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Analytical Solution of Stokes Flow in a Driven Cavity Using the Natural Boundary Element Method

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Abstract: In this paper, the natural boundary element method is used to solve two-dimensional steady-state incompressible Stokes flows in a driven cavity. The analytical functions are expressed for the Stokes problem in an exterior circular domain under single value conditions, which satisfy the Stokes equations' solutions in the form of complex functions. In order to obtain a uniform integral formula, the velocities on the boundary are expanded into Laurent series, and then compared with the analytical solutions obtained as described above. In this manner, the coefficients of the analytical solutions in the form of complex function are further confirmed. According to the formulae of Fourier series and convolutions, the boundary integral formulae related only to boundary velocities are obtained for the Stokes problems in an exterior circular domain. Similarly, the boundary integral formulae are investigated for the interior circular domain. The formulae are applied to the Stokes flows in some circular and annular cavities, and the obtained results are compared with those produced by previous works. The results prove that the current technique is in very good agreement with previous investigations, and that the natural boundary element method is an accurate and flexible method for the solution of Stokes flows.

Keywords: Natural boundary element method, Stokes flow, driven cavity, Fourier series and convolutions, boundary integral formulae.

1 Introduction

Theoretical studies of Stokes flow in a cavity with rotating boundaries provide useful information on flow behavior and distributive mixing, which has many industrial applications. Stokes flow can be considered for a range of engineering

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processes and natural phenomena. Although the flow is dominated only by shear force, the presence of obstacles and the rotation of cylinders produce eddy zones and dead zones, which affect the fluid flow and mixing process. For simple flow configurations involving only a moving wall with a stationary cylinder, the Stokes equations were solved directly by an analytical method [Jeffrey(1980)]. The problems of Stokes flow in a driven cavity possess important theoretical and practical significance. As the flow boundary is clear, the flowing state can be analyzed theoretically. Accordingly, Stokes flow of this sort can be taken as an accepted standard when testing a new computational fluid dynamics (CFD) method. The finite difference method was adopted to study Stokes flow with a single driven plate [Burggraf(1966); Pan and Acrivos (1967)]. Stokes flow in a rectangular cavity was used to test the new CFD method [Eid(2005); Fan and Young (2002)]. The Stokes flow problem with circular boundaries was considered [Chen, Hsiao and Leu(2008)], and the aim was to study biharmonic problems with circular boundaries by using direct and indirect boundary integral equations in conjunction with degenerate kernels, Fourier series, vector decomposition and the adaptive observer frame. A method of fundamental solutions was presented to solve Stokes problems based on the combination of the Laplace equation for velocity potential and vector bi-harmonic equation for stream function vector by using the Helmholtz decomposition theorem[Young Chen and Fan (2005); Young Chiu, Fan, Tsai and Lin(2006)]. The Stokes problems of circular cavity and circular cavity with eccentric rotating cylinder were calculated.

As for computational fluid dynamics, the three most well known numerical schemes are the finite difference method (FDM), finite element method (FEM)[Katsushi and Norikazu(2007)] and boundary element method (BEM) [Primo and Wrobel(2000); Curteanu, Elliott and Ingham(2007); Frangi and Tausch(2005); Frangi(2005); Frangi, Spinola and Vigna(2006); Zhu(1986); Yan(1986)]. For instance, the boundary integral equation method was proposed [Youngren and Acrivos(1975), which is based on the hydrokinetics potential theory proposed by a bas bleu from former Russia. The method can be utilized to deduce Stokes flow problems under arbitrary boundary conditions; nevertheless it must resolve a mass of integrals, the singular integrals among which make the workload increase in size. In recent years, a new computational technique, namely the natural boundary element method (NBEM), has been developed as a new BEM [Yu(1993)], which is a branch of a number of boundary element methods. Based on either the Green functions method or Fourier series method, it induces the Dirichlet value problems of differential equations into the boundary integral formula, or induces Neumann boundary value problems of differential equation into a strong singular boundary integral equation. NBEM has some advantages over traditional computational methods [Liu and Yan(2006);

Mirela(2007); Young, Jane and Lin(2004)]; Grigoriev and Dargush (2005)], as it has direct derivation, unique form, small calculation quantity and energy function being able to remain invariable before and after the boundary reduction [Yu and Zhao(2005); Dong, Li and Yu(2005)]. The method has been used to resolve some engineering problems: the semi-plane elasticity problems for the mine pressure were resolved [Peng, Dong and Li(2005)]; the stress distribution problems of surrounding rock of circular roadways were deduced [Li, Dong and Ma(2011)]; the bending deflection problems for infinite plates with a unit circle under the boundary load were obtained [Liu, Dong and Li(2009)]; the coupling method for torsion problems of the square cross-section bar with cracks were settled [Zhao, Dong and Cao(2000)]; and Stokes problems were studied preliminarily [Peng, Dong and Zhao(2006); Peng, Dong and Cao(2008); Peng, Cao, Dong and Li(2011)].

The purpose of this paper is to study Stokes problems in the circular and annular cavities using the natural boundary element method. The analytical functions are expressed for the Stokes problem in an exterior circular domain under single value conditions, which satisfy the Stokes equations' solutions in the form of complex functions. Accordingly, the boundary integral formulae related only to boundary velocities could be obtained for the Stokes problems in an exterior and interior circular domain with the Fourier series method. The formulae can be used to calculate any Stokes flow in a driven cavity. Finally, several examples are presented to show the simplicity and accuracy of the proposed scheme.

2 Analytical functions of the exterior circular domain

The Dirichlet boundary problems of Stokes equations for an incompressible flow in an exterior circular domain Ω with the smooth boundary Γ can be represented by the continuity and momentum equations as follows:

$$\begin{cases} -\mu\Delta\vec{u} + \nabla p = 0, & \text{in } \Omega, \\ \nabla\cdot\vec{u} = 0, & \text{in } \Omega, \\ \vec{u} = \vec{u}_0, & \text{on } \Gamma, \end{cases} \tag{1}$$

where μ is the dynamic viscosity; the symbols Δ, ∇ and $\nabla\cdot$ stand for the Laplacian, gradient and divergence operators, respectively; $\vec{u} = (u_x, u_y)$ denotes the velocity vector; p is the pressure; and $\vec{u}_0$ is the given function on Γ.

The velocity vectors and stress tensors of Eq.1 in Ω can be expressed as the following real and imaginary parts of the complex functions [Yu(1993)]:

$$\begin{cases} u_x(x,y) = Re[-\varphi'(z)\bar{z} + \varphi(z) - \psi(z)], \\ u_y(x,y) = Im[\varphi'(z)\bar{z} + \varphi(z) + \psi(z)], \end{cases} \tag{2}$$

$$\begin{cases} \sigma_x(x,y) = 2\mu Re[2\varphi'(z) - \varphi''(z)\bar{z} - \psi'(z)], \\ \sigma_y(x,y) = 2\mu Re[2\varphi'(z) + \varphi''(z)\bar{z} + \psi'(z)], \\ \tau_{xy}(x,y) = 2\mu Im[\varphi''(z)\bar{z} + \psi'(z)]. \end{cases} \tag{3}$$

where $z, \bar{z} \in \Omega$, $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$; $\varphi(z), \psi(z)$ are the analytical functions in Ω; and $\varphi'(z)$, $\varphi''(z)$ and $\psi'(z)$ are the derivative functions of $\varphi(z)$ and $\psi(z)$, respectively.

