

# Nonlinear Bending and Thermal Post-Buckling Analysis of FGM Beams Resting on Nonlinear Elastic Foundations

Da-Guang Zhang<sup>1,2</sup> and Hao-Miao Zhou<sup>1</sup>

**Abstract:** A model of FGM beams resting on nonlinear elastic foundations is put forward by physical neutral surface and high-order shear deformation theory. Material properties are assumed to be temperature dependent and von Kármán strain-displacement relationships are adopted. Nonlinear bending and thermal post-buckling are given by multi-term Ritz method, and influences played by different supported boundaries, thermal environmental conditions, different elastic foundations, and volume fraction index are discussed in detail. It is worth noting that the effect of nonlinear elastic foundation increases with increasing deflection.

**Keywords:** Functionally graded materials, Physical neutral surface, Nonlinear elastic foundations, Nonlinear bending, Thermal post-buckling.

## 1 Introduction

Functionally graded materials (FGMs) are a class of novel materials in which properties vary continuously in a specific direction. To withstand the high-temperature effects, generally FGM structures made of ceramic and metal constituents are addressed. Consequently, investigations on behaviors of FGM structures are identified as an interesting field of study in recent years.

Many investigations on the linear bending and buckling of FGM or laminated or homogeneous beams can be seen in [Sankar (2001); Venkataraman and Sankar (2003); Zhong and Yu (2007); Ying et al. (2008); Giunta et al. (2010); Aminbaghai et al. (2012); Giunta et al. (2013); Dong et al. (2013); Dong et al. (2014); Elgohary et al. (2014)]. And lots of studies have been also made on the nonlinear bending and post-buckling of FGM beams based on Euler-Bernoulli beam theory, Timoshenko beam theory and higher order shear deformation beam theory. However, relatively few have been made on the nonlinear analysis of FGM beams resting on elastic foundations. Among those, Zhang (2013, 2014) put forward a model of the FGM

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beams based on physical neutral surface and high order shear deformation theory, and analyzed nonlinear bending, thermal post-buckling and nonlinear vibration behaviors by Ritz method. Ma and Lee (2011) derived governing equations for both the static behavior and the dynamic response of FGM beams subjected to uniform in-plane thermal loading, based on the physical neutral surface and the first order shear deformation beam theory, then Ma and Lee (2012) gave exact solutions for nonlinear static responses of a shear deformable FGM beam under an in-plane thermal loading. Li et al. (2006) studied thermal post-buckling of functionally graded material Timoshenko beams. Almeida et al. (2011) presented geometric nonlinear analysis formulation for beams of functionally graded cross-sections by means of a total Lagrangian formulation. Rahimi and Davoodinik (2010) discussed the large deflection of a functionally graded cantilever beam under inclined end loading by fully accounting for geometric nonlinearities using analytical and Adomian decomposition methods, then Davoodinik and Rahimi (2011) extended their works to a flexible tapered functionally graded cantilever beam. Rahimi et al. (2013) investigated the post-buckling behavior of functionally graded beams by means of an exact solution method. Zhao et al. (2007) derived the non-linear differential equations of post-buckling for FGM rod subjected to thermal load. They considered rods with both ends pinned and used the shooting method to solve the equations. Fu et al. (2012) discussed nonlinear analysis of buckling, free vibration and dynamic stability for the piezoelectric functionally graded beams in thermal environment. Ke et al. (2009) studied post-buckling behavior of functionally graded material beams including an edge crack effect based on Timoshenko beam theory and von-Kármán's strain-displacement relations. They applied the Ritz method to obtain the non-linear governing equations and used the Newton-Raphson method to obtain the postbuckling load-end shortening curves and post-buckling deflection-end shortening curves. Anandrao et al. (2010) investigated the buckling and thermal post-buckling behavior of uniform slender FGM beams. The von-Kármán strain-displacement relationship is used to obtain the equations. Single-term Ritz method and finite elements method are applied to obtain the response of the beam. Then Anandrao et al. (2012) studied large amplitude free vibration and thermal post-buckling of shear flexible FGM beams using finite element formulation based on first order Timoshenko beam theory. Ansari et al. (2013) investigated thermal post-buckling characteristics of FGM microbeams undergoing thermal loads based on the modified strain gradient theory. Shegokar and Lal (2013) provided the stochastic nonlinear bending response of FGM beam with surface bonded piezoelectric layers subjected to thermo-electro-mechanical loadings with uncertain material properties. Shen and Wang (2014) dealt with the large amplitude vibration, nonlinear bending and thermal postbuckling of FGM beams resting on an elastic foundation in thermal environments. Fallah and Aghdam (2011, 2012) investigated

nonlinear free vibration and post-buckling analysis of functionally graded beams on nonlinear elastic foundation. Esfahani et al. (2013) examined thermal buckling and post-buckling analysis of FGM Timoshenko beams resting on a non-linear elastic foundation. Ghiasian et al. (2013) studied static and dynamic buckling of an FGM beam resting on a non-linear elastic foundation subjected to uniform temperature rise loading and uniform compression.

