plates [VanderWeen (1982)]; $T_{\theta\alpha}$ and $U_{\theta\alpha}$ are the fundamental solutions for plane stress [Wrobel and Aliabadi (2002)]; n_α is the unity vector normal to the boundary at field point. $\mathbf{x}' \in \Gamma$ are source points and $\mathbf{x} \in \Gamma$ and $\mathbf{X} \in \Omega$ represents field points. The value of $c_{ij}(\mathbf{x}')$ is equal to $\frac{1}{2}\delta_{ij}$ when $\mathbf{x}'$ is located on a smooth boundary. These equations represents five integral equations, the first two in (8) $(i = \alpha = 1,2)$ are for rotations, the third $(i = 3)$ is for the out-of-plane displacement and two in (9) $(\alpha = 1,2)$ for in-plane displacements, which can be used to solve shear deformable plate shallow shell bending problems.

4 Transformation of domain integrals

Only uniform distributed pressure is considered as external loads acting on the shell surface. In this way, considering a uniform load $q_3 = p(t)$, the third to last integral in (8) can be transfered to a boundary integral by applying the divergence theorem [Rashed (2000)]. For sake of simplicity, domain integrals in equations (8) and (9) related with curvature terms, are treated using the Cell Method as presented in [Dirgantara and Aliabadi (1999)].

In order to transform domain integrals related to inertial terms into boundary integrals, the Dual Reciprocity Boundary Element Method (DRM) was used [Partridge, Bebbia, and Wrobel (1992)]. Then, the two last integrals in equations (8) and (9) can be re-written as:

$$\int_\Omega W_{ij}(\mathbf{x}',\mathbf{x})\Lambda^b_{ij}\ddot{w}_j(\mathbf{X})d\Omega = \sum_{n=1}^{NDR} \ddot{\alpha}^n_l(t)\left[c_{ij}\hat{W}^n_{lj} - \int_\Gamma W_{ij}(\mathbf{x}',\mathbf{x})\hat{P}^n_{lj}d\Gamma\right.$$
$$\left. + \int_\Gamma P_{ij}(\mathbf{x}',\mathbf{x})\hat{W}^n_{lj}d\Gamma\right] \tag{10}$$

$$\int_{\Omega} W_{i\alpha}(\mathbf{x}', \mathbf{X}) \Lambda^{bm}_{i\alpha} \ddot{u}_{\alpha}(\mathbf{X}) d\Omega = \sum_{n=1}^{NDR} \ddot{\beta}^{n}_{l}(t) \Big[c_{i\alpha} \hat{W}^{n}_{l\alpha} - \int_{\Gamma} W_{i\alpha}(\mathbf{x}', \mathbf{x}) \hat{P}^{n}_{l\alpha} d\Gamma + \int_{\Gamma} P_{i\alpha}(\mathbf{x}', \mathbf{x}) \hat{W}^{n}_{l\alpha} d\Gamma \Big] \tag{11}$$

$$\int_{\Omega} W_{i\alpha}(\mathbf{x}', \mathbf{X}) \Lambda^{m}_{i\alpha} \ddot{u}_{\alpha}(\mathbf{X}) d\Omega = \sum_{n=1}^{NDR} \ddot{\gamma}^{n}_{l}(t) \Big[c_{i\alpha} \hat{W}^{n}_{l\alpha} - \int_{\Gamma} W_{i\alpha}(\mathbf{x}', \mathbf{x}) \hat{P}^{n}_{l\alpha} d\Gamma + \int_{\Gamma} P_{i\alpha}(\mathbf{x}', \mathbf{x}) \hat{W}^{n}_{l\alpha} d\Gamma \Big] \tag{12}$$

$$\int_{\Omega} U_{\theta\alpha}(\mathbf{x}', \mathbf{X}) \Lambda^{bm}_{\theta\alpha} \ddot{w}_{\alpha}(\mathbf{X}) d\Omega = \sum_{n=1}^{NDR} \ddot{\chi}^{n}_{l}(t) \Big[c_{\theta\alpha} \hat{U}^{n}_{l\alpha} - \int_{\Gamma} U_{\theta\alpha}(\mathbf{x}', \mathbf{x}) \hat{T}^{n}_{l\alpha} d\Gamma + \int_{\Gamma} T_{\theta k}(\mathbf{x}', \mathbf{x}) \hat{U}^{n}_{l\alpha} d\Gamma \Big] \tag{13}$$