Since the exterior circular domain is multi-connective, the analytical functions $\varphi(z), \psi(z)$ could be multi-values. Commonly, the multi- connectivity may process m inner boundaries. As the Stokes problems in the circular and annular cavities are studied in this paper, one inner boundary and one outer boundary are considered.

According to Eq.3, the stress components can be expressed as follows:

$$\sigma_x + \sigma_y = 8\mu Re\varphi'(z). \tag{4}$$

Under the single value condition in a multi-connectivity region, the real part of $\varphi'(z)$must be single-valued, whereas the imaginary part can be multiple valued. Suppose the increment of imaginary number is $2\pi A\mathrm{i}$ when making one circuit around the inner boundary:

$$\varphi'(z) = A\ln(z - z_0) + \varphi'_*(z), \tag{5}$$

where A is real constant coefficients, z_0 is the arbitrary point of the exterior inner boundary, and $\varphi'_*(z)$ is a single-valued and analytic function of the multi-connectivity domain.

Accordingly, the analytical functions $\varphi(z)$ can be acquired as follows:

$$\varphi(z) = Az\ln(z - z_0) + \gamma\ln(z - z_0) + \varphi_*(z), \tag{6}$$

where $\varphi_*(z)$ is a single-valued and analytic function of the multi-connectivity domain, and $\gamma = \alpha + \mathrm{i}\beta$, and α, β are arbitrary real constants.

Likewise, $\psi(z)$ can be obtained as follows:

$$\psi(z) = \gamma'\ln(z - z_0) + \psi_*(z), \tag{7}$$

where $\psi_*(z)$ is a single-valued and analytic function of the multi-connectivity region, and γ' is a constant, commonly a complex number.

According to Eq.2, the velocity variables are satisfied with the following equation:

$$u_x + \mathrm{i}u_y = \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{8}$$

Substituting Eqs.5-7 into Eq.8, and utilizing the single value conditions of the velocities, the following conditions should be tenable:

$$A = 0, \quad \gamma + \overline{\gamma'} = 0. \tag{9}$$

Therefore, $\varphi(z)$ and $\psi(z)$ can be deduced preliminarily as follows:

$$\begin{cases} \varphi(z) = (\alpha + \mathrm{i}\beta)\ln(z - z_0) + \varphi_*(z), \\ \psi(z) = (-\alpha + \mathrm{i}\beta)\ln(z - z_0) + \psi_*(z). \end{cases} \tag{10}$$

As for the infinite multi-connectivity, the analytical functions $\varphi(z), \psi(z)$ should be finite. It can be seen from Eq.10 that the following is true:

$$\ln(z - z_0) = \ln z + \ln(1 - \frac{z_0}{z}) = \ln z - \frac{z_0}{z} - \frac{1}{2}(\frac{z_0}{z})^2 - \cdots. \tag{11}$$

Correspondingly, in order to satisfy the finite property, $\varphi(z)$ and $\psi(z)$ can be written as follows:

$$\begin{cases} \varphi(z) = (\alpha + \mathrm{i}\beta)\ln(z) + \varphi_{**}(z) \\ \psi(z) = (-\alpha + \mathrm{i}\beta)\ln(z) + \psi_{**}(z) \end{cases}, \tag{12}$$

where $\varphi_{**}(z)$ and $\psi_{**}(z)$ are the complex functions in an exterior circular domain, which are analytical and can be expanded into Laurent series:

$$\varphi_{**}(z) = \sum_{-\infty}^{\infty} a_n z^n, \; \psi_{**}(z) = \sum_{-\infty}^{\infty} b_n z^n, \tag{13}$$

where $n = 0, 1, 2, \cdots, a_n$ and b_n are complex coefficients.
Substituting the first formula of Eqs.12-13 into Eq.4, we obtain the following:

$$\sigma_x + \sigma_y = 4\mu[\frac{\alpha + \mathrm{i}\beta}{z} + \frac{\overline{\alpha + \mathrm{i}\beta}}{\bar{z}} + \sum_{-\infty}^{\infty} n(a_n z^{n-1} + \overline{a_n z^{n-1}})]. \tag{14}$$

As for infinite domain, reviewing the right part of Eq.14, the terms of the high-level power of z rapidly become increscent. The stress tensors could not be infinite, thus the following terms must be zero:

$$a_n = 0, \quad b_n = 0, (n \geq 2). \tag{15}$$

After this stage, the complex functions $\varphi(z)$ and $\psi(z)$ can be expressed as follows:

$$\begin{cases} \varphi(z) = (\alpha + \mathrm{i}\beta)\ln(z) + (\alpha' + \mathrm{i}\beta')z + \varphi_0(z), \\ \psi(z) = (-\alpha + \mathrm{i}\beta)\ln(z) + (\alpha'' + \mathrm{i}\beta'')z + \psi_0(z), \end{cases} \tag{16}$$

where $\alpha', \alpha'', \beta'and\beta''$ are real constants, and $\varphi_0(z)$ and $\psi_0(z)$ are analytical functions:

$$\begin{cases} \varphi_0(z) = \sum\limits_0^\infty a_{-n} z^{-n}, \\ \psi_0(z) = \sum\limits_0^\infty b_{-n} z^{-n}, \end{cases} \tag{17}$$

where a_{-n} and b_{-n} are complex coefficients, $n = 0, 1, 2, \cdots$. The following can be yielded by Eq.2:

$$\begin{aligned} u_x + \mathrm{i}u_y &= \phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \\ &= (\alpha + \mathrm{i}\beta)\ln(\mathrm{z}) + (\alpha' + \mathrm{i}\beta')z + \phi_0(z) \\ &\quad - z\overline{[\frac{\alpha + \mathrm{i}\beta}{z} + (\alpha' + \mathrm{i}\beta') + \phi_0'(z)]} \\ &\quad - \overline{[(-\alpha + \mathrm{i}\beta)\ln(z) + (\alpha'' + \mathrm{i}\beta'')z + \psi_0(z)]}. \end{aligned} \tag{18}$$

The velocity vectors could not be infinite, thus the coefficients α, β, β', and $\alpha'' + \mathrm{i}\beta''$ should be zero in Eq.18. Accordingly, the analytical functions of the infinite exterior circular region can be deduced, which satisfy the complex functions of the Stokes equations' solution:

$$\begin{cases} \varphi(z) = \alpha' z + \varphi_0(z), \\ \psi(z) = \psi_0(z). \end{cases} \tag{19}$$

As for a finite region, the velocity vectors and stress tensors satisfy the characteristic of finiteness. Then, the analytical functions of finite exterior circular domain can be deduced additionally as follows [Peng, Dong and Cao(2008)]:

$$\begin{cases} \varphi(z) = \alpha' z + \varphi_0(z) + \varphi_1(z), \\ \psi(z) = \psi_0(z) \ + \psi_1(z). \end{cases} \tag{20}$$

where $\varphi_1(z) = \alpha \ln(z) + \alpha_1 z^2$; $\psi_1(z) = -\alpha \ln(z)$; and α, α_1 are real constants.