The present paper extends the previous works [Zhang (2013); Zhang (2014)] to nonlinear bending and thermal post-buckling analysis of FGM beams resting on nonlinear elastic foundations based on physical neutral surface and high order shear deformation theory. Temperature dependent material properties and von Kármán strain-displacement relationship are taken into consideration. The material properties of FGMs are assumed to be graded in thickness direction according to a volume fraction power law distribution and are expressed as a nonlinear function of temperature. Approximate solutions of FGM beams are obtained by Ritz method.

**2 Temperature dependent material properties of FGM beams**

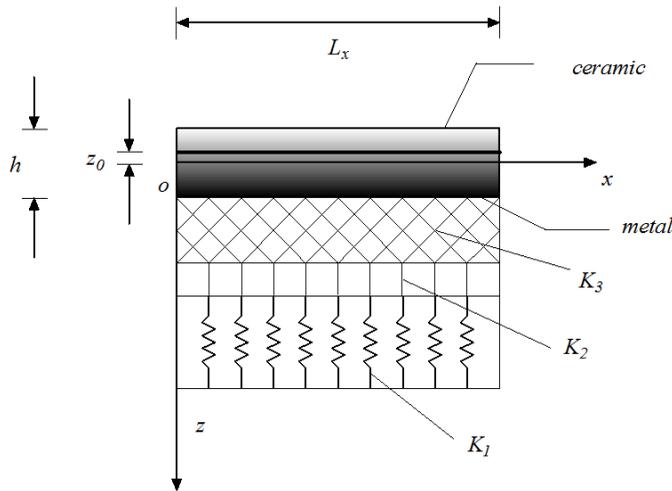


Figure 1: Geometry and coordinates of a FGM beam resting on nonlinear elastic foundations.

Consider a FGM beam (thickness  $h$  and length  $L_x$ ) with rectangular cross-section, which is made from a mixture of metals and ceramics. The coordinate system is illustrated in Fig. 1. It can be assumed that the effective material properties  $P$  of FGMs, such as Young’s modulus  $E$ , Poisson’s ratio  $\nu$ , thermal conductivity  $\kappa$  and

thermal expansion coefficient  $\alpha$ , can be expressed as

$$P = P_c V_c + P_m V_m \quad (1)$$

in which  $V_m$  and  $V_c$  are the metal and ceramic volume fractions and are related by  $V_m + V_c = 1$ ,  $P_m$  and  $P_c$  denote the temperature-dependent properties of metal and ceramic beam, respectively, and may be expressed as a nonlinear function of temperature [Touloukian (1967)]

$$P = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3) \quad (2)$$

in which  $T = T_0 + \Delta T$  and  $T_0 = 300$  K (room temperature),  $P_{-1}$ ,  $P_0$ ,  $P_1$ ,  $P_2$  and  $P_3$  are the coefficients of temperature  $T$  (K) and are unique to the constituent materials.

### 3 Modeling of FGM beams resting on nonlinear elastic foundations based on physical neutral surface and high order shear deformation theory

As is customary, the foundation is assumed to be a compliant foundation, meaning no part of the beam lifts off the foundation in the deformed region. The load-displacement relationship of the foundation is assumed to be  $p = K_1 w - K_2 d^2 w / dx^2 + K_3 w^3$ , where  $p$  is the force per unit area,  $K_1$  is the Winkler foundation stiffness,  $K_2$  is a constant showing the effect of the shear interactions of the vertical elements and  $K_3$  is nonlinear elastic foundation coefficients.

According to model of FGM beams [Zhang (2013); Zhang (2014)] based on physical neutral surface and high order shear deformation theory, the displacements, the strains and the stresses have the same form as the previous works [Zhang (2013); Zhang (2014)], see Appendix A.1-8. For the sake of brevity, the deducing process of the formulae is omitted, and the governing equations can be derived according to energy variational principle.

$$\tilde{D}_{11} \frac{d^2 \psi_x}{dx^2} - \frac{4\bar{F}_{11}}{3h^2} \frac{d^3 w}{dx^3} - \tilde{A}_{44} \left( \psi_x + \frac{dw}{dx} \right) - \frac{d\bar{M}_T}{dx} = 0 \quad (3a)$$

$$\begin{aligned} \tilde{A}_{44} \left( \frac{d\psi_x}{dx} + \frac{d^2 w}{dx^2} \right) + \frac{4\bar{F}_{11}}{3h^2} \frac{d^3 \psi_x}{dx^3} - \frac{16H_{11}}{9h^4} \frac{d^4 w}{dx^4} - \frac{4}{3h^2} \frac{d^2 P_T}{dx^2} + N_x \frac{d^2 w}{dx^2} \\ - K_1 w + K_2 \frac{d^2 w}{dx^2} - K_3 w^3 + q = 0 \end{aligned} \quad (3b)$$