In these equations, *NDR* represents the number of total dual reciprocity collocations points used in the plate; $\hat{W}^{m}_{lk}$, $\hat{P}^{m}_{lk}$ are the particular solutions to equivalent homogeneous equation (1) and $\hat{U}^{m}_{l\alpha}$, $\hat{T}^{m}_{l\alpha}$ are the particular solutions to equivalent homogeneous equation (2). These particular solutions were obtained considering the function $f_m = 1 - \lambda^2 r_m^3/9$ for the approximation of angular accelerations and the function $f_m = 1 + r_m$ for the approximation of the transversal and in-plane accelerations. Coefficients $\ddot{\alpha}^{m}_{l}$, $\ddot{\beta}^{m}_{l}$, $\ddot{\gamma}^{m}_{l}$ and $\ddot{\chi}^{m}_{l}$ are related to $\Lambda^{b}_{ij}\ddot{w}_k$, $\Lambda^{bm}_{i\alpha}\ddot{u}_\alpha$, $\Lambda^{m}_{i\alpha}\ddot{u}_\alpha$ and $\Lambda^{bm}_{\theta\alpha}\ddot{w}_\alpha$ respectively, through expressions:

$$\Lambda^{b}_{ij}\ddot{w}_k(t) = A_{kl}\ddot{\alpha}^{m}_{l}(t); \quad \Lambda^{bm}_{i\alpha}\ddot{u}_\alpha(t) = B_{\alpha l}\ddot{\beta}^{m}_{l}(t) \tag{14}$$

$$\Lambda^{m}_{i\alpha}\ddot{u}_\alpha(t) = D_{\alpha l}\ddot{\gamma}^{m}_{l}(t); \quad \Lambda^{bm}_{\theta\alpha}\ddot{w}_\alpha(t) = C_{\alpha l}\ddot{\chi}^{m}_{l}(t) \tag{15}$$

with $l = 1, 2 \ldots, NDR$ and A_{kl}, $B_{\alpha l}$, $C_{\alpha l}$ and $D_{\alpha l}$ are matrices of coefficients, obtained by taking the value of $\ddot{w}_k(t)$ and $\ddot{u}_\alpha(t)$ at different DRM points.

5 Boundary element formulation

Quadratic isoparametric boundary elements were used to describe the geometry and the unknows functions along the boundary. Discontinuous elements are used to avoid difficulties with discontinuity of the traction between elements. In this way, equations (8) and (9) can be rewritten in a discretised way as:

$$
\begin{aligned}
& c_{ij}(\mathbf{x}')w_j(\mathbf{x}') + \sum_{n=1}^{N_e}\sum_{m=1}^{3} w_j^{nm} \int_{\xi=-1}^{\xi=+1} P_{ij}(\mathbf{x}',\mathbf{x})\Phi^m(\xi)J_n(\xi)d\xi \\
& = \sum_{n=1}^{N_e}\sum_{m=1}^{3} p_j^{nm} \int_{\xi=-1}^{\xi=+1} W_{ij}(\mathbf{x}',\mathbf{x})\Phi^m(\xi)J_n(\xi)d\xi \\
& - \sum_{n=1}^{N_e}\sum_{m=1}^{3} \kappa_{\alpha\beta}B\frac{1-\nu}{2}\left(u_\alpha^{nm}n_\beta^{nm} + u_\beta^{nm}n_\alpha^{nm} + \frac{2\nu}{1-\nu}u_\gamma^{nm}n_\gamma^{nm}\delta_{\alpha\beta}\right) \\
& \times \int_{\xi=-1}^{\xi=+1} W_{i3}(\mathbf{x}',\mathbf{x})\Phi^m(\xi)J_n(\xi)d\xi \\
& + \sum_{k=1}^{N_c} \kappa_{\alpha\beta}B\frac{1-\nu}{2}u_\alpha^k \int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3,\beta}(\mathbf{x}',\mathbf{x})J_k^c(\xi,\eta)d\xi d\eta \\
& + \sum_{k=1}^{N_c} \kappa_{\alpha\beta}B\frac{1-\nu}{2}u_\beta^k \int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3,\alpha}(\mathbf{x}',\mathbf{x})J_k^c(\xi,\eta)d\xi d\eta \\
& + \sum_{k=1}^{N_c} \kappa_{\alpha\beta}B\nu u_\gamma^k \int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3,\gamma}(\mathbf{x}',\mathbf{x})J_k^c(\xi,\eta)d\xi d\eta \\
& - \sum_{k=1}^{N_c} \kappa_{\alpha\beta}B[(1-\nu)\kappa_{\alpha\beta} + \nu\delta_{\alpha\beta}\kappa_{\gamma\gamma}]w_3^k \int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} W_{i3}(\mathbf{x}',\mathbf{X})J_k^c(\xi,\eta)d\xi d\eta \\
& + p(t)\sum_{n=1}^{N_e}\int_{\xi=-1}^{\xi=+1} V_{i,\alpha}(\mathbf{x}',\mathbf{x})n_\alpha(\xi)J_n(\xi)d\xi \\
& + \sum_{n=1}^{NDR} \ddot{\alpha}_l^n(t)\Big[c_{ij}\hat{W}_{lj}^n - \sum_{k=1}^{N_e}\sum_{r=1}^{3}\int_{\xi=-1}^{\xi=+1} W_{ij}(\mathbf{x}',\mathbf{x})\hat{P}_{lj}^n\Phi^r(\xi)J_k(\xi)d\Gamma \\
& + \sum_{k=1}^{N_e}\sum_{r=1}^{3}\int_{\xi=-1}^{\xi=+1} P_{ij}(\mathbf{x}',\mathbf{x})\hat{W}_{lj}^n\Phi^r(\xi)J_k(\xi)d\Gamma\Big] \\
& + \sum_{n=1}^{NDR} \ddot{\gamma}_l^n(t)\Big[c_{i\alpha}\hat{W}_{l\alpha}^n - \sum_{k=1}^{N_e}\sum_{r=1}^{3}\int_{\xi=-1}^{\xi=+1} W_{i\alpha}(\mathbf{x}',\mathbf{x})\hat{P}_{l\alpha}^n\Phi^r(\xi)J_k(\xi)d\Gamma \\
& + \sum_{k=1}^{N_e}\sum_{r=1}^{3}\int_{\xi=-1}^{\xi=+1} P_{i\alpha}(\mathbf{x}',\mathbf{x})\hat{W}_{l\alpha}^n\Phi^r(\xi)J_k(\xi)d\Gamma\Big]
\end{aligned}
\tag{16}
$$

Similarly, equation (9) can be written as:

$$
\begin{aligned}
& c_{\theta\alpha}(\mathbf{x}')u_{\alpha}(\mathbf{x}') + \sum_{n=1}^{N_e}\sum_{m=1}^{3} u_{\alpha}^{nm}\int_{\xi=-1}^{\xi=+1} T_{\theta\alpha}(\mathbf{x}',\mathbf{x})\Phi^{m}(\xi)J_n(\xi)d\xi \\
& = -\sum_{k=1}^{N_c} B[\kappa_{\alpha\beta}(1-\nu)+\nu\delta_{\alpha\beta}\kappa_{\gamma\gamma}]w_3^k \int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1} U_{\theta\alpha,\beta}(\mathbf{x}',\mathbf{X})J_k^c(\xi,\eta)d\xi d\eta \\
& + \sum_{n=1}^{NDR}\ddot{\gamma}_l^n(t)\Big[c_{i\alpha}\hat{W}_{l\alpha}^n - \sum_{k=1}^{N_e}\sum_{r=1}^{3}\int_{\xi=-1}^{\xi=+1} W_{i\alpha}(\mathbf{x}',\mathbf{x})\hat{P}_{l\alpha}^n\Phi^r(\xi)J_k(\xi)d\Gamma \\
& + \sum_{n=1}^{NDR}\ddot{\chi}_l^n(t)\Big[c_{\theta\alpha}\hat{U}_{l\alpha}^n - \sum_{k=1}^{N_e}\sum_{r=1}^{3}\int_{\xi=-1}^{\xi=+1} U_{\theta\alpha}(\mathbf{x}',\mathbf{x})\hat{T}_{l\alpha}^n\Phi^r(\xi)J_k(\xi)d\Gamma \\
& + \sum_{k=1}^{N_e}\sum_{r=1}^{3}\int_{\xi=-1}^{\xi=+1} T_{\theta k}(\mathbf{x}',\mathbf{x})\hat{U}_{l\alpha}^n\Phi^r(\xi)J_k(\xi)d\Gamma\Big]
\end{aligned}
\tag{17}
$$

In these equations, N_e and N_c are number of boundary elements and internal cells respectively; n_α are the boundary normal vector components; J represents the Jacobian of the transformation $d\Gamma = J(\xi)d\xi$ for boundary elements; J_r^c is the Jacobian of the transformation $d\Omega_c = J(\xi,\eta)d\xi d\eta$ for cells; ξ and η are the local coordinates; Φ^r with $(r = 1,2,3)$ are the quadratic shape functions. In order to obtain $\ddot{\alpha}_l^m$, $\ddot{\beta}_l^m$, $\ddot{\gamma}_l^m$ and $\ddot{\chi}_l^m$ relation (15) are inverted and replaced into Equations (**??**) and (17). Applying this equation at each collocation point, these equations will give the following linear system of equations:

$$
\begin{bmatrix} \mathbf{M}^b & \mathbf{M}^u \\ \mathbf{M}^w & \mathbf{M}^p \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{w}} \\ \ddot{\mathbf{u}} \end{Bmatrix} + \begin{bmatrix} \mathbf{H}^b & \mathbf{H}^u \\ \mathbf{H}^w & \mathbf{H}^p \end{bmatrix} \begin{Bmatrix} \mathbf{w} \\ \mathbf{u} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^b & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^p \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ \mathbf{t} \end{Bmatrix} + \begin{Bmatrix} \mathbf{q} \\ \mathbf{0} \end{Bmatrix} \tag{18}
$$

where $\mathbf{u} = \{u_1,u_2\}^T$, $\mathbf{w} = \{w_1,w_2,w_3\}^T$, $\mathbf{p} = \{p_1,p_2,p_3\}^T$ and $\mathbf{t} = \{t_1,t_2\}^T$ are displacement and traction vectors for plane stress and plate bending formulations, respectively; $\ddot{\mathbf{u}} = \{\ddot{u}_1,\ddot{u}_2\}^T$, $\ddot{\mathbf{w}} = \{\ddot{w}_1,\ddot{w}_2,\ddot{w}_3\}^T$ are the in-plane and bending acceleration vectors; $\mathbf{q} = \{0,0,q_0\}^T$ is the domain load vectors; $\mathbf{M}^s$, $\mathbf{M}^p$ are mass matrices for plane stress and plate bending formulations, respectively; $\mathbf{H}^p$, $\mathbf{H}^p$, $\mathbf{G}^b$ and $\mathbf{G}^b$ are boundary element influence matrices for plane stress and plate bending formulations, respectively, and $\mathbf{M}^u$, $\mathbf{M}^w$, $\mathbf{H}^u$, $\mathbf{H}^w$ are matrices which contain coupled terms between shell bending and plane stress formulations.