3 Application of the natural boundary element method (NBEM)

3.1 Boundary integral formula of an annular domain

In order to obtain uniform boundary integral formula for Stokes problems of the exterior circular region, the Stokes equations are calculated according to the analytic functions $\varphi(z)$ and $\psi(z)$ of the finite region. For briefness, Let $z = r\mathrm{e}^{\mathrm{i}\theta}$ hereinafter, in which r and θ stand for the radial and circumferential coordinates. As Eq.2 shows, the solutions of Stokes equations are described as the real and image parts of the complex functions. The sum of a pair complex functions conjugated divided by 2 is their real part, and the difference divided by 2i is their image part. Therefore, suppose that $a_n = \bar{a}_{-n} b_n = \overline{b}_{-n}$ of Eq.17, and substitute Eq.20 into Eq.2, then the velocity equations in an exterior circular domain can be obtained according to the real and image part characteristics of the complex functions:

$$\begin{aligned} u_x(r,\theta) =& Re(a_0 - b_0) + \frac{1}{2r}[(a_{-1} - b_{-1})e^{-\mathrm{i}\theta} + (a_1 - b_1)e^{\mathrm{i}\theta}] \\ &+ \frac{1}{2}\sum_{\substack{-\infty \\ n\neq 0,\pm 1}}^{\infty}[(|n|\text{-}2)a_{(|n|-2)signn}r^2 + (a_n - b_n)]r^{-|n|}e^{\mathrm{i}n\theta} \\ &+ Re[-\phi_1'(z)\bar{z} + \phi_1(z) - \psi_1(z)], \end{aligned} \tag{21}$$

$$\begin{aligned} u_y(r,\theta) =& Im(a_0 + b_0) + \frac{\mathrm{i}}{2r}[(a_1 + b_1)e^{\mathrm{i}\theta} - (a_{-1} + b_{-1})e^{-\mathrm{i}\theta}] \\ &+ \frac{\mathrm{i}}{2}\sum_{\substack{-\infty \\ n\neq 0,\pm 1}}^{\infty}[(2 - |n|)\,a_{(|n|-2)signn}r^2 + (a_n + b_n)]r^{-|n|}e^{\mathrm{i}n\theta} \\ &+ Im[\phi_1'(z)\bar{z} + \phi_1(z) + \psi_1(z)], \end{aligned} \tag{22}$$

where

$$Re[-\varphi_1'(z)\bar{z} + \varphi_1(z) - \psi_1(z) = [\alpha_1 r^2 - \alpha]\cos(2\theta) + \alpha\ln(r^2) - 2\alpha_1 r^2;$$

$$Im[\varphi_1'(z)\bar{z} + \varphi_1(z) + \psi_1(z)] = (\alpha_1 r^2 - \alpha)\sin(2\theta).$$

Substituting $r = 1$ into Eqs.21-22, the expressions of the boundary velocities in the exterior unit circle are deduced. Furthermore, suppose the boundary velocities in the exterior unit circle are described as follows:

$$\begin{cases} u_x(1,\theta) = \sum\limits_{-\infty}^{\infty} c_n e^{\mathrm{i}n\theta},\ c_n = \overline{c}_{-n}, \\ u_y(1,\theta) = \sum\limits_{-\infty}^{\infty} d_n e^{\mathrm{i}n\theta},\ d_n = \overline{d}_{-n}, \end{cases} \tag{23}$$

where c_0, d_0 are real constants; c_i, d_i are complex constants; and $i \neq 0$. Comparing Eq.23 with the boundary velocity of an exterior unit circle, and noting that $\cos 2\theta = \frac{e^{\mathrm{i}2\theta} + e^{-\mathrm{i}2\theta}}{2}$, $\sin 2\theta = \frac{e^{\mathrm{i}2\theta} - e^{-\mathrm{i}2\theta}}{2\mathrm{i}}$, the following equations are obtained:

$$\begin{cases} c_0 = Re(a_0 - b_0) - 2\alpha_1, \\ c_1 = \frac{a_1}{2} - \frac{b_1}{2}, \\ c_2 = \frac{a_2}{2} - \frac{b_2}{2} + \frac{\alpha_1 - \alpha}{2}, \\ c_n = \frac{1}{2}[(|n| - 2)a_{(|n|-2)\mathrm{sign}n} + a_n - b_n], \end{cases}$$

$$\begin{cases} d_0 = Im(a_0 - b_0), \\ d_1 = \mathrm{i}\frac{a_1}{2} + \mathrm{i}\frac{b_1}{2}, \\ d_2 = \mathrm{i}\frac{a_2}{2} + \mathrm{i}\frac{b_2}{2} + \frac{\alpha_1 - \alpha}{2\mathrm{i}}, \\ d_n = \frac{\mathrm{i}}{2}[(2-|n|)a_{(|n|-2)\mathrm{sign}n} + a_n + b_n]. \end{cases}$$

By the above two equations, the coefficients can be obtained as follows:

$$\begin{cases} Re(a_0 - b_0) = c_0 + 2\alpha_1, \\ a_1 = c_1 - \mathrm{i}d_1, \\ a_2 = c_2 - \mathrm{i}d_2, \\ a_n = c_n - \mathrm{i}d_n, \end{cases}$$

$$\begin{cases} Im(a_0 + b_0) = d_0, \\ b_1 = -(c_1 + \mathrm{i}d_1), \\ b_2 = -(c_1 + \mathrm{i}d_1) + (\alpha_1\text{-}\alpha), \\ b_n = [(n-2)c_{n-2} - c_n] - \mathrm{i}[d_n + (n-2)d_{n-1}], \end{cases}$$

where $n = 3, 4, \cdots$; $a_n = \bar{a}_{-n}$; $b_n = \overline{b}_{-n}$; $c_n = \overline{c}_{-n}$; $d_n = \overline{d}_{-n}$.