And if the boundaries have no in-plane displacements on the geometric middle plane, i.e. prevented from moving in the  $x$ -direction,  $N_x$  can then be written in integral form as

$$N_x = \frac{1}{L_x} \int_0^{L_x} \left\{ A_{11} \left[ z_0 \frac{d\psi_x}{dx} - \frac{4c_0}{3h^2} \left( \frac{d^2 w}{dx^2} + \frac{d\psi_x}{dx} \right) + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] - N_T \right\} dx \quad (3c)$$

All symbols used in Eq. (3) are defined by Zhang (2013, 2014), see Appendix A.9. In the following analysis, two cases of boundaries will be considered.

$$w = 0, \quad M_x = -z_0 N_x, \quad P_x = -c_0 N_x, \quad (\text{for immovable simply supported ends}) \quad (4a)$$

$$w = \frac{dw}{dx} = \psi_x = 0, \quad (\text{for immovable clamped ends}) \quad (4b)$$

#### 4 Ritz method for approximate solutions of nonlinear problems of FGM beams

Ritz method is adopted in this section to obtain approximate solutions of FGM beams, and in symmetrical problems about the beams with immovable simply supported ends, it can be assumed that

$$w = \sum_{i=1,2,\dots}^M a_i \sin \frac{(2i-1)\pi x}{L_x} \quad (5a)$$

where  $M$  is total number of series, and where  $a_i$  are undetermined coefficients. We assume that the temperature variation is uniform and occurs in the thickness direction only, i.e.  $N_T$ ,  $M_T$  and  $P_T$  are constants for FGM beams, so substituting Eq. (5a) into Eq. (3a),  $\psi_x$  can be determined as

$$\psi_x = \sum_{i=1,2,\dots}^M c_i \cos \frac{(2i-1)\pi x}{L_x} \quad (5b)$$

where

$$c_i = \frac{\frac{4\bar{F}_{11}}{3h^2} \left[ \frac{(2i-1)\pi}{L_x} \right]^3 - \tilde{A}_{44} \frac{(2i-1)\pi}{L_x}}{\tilde{A}_{44} + \tilde{D}_{11} \left[ \frac{(2i-1)\pi}{L_x} \right]^2} a_i \quad (6)$$

For symmetrical problems about the beam with immovable clamped ends, it can be assumed that

$$w = \sum_{i=1,2,\dots}^M a_i \left( 1 - \cos \frac{2i\pi x}{L_x} \right) \quad (7a)$$

Similarly,  $\psi_x$  can be determined as

$$\psi_x = \sum_{i=1,2,\dots}^M c_i \sin \frac{2i\pi x}{L_x} \quad (7b)$$

where

$$c_i = \frac{\frac{4\tilde{F}_{11}}{3h^2} \left(\frac{2i\pi}{L_x}\right)^3 - \tilde{A}_{44} \frac{2i\pi}{L_x}}{\tilde{A}_{44} + \tilde{D}_{11} \left(\frac{2i\pi}{L_x}\right)^2} a_i \tag{8}$$

Cubic algebraic equations about  $a_i$  can be obtained by substituting  $w$  and  $\psi_x$  into the following expression.

$$\frac{\partial \Pi}{\partial a_i} = 0 \tag{9}$$

in which  $\Pi = U + V$ , in which the strain energy  $U$  is

$$U = \frac{1}{2} \int_{\Omega} (\sigma_x \varepsilon_x + \tau_{xz} \gamma_{xz} - \sigma_x \alpha \Delta T) d\Omega + \frac{1}{2} \int_0^{L_x} \left[ K_1 w^2 + K_2 \left(\frac{dw}{dx}\right)^2 + \frac{1}{2} K_3 w^4 \right] dx \tag{10a}$$

where  $\Omega$  denotes domain of FGM beams. Work done by applied forces  $V$  is

$$V = - \int_0^{L_x} q w dx \tag{10b}$$

As for FGM beams with given loads (like transverse uniformly distributed loads  $q_0$  and thermal loads  $N_T, M_T, P_T$ ) and other known coefficients,  $a_i$  can be solved by Newton-Raphson iterative method or other equivalent methods. For the sake of brevity, nonlinear algebraic equations and the solving process are omitted. Substituting these coefficients back into Eqs. (5) and (7),  $w$  and  $\psi_x$  may then be completely determined. In addition, if critical thermal buckling loads exist, critical value can be easily obtained by making solutions of deflection coefficients  $a_i$  approach to zero.

In addition, the present paper extends convergence studies of the previous works [Zhang (2013)] to nonlinear bending and thermal post-buckling analysis of FGM beams resting on nonlinear elastic foundations, so  $M = 3$  is also used in all the following calculations.