Equations (18) can be rewritten:

$$
\mathbf{M}\ddot{\mathbf{w}}_s + \mathbf{H}\mathbf{w}_s = \mathbf{G}\mathbf{p}_s + \mathbf{f}_s \tag{19}
$$

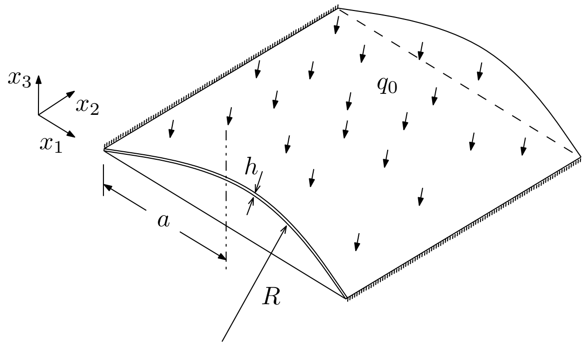

Figure 2: Rectangular shallow shell geometry

where, $\ddot{\mathbf{w}}_s = \{\ddot{\mathbf{w}}, \ddot{\mathbf{u}}\}^T$, $\mathbf{w}_s = \{\mathbf{w}, \mathbf{u}\}^T$, $\mathbf{p}_s = \{\mathbf{p}, \mathbf{t}\}^T$ and $\mathbf{f}_s = \{\mathbf{q}, \mathbf{0}\}^T$, where subindex s intends for shell.

Boundary integral equations discussed above contain integrands with several different orders of singularities. These singular integrals are treated separately based on their order of singularity [Dirgantara (2002)]. In this work, all of the regular integrals are evaluated numerically using the standard Gauss quadrature formula. The influence matrix $\mathbf{G}$ contains weakly singular integrals, which are treated using Telles's nonlinear coordinate transformation method. However, for better numerical accuracy, a suitable number of element sub divisions must be used with the Telles transformation. In this work, four element subdivisions are used. The influence matrix $\mathbf{H}$ contains strongly singular integrals, and in this work these integrals are computed indirectly by considering the generalised rigid body movements.

In order to obtain the harmonic response using equation (19), the system was initially partitioned according to the type of applied boundary condition, and statically condensed in such way that the final equation system could be solved for unknown displacements only [Partridge, Bebbia, and Wrobel (1992)]. Partitioning the boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$, and considering $\mathbf{u}_s = \mathbf{w}_s = 0$ and $\ddot{\mathbf{u}}_s = \ddot{\mathbf{w}}_s = 0$ in Γ_1 and $\mathbf{p}_s = 0$ in Γ_2, equation (19) can be written as:

$$\hat{\mathbf{M}}\ddot{\mathbf{w}}_2 + \hat{\mathbf{H}}\mathbf{w}_2 = \mathbf{f} \tag{20}$$

where,

$$\begin{aligned} \hat{\mathbf{M}} &= \mathbf{M}_{22} - \mathbf{G}_{12}\mathbf{G}_{11}^{-1}\mathbf{M}_{12} \\ \hat{\mathbf{H}} &= \mathbf{H}_{22} - \mathbf{G}_{12}\mathbf{G}_{11}^{-1}\mathbf{H}_{12} \end{aligned} \tag{21}$$

(In these equations subindex s was dropped for the sake of simplicity). Assuming that displacements and distributed forces are harmonic functions of time, τ, they