Substituting the coefficients into Eqs.21-22, the velocities are obtained as follows:

$$\begin{aligned} u_x(r,\theta) = & \sum_{-\infty}^{\infty} c_n r^{-|n|} e^{\mathrm{i}n\theta} + (1 - \frac{1}{r^2})\{\frac{1}{2}\cos 2\theta \sum_{-\infty}^{\infty} (|n| c_n - \mathrm{i}nd_n) r^{-|n|} e^{\mathrm{i}n\theta} \\ & + \frac{1}{2}\sin 2\theta \times \sum_{-\infty}^{\infty} (\mathrm{i}nc_n + |n| d_n) r^{-|n|} e^{\mathrm{i}n\theta}\} + u'_x(r,\theta), \end{aligned} \tag{24}$$

$$\begin{aligned} u_y(r,\theta) = & \sum_{-\infty}^{\infty} d_n r^{-|n|} e^{\mathrm{i}n\theta} + (1 - \frac{1}{r^2})\{\frac{1}{2}\sin 2\theta \sum_{-\infty}^{\infty} (|n| c_n - \mathrm{i}nd_n) r^{-|n|} e^{\mathrm{i}n\theta} \\ & - \frac{1}{2}\cos 2\theta \times \sum_{-\infty}^{\infty} (\mathrm{i}nc_n + |n| d_n) r^{-|n|} e^{\mathrm{i}n\theta}\} + u'_y(r,\theta), \end{aligned} \tag{25}$$

where

$$u'_x(r,\theta) = [(\alpha - \alpha_1) r^{-2} + \alpha_1 r^2 - \alpha] \cos(2\theta) + 2\alpha_1(1 - r^2) + \alpha \ln(r^2);$$

$$u'_y(r,\theta) = [(\alpha - \alpha_1) r^{-2} + \alpha_1 r^2 - \alpha] \sin(2\theta).$$

Utilizing the following convolution integral properties: if $u = \sum_{-\infty}^{\infty} a_n e^{\mathrm{i}n\theta}$, $v = \sum_{-\infty}^{\infty} b_n e^{\mathrm{i}n\theta}$, that $u * v = \sum_{-\infty}^{\infty} (2\pi a_n b_n) e^{\mathrm{i}n\theta}$, and using the basic formula of the Fourier series:

$$\frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{-|n|} e^{\mathrm{i}n\theta} = \frac{r^2 - 1}{2\pi(1 + r^2 - 2r\cos\theta)} \equiv P(r,\theta), r > 1,$$

the boundary integral formulae in an exterior circular domain with radius Rin a Cartesian coordinate system are deduced as follows:

$$\begin{aligned} u_x(r,\theta) =& P(r,\theta) * u_x(R,\theta) + \frac{r^2-R^2}{2r^2}\{\cos 2\theta[(-r\frac{\partial}{\partial r}P(r,\theta)) \\ & * u_x(R,\theta) - \frac{\partial}{\partial\theta}P(r,\theta) * u_y(R,\theta)] + \sin 2\theta[\frac{\partial}{\partial\theta}P(r,\theta) \\ & * u_x(R,\theta) + (-r\frac{\partial}{\partial r}P(r,\theta)) * u_y(R,\theta)]\} + [(\alpha-\alpha_1)r^{-2} \\ & + \alpha_1 r^2 - \alpha]\cos(2\theta) + 2\alpha_1(1-r^2) + \alpha\ln(r^2), \end{aligned} \tag{26}$$

$$\begin{aligned} u_y(r,\theta) =& P(r,\theta) * u_y(R,\theta) + \frac{r^2-R^2}{2r^2}\{\sin 2\theta[(-r\frac{\partial}{\partial r}P(r,\theta)) \\ & * u_x(R,\theta) - \frac{\partial}{\partial\theta}P(r,\theta) * u_y(R,\theta)] - \cos 2\theta[\frac{\partial}{\partial\theta}P(r,\theta) \\ & * u_x(R,\theta) + (-r\frac{\partial}{\partial r}P(r,\theta)) * u_y(R,\theta)]\} \\ & + [(\alpha-\alpha_1)r^{-2} + \alpha_1 r^2 - \alpha]\sin(2\theta). \end{aligned} \tag{27}$$

Finally, using coordinate transforming expressions, the boundary integral formula in an exterior circular domain with radius R in a polar coordinate system are easily obtained as follows:

$$\begin{aligned} u_r(r,\theta) =& \{\cos\theta P(r,\theta) + \frac{r^2-R^2}{2r^2}[\cos\theta(-r\frac{\partial}{\partial r}P(r,\theta)) \\ & + \sin\theta\frac{\partial}{\partial\theta}P(r,\theta)]\} * u_r(R,\theta) + \{\sin\theta P(r,\theta) + \frac{r^2-R^2}{2r^2} \\ & \times [\sin\theta(-r\frac{\partial}{\partial r}P(r,\theta)) - \cos\theta\frac{\partial}{\partial\theta}P(r,\theta)]\} * u_\theta(R,\theta) \\ & + [(\alpha-\alpha_1)r^{-2} - \alpha_1 r^2 - \alpha + 2\alpha_1 + \alpha\ln(r^2)]\cos(\theta), \end{aligned} \tag{28}$$

$$\begin{aligned} u_\theta(r,\theta) =& \{-\sin\theta P(r,\theta) + \frac{r^2-R^2}{2r^2}[\sin\theta(-r\frac{\partial}{\partial r}P(r,\theta)) \\ & - \cos\theta\frac{\partial}{\partial\theta}P(r,\theta)]\} * u_r(R,\theta) + \{\cos\theta P(r,\theta) - \frac{r^2-R^2}{2r^2} \\ & \times [\cos\theta(-r\frac{\partial}{\partial r}P(r,\theta)) + \sin\theta\frac{\partial}{\partial\theta}P(r,\theta)]\} * u_\theta(R,\theta) \\ & - [(\alpha_1-\alpha)r^{-2} - 3\alpha_1 r^2 + \alpha + 2\alpha_1 + \alpha\ln(r^2)]\sin(\theta), \end{aligned} \tag{29}$$

where * is the convolution integral.

According to the above two formulae, the Stokes problem can be calculated in an annular domain, and the coefficients α, α_1 can be computed by the boundary conditions.

3.2 Boundary integral formula of a circular domain

Likewise, the boundary integral formulae of the interior circular domain with radius R in a Cartesian coordinate system can be obtained with the Fourier series method:

$$\begin{aligned}u_x(r,\theta) =&P'(r,\theta)*u_x(R,\theta)+\frac{R^2-r^2}{2r^2}\{\cos 2\theta[(r\frac{\partial}{\partial r}P'(r,\theta))\\&*u_x(R,\theta)+\frac{\partial}{\partial\theta}P'(r,\theta)*u_y(R,\theta)]+\sin 2\theta[-\frac{\partial}{\partial\theta}P'(r,\theta)\\&*u_x(R,\theta)+(r\frac{\partial}{\partial r}P'(r,\theta))*u_y(R,\theta)]\}-\frac{R^2-r^2}{2\pi Rr}\times\\&\int_0^{2\pi}[u_x(R,\theta')\cos(\theta+\theta')+u_y(R,\theta')\sin(\theta+\theta')]\mathrm{d}\theta',\end{aligned} \tag{30}$$

$$\begin{aligned}u_y(r,\theta) =&P'(r,\theta)*u_y(R,\theta)+\frac{R^2-r^2}{2r^2}\{\sin 2\theta[(r\frac{\partial}{\partial r}P'(r,\theta))\\&*u_x(R,\theta)+\frac{\partial}{\partial\theta}P'(r,\theta)*u_y(R,\theta)]-\cos 2\theta[-\frac{\partial}{\partial\theta}P'(r,\theta)\\&*u_x(R,\theta)+(r\frac{\partial}{\partial r}P'(r,\theta))*u_y(R,\theta)]\}+\frac{R^2-r^2}{2\pi Rr}\times\\&\int_0^{2\pi}[-u_x(R,\theta')\sin(\theta+\theta')+u_y(R,\theta')\cos(\theta+\theta')]\mathrm{d}\theta'.\end{aligned} \tag{31}$$