### 5 Results and discussions

Numerical results are presented in this section for nonlinear bending and thermal post-buckling of FGM beams. A  $\text{Si}_3\text{N}_4/\text{SUS304}$  is selected as an example. Typical values for Young's modulus  $E$  (in Pa), Poisson's ratio  $\nu$ , thermal expansion coefficient  $\alpha$  (in  $\text{K}^{-1}$ ) and thermal conductivity  $\kappa$  (in  $\text{W/mK}$ ) are listed in Table 1

from Reddy and Chin (1998). One dimensional temperature field is assumed to be constant in the  $x - y$  plane of the layer. In such a case, the temperature distribution along the thickness can be obtained by solving a steady-state heat transfer equation

$$-\frac{d}{dz} \left[ \kappa(z, T) \frac{dT}{dz} \right] = 0 \tag{11}$$

This equation can be solved by imposing boundary condition of  $T = T_t$  at the top surface ( $z = -h/2$ ) and  $T = T_b$  at bottom surface ( $z = h/2$ ). The solution of this equation is

$$T = T_t - (T_t - T_b) \frac{\int_{-h/2}^z \frac{1}{\kappa(z, T)} dz}{\int_{-h/2}^{h/2} \frac{1}{\kappa(z, T)} dz} \tag{12}$$

Note that the temperature field is uniform when  $T_t = T_b$ .

Table 1: Temperature-dependent coefficients for ceramic and metals [Reddy and Chin (1998)].

Material	Properties	$P_{-1}$	$P_0$	$P_1$	$P_2$	$P_3$
Si <sub>3</sub> N <sub>4</sub>	$E$ (Pa)	0	348.43e+9	-3.070e-4	2.160e-7	-8.946e-11
	$\nu$	0	0.24	0	0	0
	$\alpha$ (1/K)	0	5.8723e-6	9.095e-4	0	0
	$\kappa$ (W/mK)	0	13.723	-1.032e-3	5.466e-7	-7.876e-11
SUS304	$E$ (Pa)	0	201.04e+9	3.079e-4	-6.534e-7	0
	$\nu$	0	0.3263	-2.002e-4	3.797e-7	0
	$\alpha$ (1/K)	0	12.330e-6	8.086e-4	0	0
	$\kappa$ (W/mK)	0	15.379	-1.264e-3	2.092e-6	-7.223e-10

### 5.1 Comparison studies

To ensure the accuracy and effectiveness of the present method, two examples are solved for nonlinear bending and thermal post-buckling analysis of isotropic and FGM beams.

**Example 1.** The central deflection-load curves for a beam with immovable simply supported ends subjected to a transverse uniform distributed load  $q_0 L_x^3 / D_{11}$  are calculated and compared in Table 2 with results of Horibe and Asano (2001) using the boundary integral equation method. In this example,  $r = \sqrt{D_{11} / A_{11}}$  is the radius of gyration, and  $L_x / r = 100$ , Pasternak type  $(k_1, k_2) = (100, 50)$  for the Pasternak

Table 2: Comparisons of nonlinear bending for isotropic beams resting on different elastic foundations.

$q_0L_x^3/D_{11}$	$w_{center}/L_x$			
	Horibe and Asano (2001)		Present	
	(100, 0)	(100, 50)	(100, 0)	(100, 50)
40	0.053751	0.048414	0.05370	0.04295
80	0.069528	0.060816	0.06948	0.06076
120	0.080385	0.072745	0.08034	0.07269
160	0.088954	0.082007	0.08891	0.08196
200	0.096154	0.089705	0.09611	0.08966

elastic foundation and  $(k_1, k_2)=(100, 0)$  for the Winkler elastic foundation, where  $k_1 = K_1L_x^4/D_{11}$  and  $k_2 = K_2L_x^2/D_{11}$ . The present results agree well with results of Horibe and Asano (2001) except at one point, and the result of Horibe and Asano (2001) may be incorrect in the case of  $(k_1, k_2)=(100, 50)$  with a dimensionless uniform distributed load of 40.

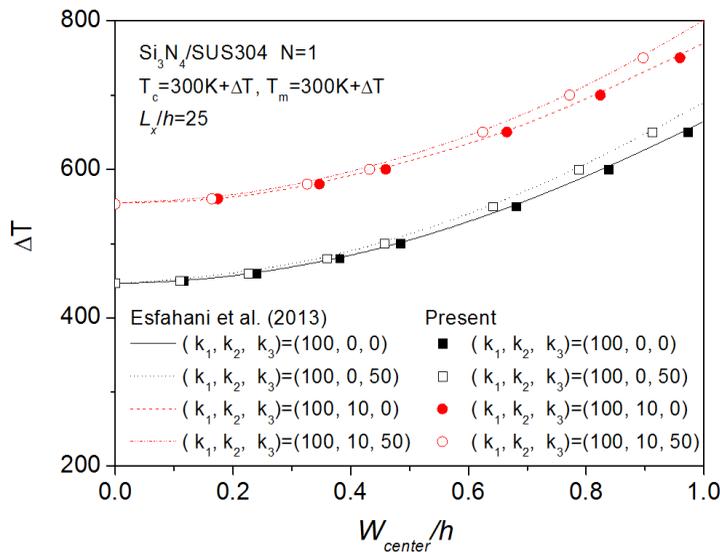


Figure 2: Comparisons of thermal post-buckling for Si<sub>3</sub>N<sub>4</sub>/SUS304 beams with immovable clamped ends resting on different elastic foundations subjected to uniform temperature rise.