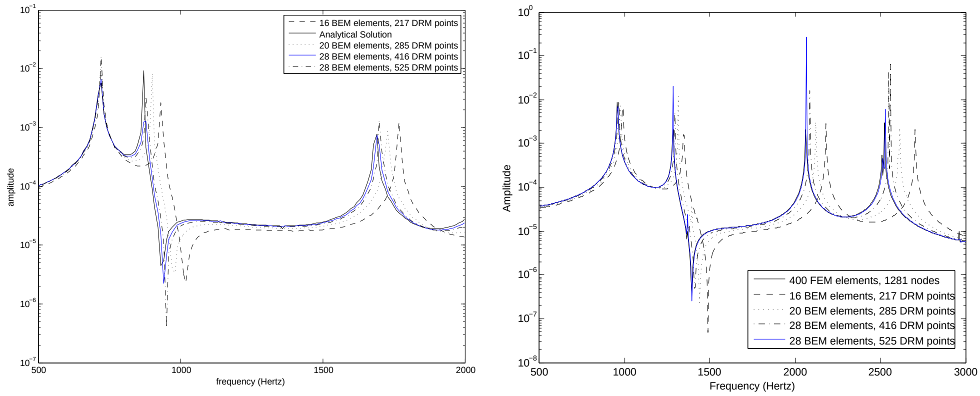

Figure 3: Harmonic analysis of a cylindrical shallow shell. Left: Harmonic response for SSSS boundary conditions. Right: Harmonic response for CCCC boundary conditions.

can be expanded in the form:

$$\mathbf{w}(\tau) = \mathbf{\Phi} e^{-i\omega\tau}, \quad \ddot{\mathbf{w}}(\tau) = -\omega^2 \mathbf{\Phi} e^{-i\omega\tau}, \quad \mathbf{f}(\tau) = \mathbf{F} e^{-i\omega\tau} \tag{22}$$

with ω being frequency of excitation, $\mathbf{\Phi}$ and $\mathbf{F}$ are the amplitude of displacements and distributed forces, respectively. Replacing these expressions into equation (20) we obtain:

$$(\hat{\mathbf{H}} - \omega^2 \hat{\mathbf{M}})\mathbf{\Phi} = \mathbf{F} \tag{23}$$

This equation is solved by incrementing ω in small steps from an initial value. At each step, equation (23) is solved and the plate displacements and rotations for each frequency are obtained.

6 Numerical examples

6.1 Analysis of a cylindrical shallow shell

In this example a cylindrical shallow shell as shown in figure 2 is analysed. The geometric properties of shell are as follows: $a/b = 0.75$, $h/a = 0.0043$, $a/R = 0.1$, $\kappa_{11} = 1/R_{11} = 0.033$, $= \kappa_{22} = 1/R_{22} = 0$ with elastic constants $E = 61670$ MPa, $\nu = 0.3$ and mass density ρ=2.375 $\times 10^{-3}$ Kgm/m^3. Two kinds of boundary conditions were considered here: simply supported (SSSS) and fully clamped (CCCC) at four edges.

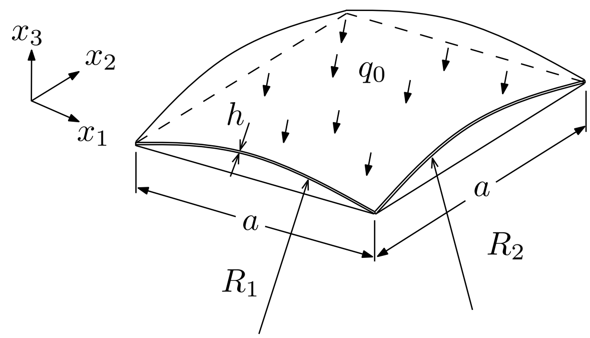

Figure 4: Square shallow spherical shell geometry

A uniform load of 1 N/m^2 is considered as the applied load. Figure 3 shows the convergence analysis for the FRS in the range from 500 to 2000 Hz for SSSS boundary condition and from 500 to 3000 Hz for CCCC condition. FRS were obtained plotting deflection at the center of the shell versus frequency of excitation. The boundary element convergent solution for the first two resonance frequencies was obtained using 28 boundary elements with a total of 525 DRM points. Again, the analysis was also carried out using the commercial code ANSYS®. A finite element convergent solution for the FRS was found with a mesh of 400 elements and 1281 nodes.