Using coordinate transforming expressions, the boundary integral formulae of the interior circular region with radius R in a polar coordinate system are acquired:

$$\begin{aligned}u_r(r,\theta) =&\{\cos\theta P'(r,\theta)+\frac{R^2-r^2}{2r^2}[\cos\theta(r\frac{\partial}{\partial r}P'(r,\theta))\\&-\sin\theta\frac{\partial}{\partial\theta}P'(r,\theta)]-\frac{R^2-r^2}{2\pi Rr}\}*u_r(R,\theta)+\{\sin\theta P'(r,\theta)\\&+\frac{R^2-r^2}{2r^2}[\sin\theta(r\frac{\partial}{\partial r}P'(r,\theta))+\cos\theta\frac{\partial}{\partial\theta}P'(r,\theta)]\}*u_\theta(R,\theta),\end{aligned} \tag{32}$$

$$\begin{aligned}u_\theta(r,\theta) =&\{-\sin\theta P'(r,\theta)+\frac{R^2-r^2}{2r^2}[\sin\theta(r\frac{\partial}{\partial r}P'(r,\theta))\\&+\cos\theta\frac{\partial}{\partial\theta}P'(r,\theta)]\}*u_r(R,\theta)+\{\cos\theta P'(r,\theta)-\frac{R^2-r^2}{2r^2}\\&\times[\cos\theta(r\frac{\partial}{\partial r}P'(r,\theta))-\sin\theta\frac{\partial}{\partial\theta}P'(r,\theta)]+\frac{R^2-r^2}{2\pi Rr}\}*u_\theta(R,\theta),\end{aligned} \tag{33}$$

where $P'(r,\theta)=\frac{R^2-r^2}{2\pi[R^2+r^2-2rR\cos\theta]}$.

4 Model verifications and application

4.1 Model verifications

The boundary integral formulae related only to boundary velocities with the NBEM obtained above are applied to test two problems to verify the feasibility and accuracy of the method. The test problems are the classical problems for which many analytical and numerical model results are available in literature concerning 2D Stokes flows. All of the examples adopt the international system of units.

4.1.1 Stokes flow in an annular cavity with rotating boundaries

The first model problem is a rotating cylinder with unit velocity in the incompressible viscous fluid. For the sake of convenience, the radius of the cylinder is taken as 1. This is a Stokes problem in an annular cavity, and the velocity boundary conditions of the inner boundary can be described as $u_r(1,\theta)=0$, $u_\theta(1,\theta)=1$. Thereby, the Stokes problem can be calculated directly using the boundary integral formulae (28)-(29). The velocity fields can be expressed analytically as follows:

$$\begin{aligned}
u_r(r,\theta) =&\{\sin\theta P(r,\theta)+\frac{r^2-1}{2r^2}[\sin\theta(-r\frac{\partial}{\partial r}P(r,\theta))-\cos\theta\frac{\partial}{\partial\theta}P(r,\theta)]\}*1\\
&=\int_0^{2\pi}\sin(\theta-\theta')\cdot\frac{r^2-1}{2\pi(1+r^2-2r\cos(\theta-\theta'))}\mathrm{d}\theta'\\
&-\frac{r^2-1}{2r}\cdot\int_0^{2\pi}\sin(\theta-\theta')\cdot\left\{\frac{r}{\pi[r^2-2r\cos(\theta-\theta')+1]}\right.\\
&\left.+\frac{[2\cos(\theta-\theta')-2r]\cdot(r^2-1)}{2\pi[r^2-2r\cos(\theta-\theta')+1]^2}\right\}\mathrm{d}\theta'+\frac{r^2-1}{2r^2}\cdot\\
&\int_0^{2\pi}\cos(\theta-\theta')\cdot\frac{r\sin(\theta-\theta')\cdot(r^2-1)}{\pi[r^2-2r\cos(\theta-\theta')+1]^2}\mathrm{d}\theta'\ ,
\end{aligned}\tag{34}$$

$$\begin{aligned}
u_\theta(r,\theta) =&\{\cos\theta P(r,\theta)-\frac{r^2-1}{2r^2}[\cos\theta(-r\frac{\partial}{\partial r}P(r,\theta))+\sin\theta\frac{\partial}{\partial\theta}P(r,\theta)]\}*1\\
&=\int_0^{2\pi}\cos(\theta-\theta')\cdot\frac{r^2-1}{2\pi(1+r^2-2r\cos(\theta-\theta'))}\mathrm{d}\theta'\\
&+\frac{r^2-1}{2r}\cdot\int_0^{2\pi}\cos(\theta-\theta')\cdot\left\{\frac{r}{\pi[r^2-2r\cos(\theta-\theta')+1]}\right.\\
&\left.+\frac{[2\cos(\theta-\theta')-2r]\cdot(r^2-1)}{2\pi[r^2-2r\cos(\theta-\theta')+1]^2}\right\}\mathrm{d}\theta'-\frac{r^2-1}{2r^2}\cdot\\
&\int_0^{2\pi}\sin(\theta-\theta')\cdot\frac{r\sin(\theta-\theta')\cdot(r^2-1)}{\pi[r^2-2r\cos(\theta-\theta')+1]^2}\mathrm{d}\theta'.
\end{aligned}\tag{35}$$

Based on Eqs.34-35, the velocity vector near the inner boundary is shown in Fig. 1, and the distribution of tangential velocity varied with the radial position r is depicted in Fig. 2. In order to make the figure clear, the range of the radial position in this figure ranges from 1 to 50. Good agreement is observed when comparing with the analytical solution w^2R/r, where w is the rotational angular velocity of the cylinder.

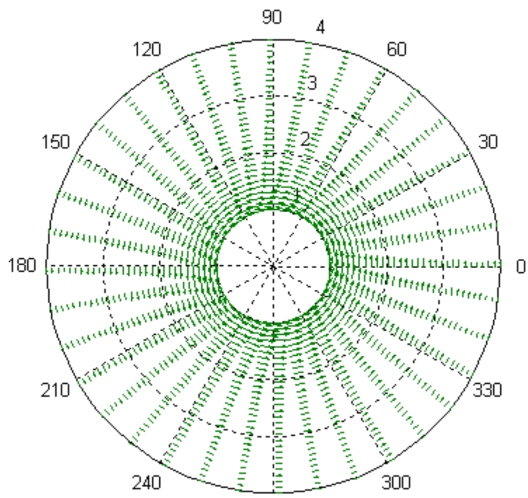

Figure 1: Velocity vector.

4.1.2 *Stokes flow in a circular cavity with a counter-clockwise driving*

The second model problem consists of a recirculating flow in a 2D circular cavity. The radius of the circular cavity is assumed as 1. The configuration and boundary conditions of this problem are depicted in Fig. 3. In the upper half of the boundary, the velocity $u_\theta = 1$ in a counter-clockwise direction and $u_r = 0$are prescribed. In the lower half, the tangential velocity and radial velocity are taken as 0.