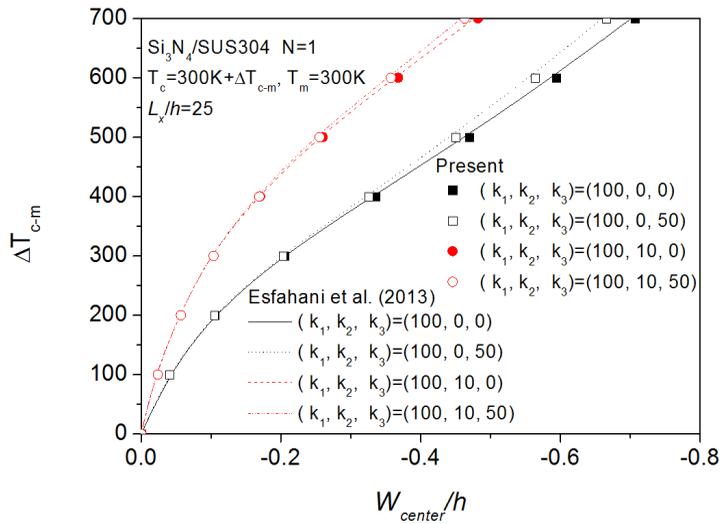


Figure 3: Comparisons of thermal post-buckling for  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable simply supported ends resting on different elastic foundations subjected to heat conduction.

**Example 2.** Thermal post-buckling for  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with different supported ends resting on different elastic foundations are calculated and compared in Figs. 2 and 3 with results of Esfahani et al. (2013) using generalized differential quadrature method. In this example, length to thickness ratio  $L_x/h = 25$ , volume fraction index  $N = 1$ , and the dimensionless foundation stiffnesses are  $(k_1, k_2, k_3) = (100, 0, 0)$  for the Winkler elastic foundation,  $(k_1, k_2, k_3) = (100, 10, 0)$  for the Pasternak elastic foundation,  $(k_1, k_2, k_3) = (100, 0, 50)$  and  $(k_1, k_2, k_3) = (100, 10, 50)$  for the nonlinear elastic foundation, where  $k_1 = 12K_1L_x^4/E_0h^3$ ,  $k_2 = 12K_2L_x^2/E_0h^3$  and  $k_3 = 12K_3L_x^4/E_0h$ ,  $E_0$  is Young's modulus of  $\text{Si}_3\text{N}_4$  at reference temperature. Excellent agreements can be seen from Figs. 2 and 3.

### 5.2 Parametric studies

A parametric study was undertaken for nonlinear bending and thermal post-buckling of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with  $L_x/h = 50$ . The volume fraction  $V_c$  is defined by  $V_c = (1/2 - z/h)^N$ , and the dimensionless foundation stiffnesses are  $(k_1, k_2, k_3) = (50, 0, 0)$  for the Winkler elastic foundation,  $(k_1, k_2, k_3) = (50, 5, 0)$  for the Pasternak elastic foundation,  $(k_1, k_2, k_3) = (50, 0, 10)$  and  $(k_1, k_2, k_3) = (50, 5, 10)$  for the nonlinear elastic foundation, where  $k_1 = K_1L_x^4/E_0h^3$ ,  $k_2 = K_2L_x^2/E_0h^3$  and  $k_3 = K_3L_x^4/E_0h$ ,  $E_0$  is Young's modulus of SUS304 at reference temperature. The top surface is

ceramic-rich, whereas the bottom surface is metal-rich, hence  $T_t = T_c$  and  $T_b = T_m$  for heat conduction.

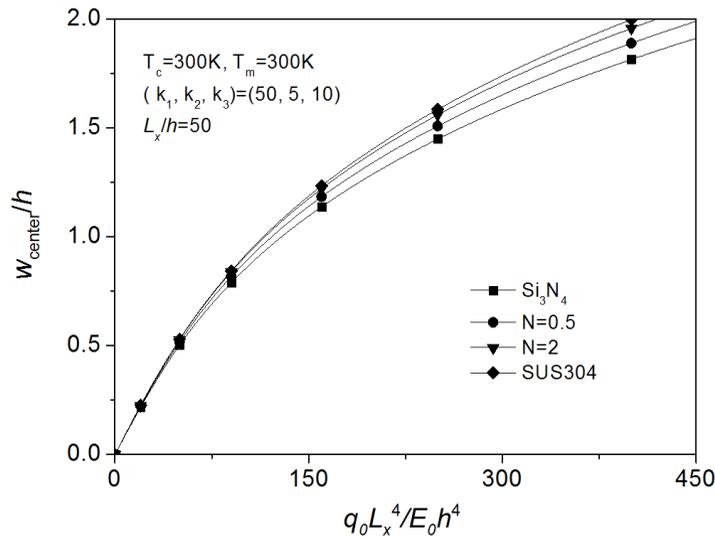


Figure 4: Effect of volume fractions on nonlinear bending of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable simply supported ends resting on nonlinear elastic foundations.