An inspection of figure 3 demonstrates a good correlation for the first and second resonance frequencies, between convergent boundary element and finite element solutions for both SSSS and CCCC conditions. For the third resonance frequency a convergent behaviour in BEM solution is observed, however mesh refinement is needed for the SSSS condition.Results obtained using BEM show good agreement with those obtained using the FEM and with the analytical solution presented in [Reddy (2004)].

6.2 Analysis of a square shallow spherical shell

In this example a square shallow spherical shell as shown in figure 4 is analysed. Again, an harmonic uniform load of 1 N/m^2 is considered. The properties of the shell are. $a = b = 1$, $h/b = 0.1$, $a/R = 0.5$, $k_{11} = k_{22} = 0.5$, $R_1/R_2 = 1$. Elastic constants are $E = 210000$ MPa, $\nu = 0.3$ and mass density ρ=5 $\times 10^{-4}$ Kgm/m^3. As in previous examples, two kinds of boundary conditions were considered: simply supported (SSSS) and fully clamped (CCCC) at four edges.

Figure 5 shows the convergence analysis in the range from 0 to 6000 Hz for the FRS. The boundary element convergent solution was obtained using 32 bound-

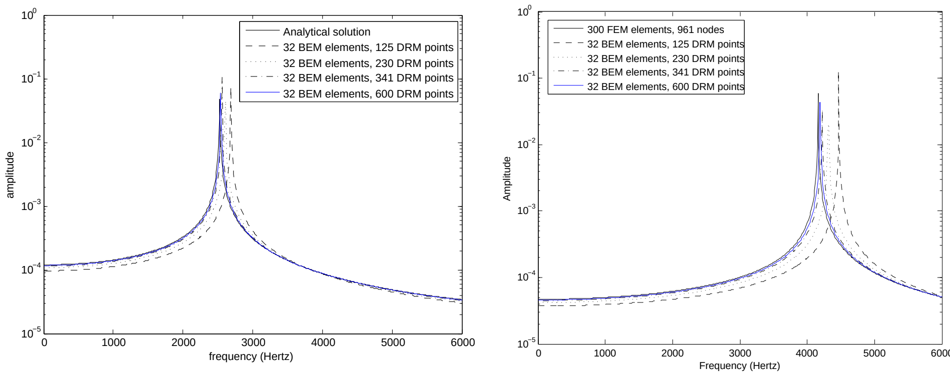

Figure 5: Harmonic analysis of a square shallow spherical shell. Left: Harmonic response for SSSS boundary conditions. Right: Harmonic response for CCCC boundary conditions.

ary elements with a total of 600 DRM points for both SSSS and CCCC boundary conditions. A finite element convergent solution, using ANSYS®, for the FRS was found with a mesh of 300 elements and 961 nodes. An inspection of figure 5 demonstrates a good correlation for the resonance frequency, between convergent boundary element and finite element solutions.BEM results show good agreement with those obtained using the FEM and and with the analytical in [Reddy (2004)].

7 Conclusions

The Boundary Element Method applied to harmonic and modal analysis of shear deformable shallow shells was presented. A boundary element formulation based on a direct time-domain formulation using the elastostatic fundamental solution of the problem was used. Shell was modeled coupling the classical Reissner plate theory and the two-dimensional plane stress elasticity. Effects of shear deformation and rotatory inertia were included in the formulation. The Dual Reciprocity Boundary Element Method was used to treat domain integrals involving inertial mass forces. Integrals involving curvature terms were treated using the Cell Integration Method. Divergence Theorem was used to transform domain integrals related with distributed domain forces. Results show good agreement with those obtained from analytical and finite element models, turning the proposed formulation on an alternative numerical engineering tool for the dynamic analysis of shear deformable shallow shells.

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