The velocity vector is observed in Fig. 4. The present computations give very good results as compared to the other solutions. The curves of the tangential velocities along the radial position r at different directions are depicted as Fig. 5, and the tangential velocities varied with the direction θ at different radial positions are depicted as Fig. 6. Likewise, Fig. 7 depicts the radial velocity profile varied with the radial position r at different directions, and Fig. 8 describes the tangential velocity profile varied with the direction θ at different radial positions.

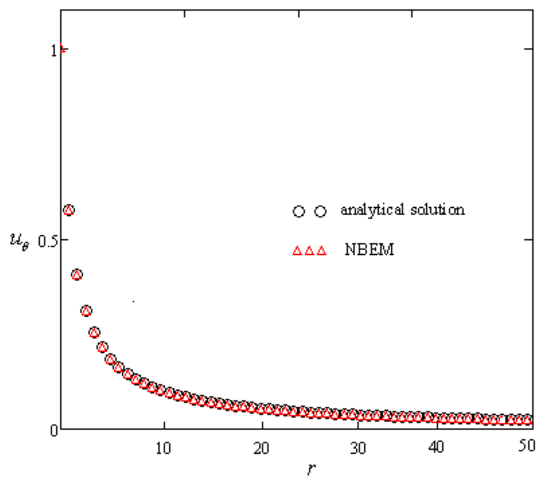

Figure 2: Tangential velocity varied with the radial position.

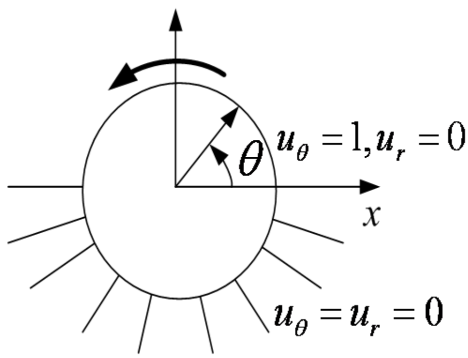

Figure 3: Boundary conditions.

Fig. 9 depicts the *x*-component velocity profile on the vertical centerline, and Fig. 10 describes the *y*-component velocity profile on the horizontal centerline. The results are compared with the numerical solution [Young, Jane and Lin(2004)], which also show good agreements.

4.2 Model application

The third model problem consists of a 2D circular cavity filled with incompressible viscous fluid moving on the upper half of the boundary. The moving velocity is equal in size and opposite in direction. The other configuration and boundary conditions of the problem are the same as 4.1.2, which are depicted in Fig. 11. The

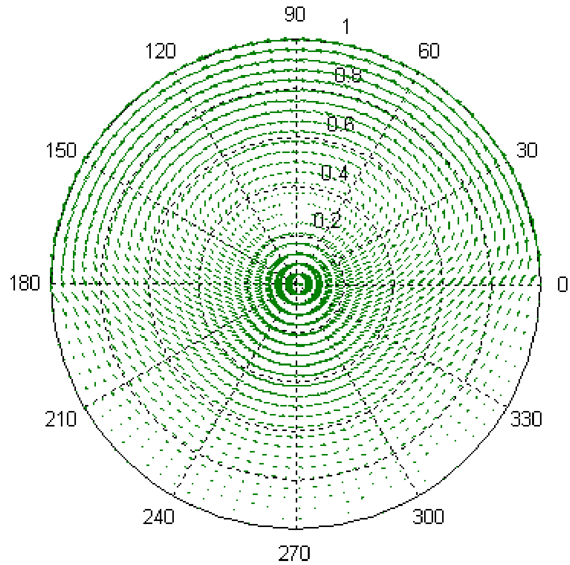

Figure 4: Velocity vector.

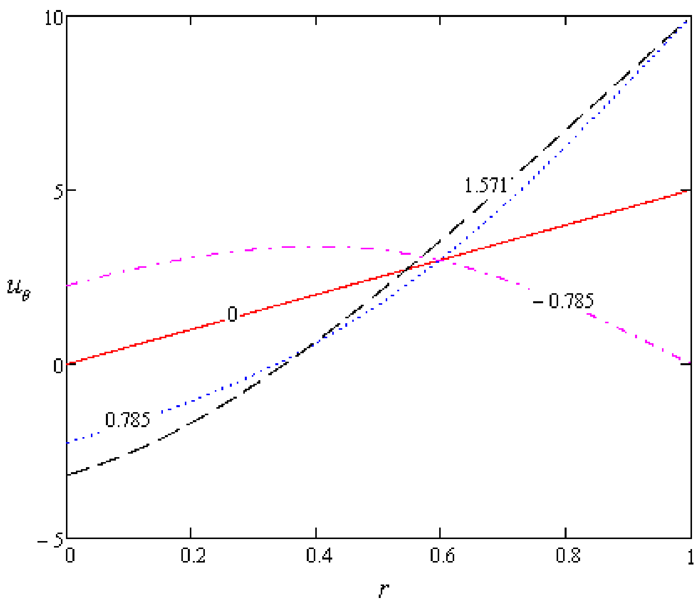

Figure 5: Tangential velocity varied with the radial position.

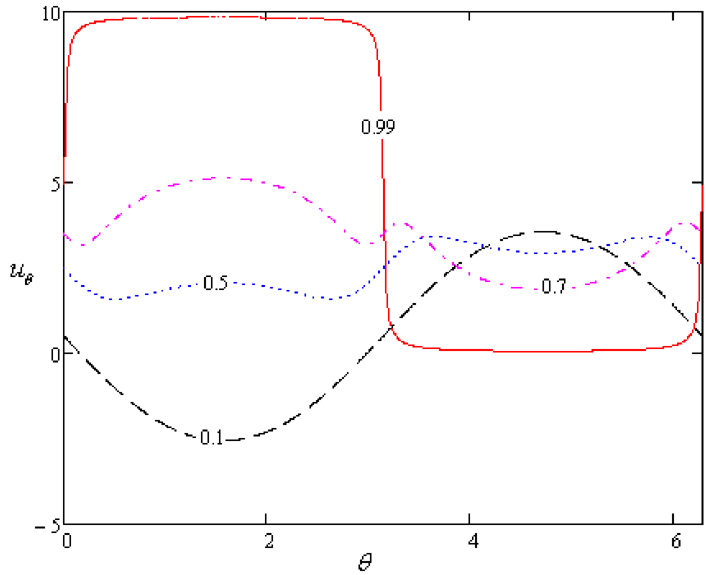

Figure 6: Tangential velocity varied with the direction.