The central dimensionless deflections of  $\text{Si}_3\text{N}_4/\text{SUS304}$  FGM beams with immovable simply supported ends and clamped ends subjected to transverse uniformly distributed loads in different temperature fields are calculated, see Figs. 4-9. The effect of volume fractions on nonlinear bending of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with different supported ends resting on nonlinear elastic foundations can be seen in Figs. 4, 5, it can be observed that the deflections increase with increasing value of volume fraction index  $N$ . The effect of different elastic foundations on nonlinear bending of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with different supported ends and volume fraction index  $N = 2$  can be seen in Figs. 6, 7, it can be observed that the effect of nonlinear elastic foundation increase with increasing deflection. The effect of different temperature fields on nonlinear bending of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with different supported ends and volume fraction index  $N = 2$  resting on nonlinear elastic foundations can be seen in Figs. 8, 9, downward initial deflections under uniform temperature rise fields and the upward initial deflections under heat conduction fields can be observed for the beams with immovable simply supported ends, while no initial deflections under both uniform temperature rise and heat conduction fields can be observed for the beams with immovable clamped ends.

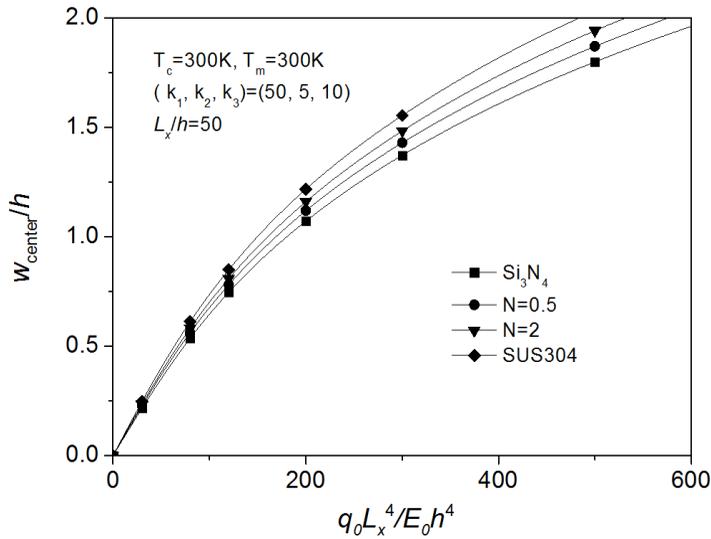


Figure 5: Effect of volume fractions on nonlinear bending of  $Si_3N_4/SUS304$  beams with immovable clamped ends resting on nonlinear elastic foundations.

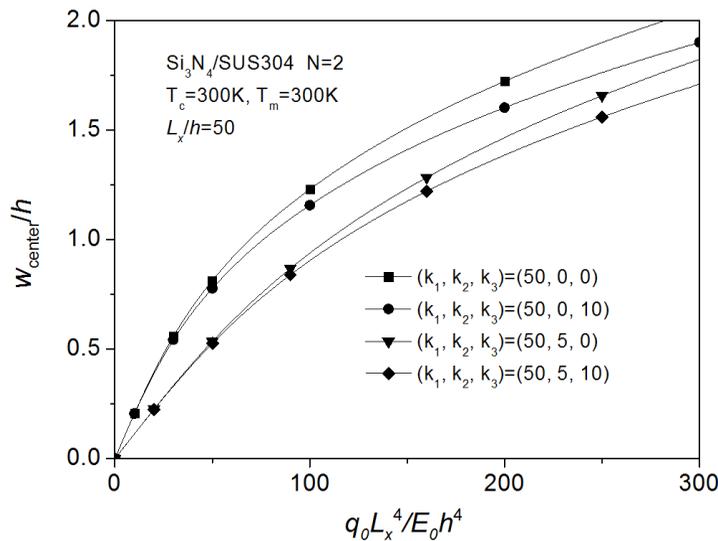


Figure 6: Effect of elastic foundations on nonlinear bending of  $Si_3N_4/SUS304$  beams with immovable simply supported ends.