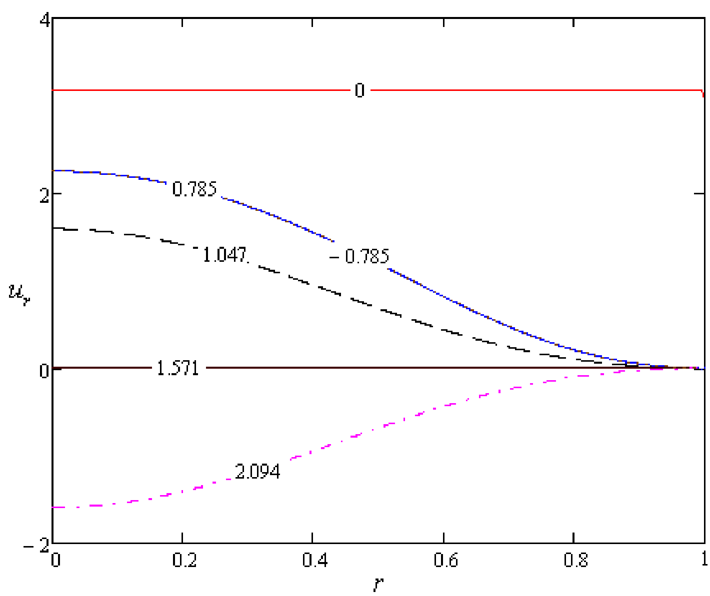

Figure 7: Radial velocity varied with the radial position.

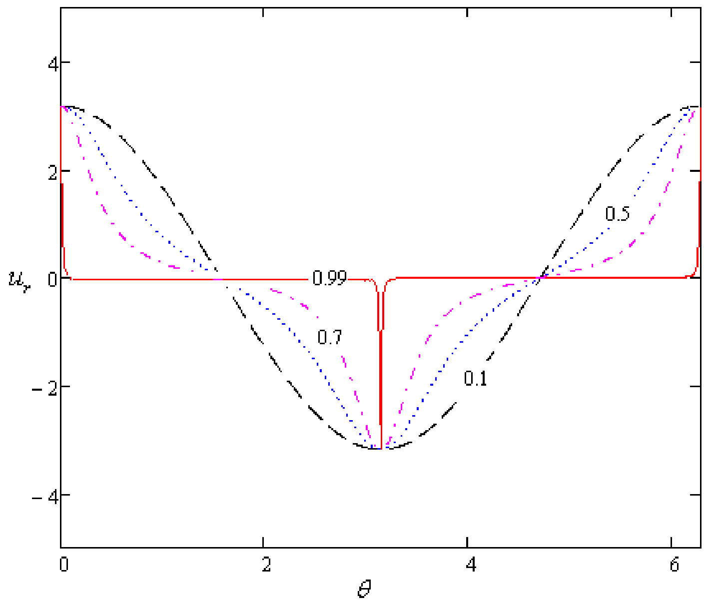

Figure 8: Radial velocity varied with the direction.

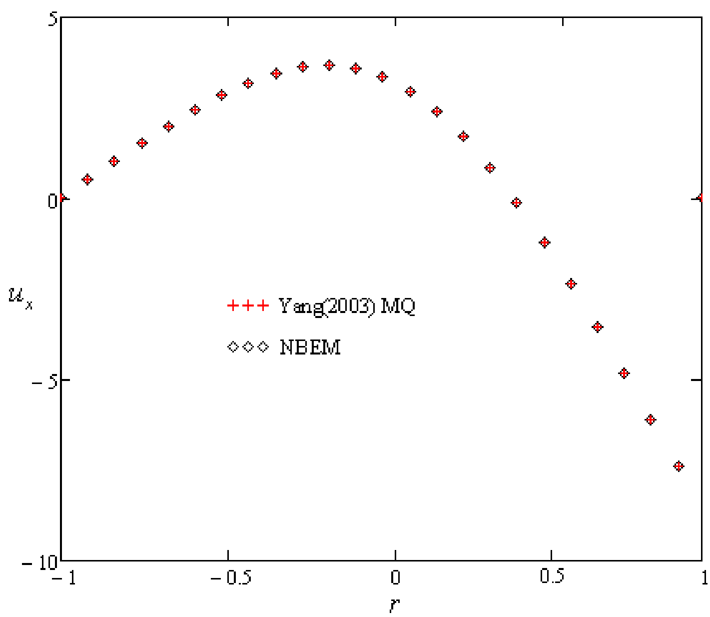

Figure 9: Comparison of the x-component velocity profile.

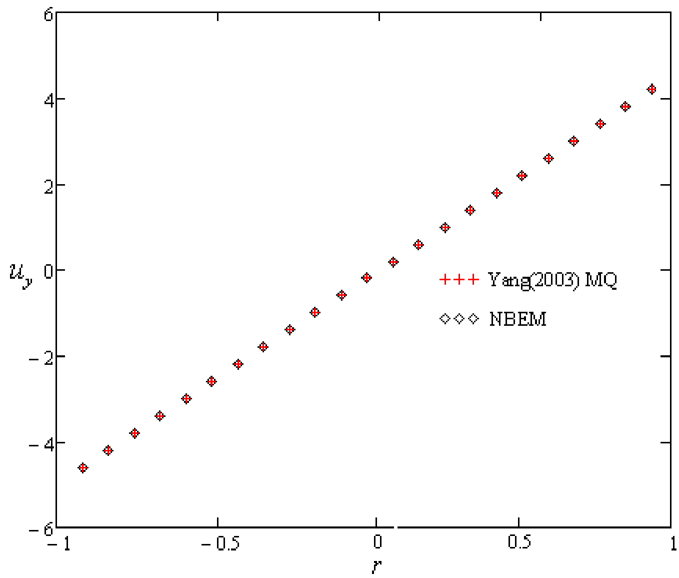

Figure 10: Comparison of the y-component velocity profile.

velocity vector is observed in Fig. 12.

The tangential velocities varied with the radial position r and direction θ are depicted as Figs. 13 and 14. Likewise, Fig. 15 depicts the radial velocity profile varied with the radial position r, and Fig. 26 describes the tangential velocity profile varied with the direction θ. The natural boundary element method may be used to easily compute the Stokes flow in a circular cavity with arbitrary velocity boundary conditions.

5 Conclusions

The boundary integral formulae of velocities for Stokes equations in interior and exterior circular domains are deduced by the natural boundary element method. The formulae are successfully used to solve Stokes flow problems in the circular and annular cavity, provided that the velocity values on the boundary are given. As the boundary integral formulae of the velocities are analytical, the results have a high computational accuracy. In addition, owing to the brevity of the formulae, the calculating process is simple and convenient compared with analytical solutions and other numerical solutions. The good performance of the natural boundary element method shows that it is a powerful tool for the numerical solution of incompressible viscous fluid flows. Although in this study mainly 2D Stokes flow problems

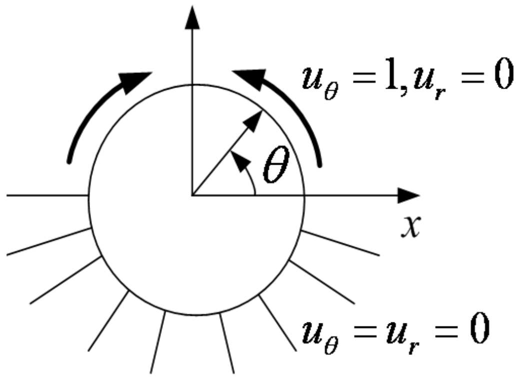

Figure 11: Boundary conditions.

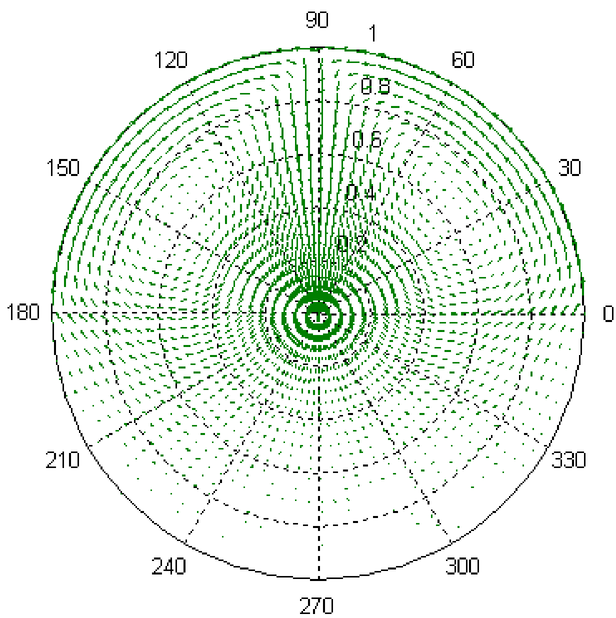

Figure 12: Velocity vector.

are solved, we expect that the natural boundary element method will have great potential for solving 3D incompressible viscous flow problems which are currently under study.

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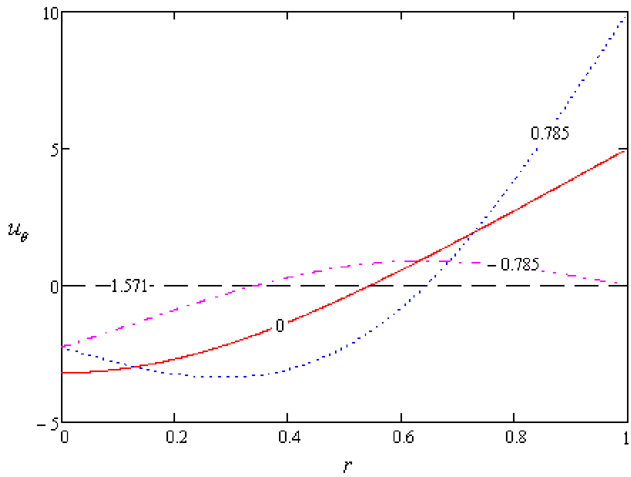

Figure 13: Tangential velocity varied with the radial position.

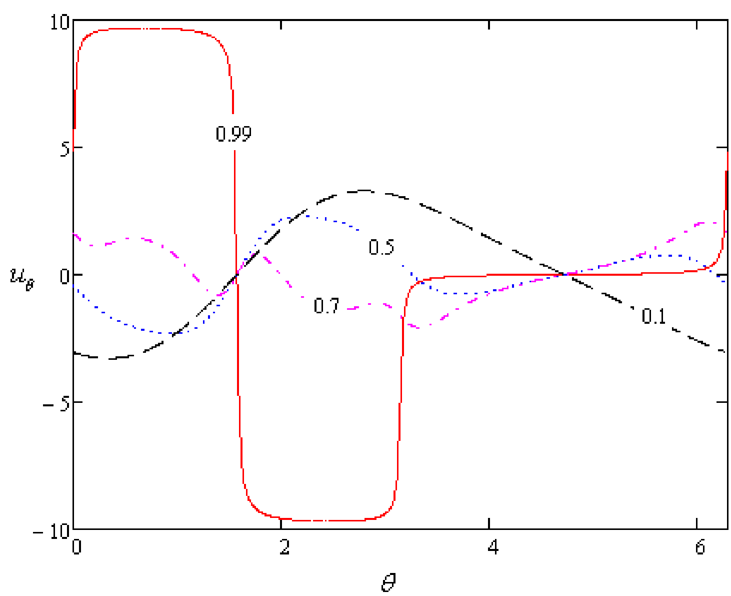

Figure 14: Tangential velocity varied with the direction.

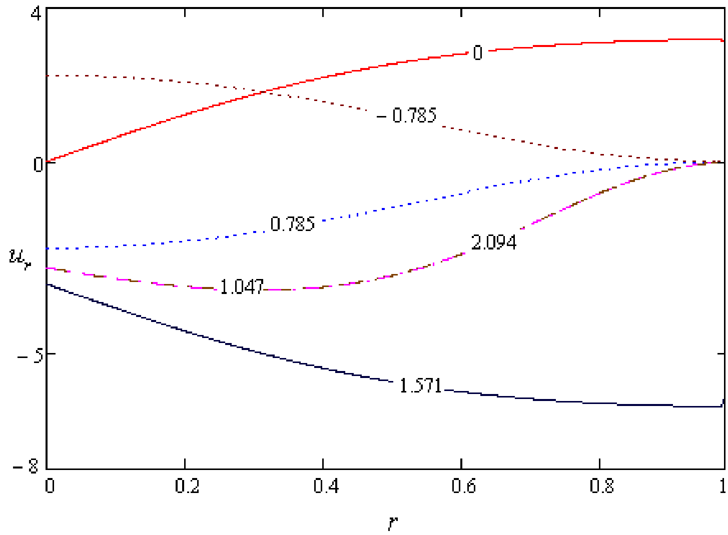

Figure 15: Radial velocity varied with the radial position.

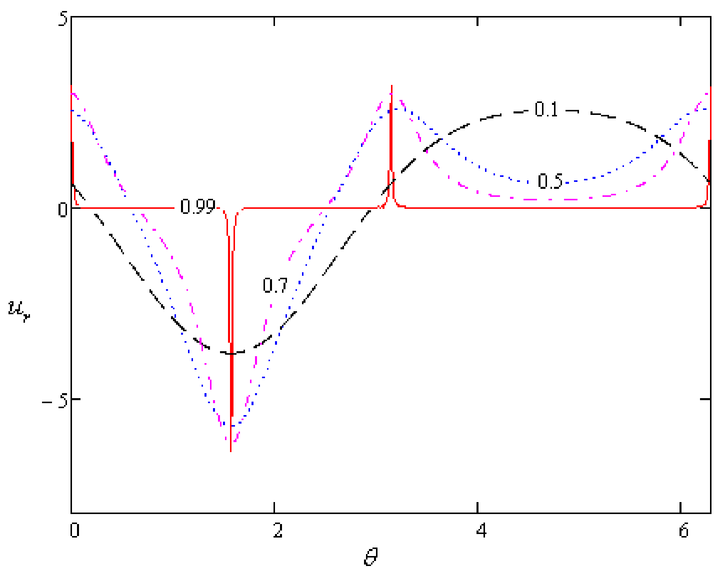

Figure 16: Radial velocity varied with the direction.

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