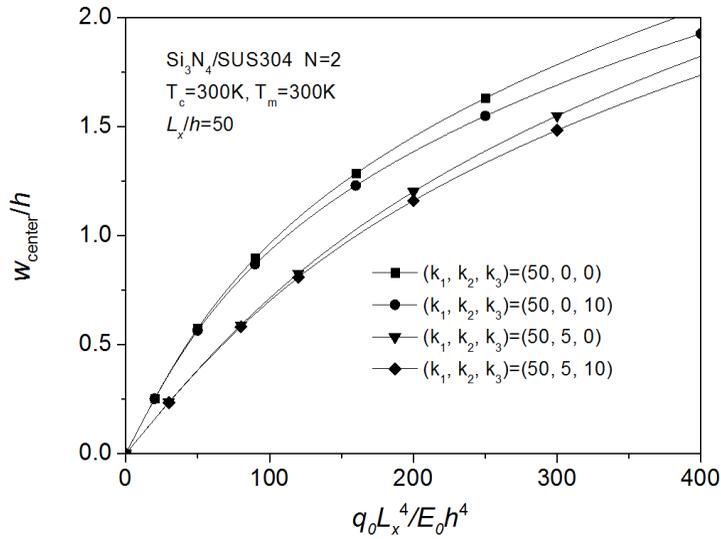


Figure 7: Effect of elastic foundations on nonlinear bending of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable clamped ends.

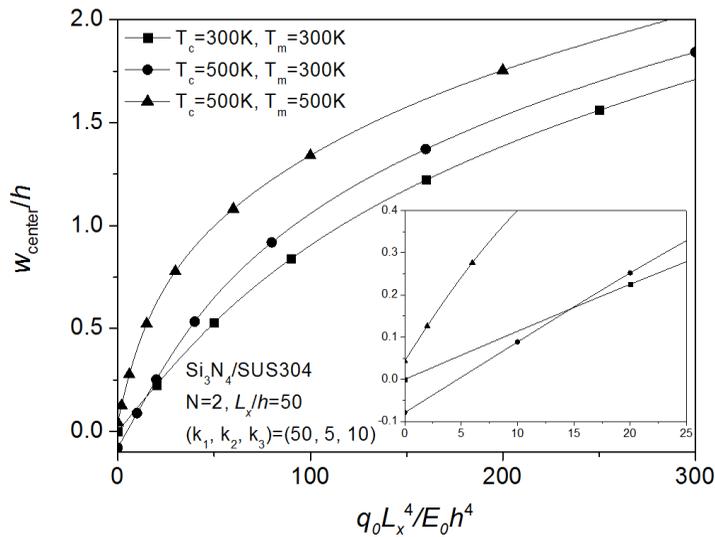


Figure 8: Effect of different temperature fields on nonlinear bending of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable simply supported ends.

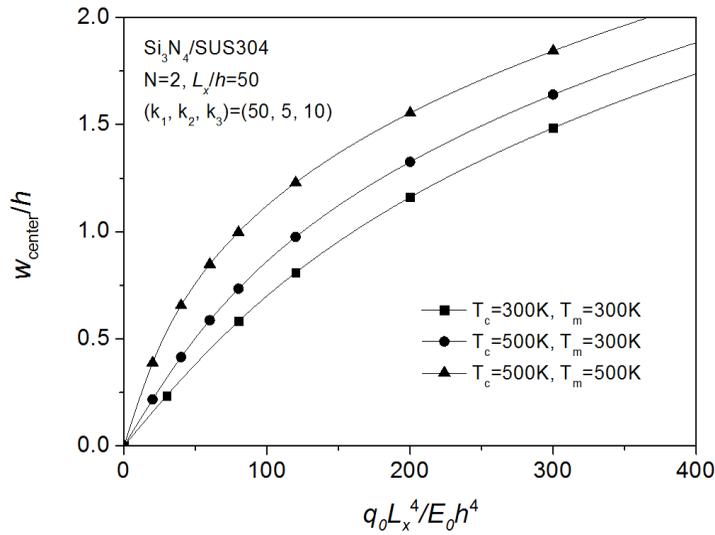


Figure 9: Effect of different temperature fields on nonlinear bending of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable clamped ends.

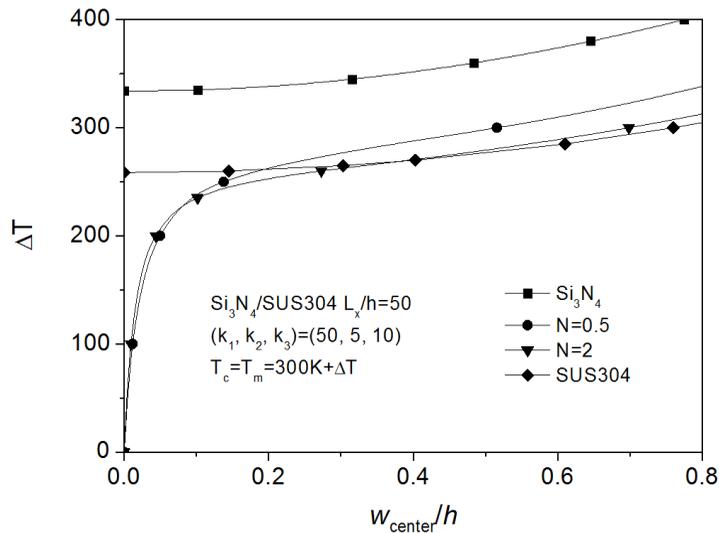


Figure 10: Thermal post-buckling behaviors for  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable simply supported ends resting on nonlinear elastic foundations subjected to uniform temperature rise.

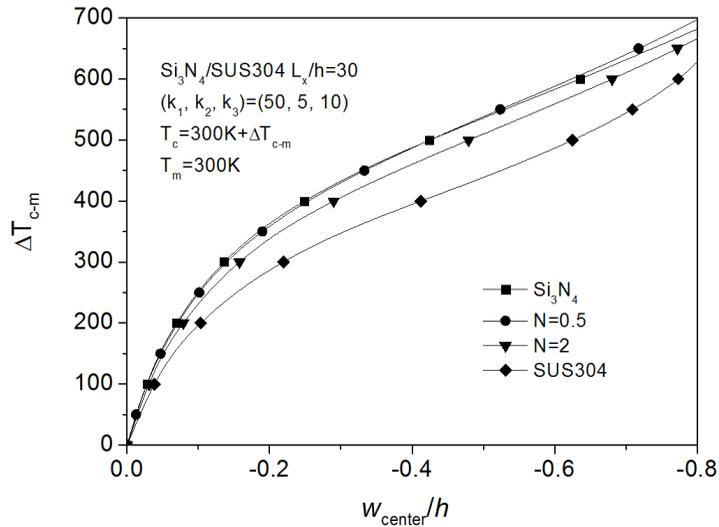


Figure 11: Thermal post-buckling behaviors for  $Si_3N_4/SUS304$  beams with immovable simply supported ends resting on nonlinear elastic foundations subjected to heat conduction.

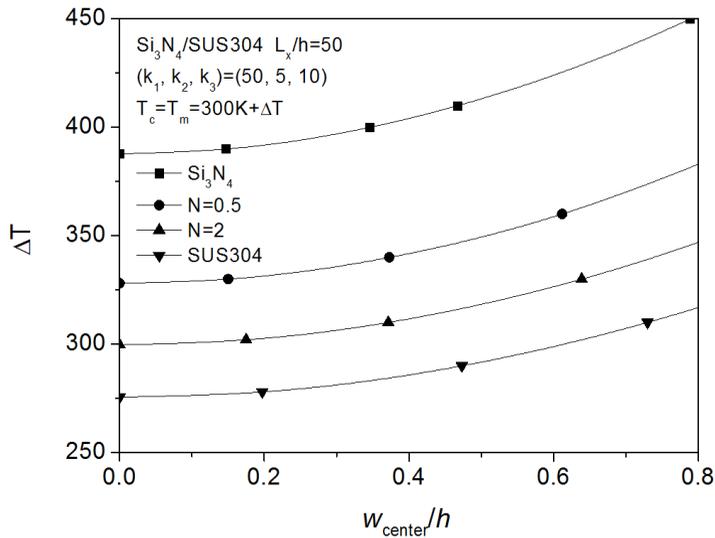


Figure 12: Thermal post-buckling behaviors for  $Si_3N_4/SUS304$  beams with immovable clamped ends resting on nonlinear elastic foundations subjected to uniform temperature rise.

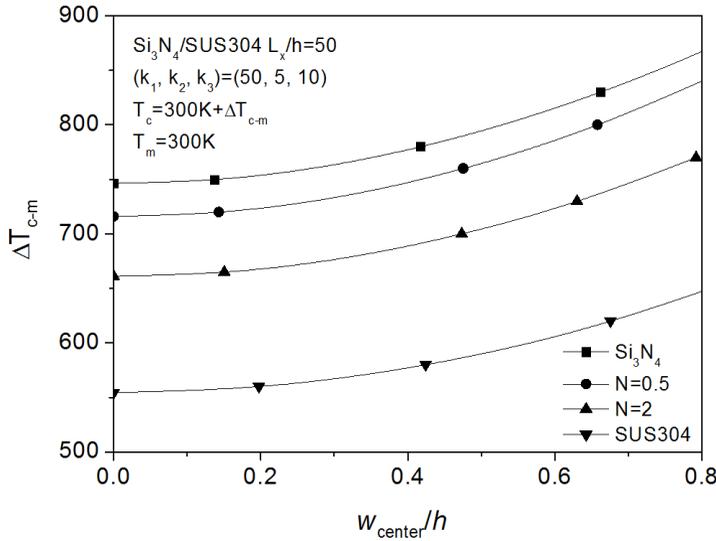


Figure 13: Thermal post-buckling behaviors for  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable clamped ends resting on nonlinear elastic foundations subjected to heat conduction.

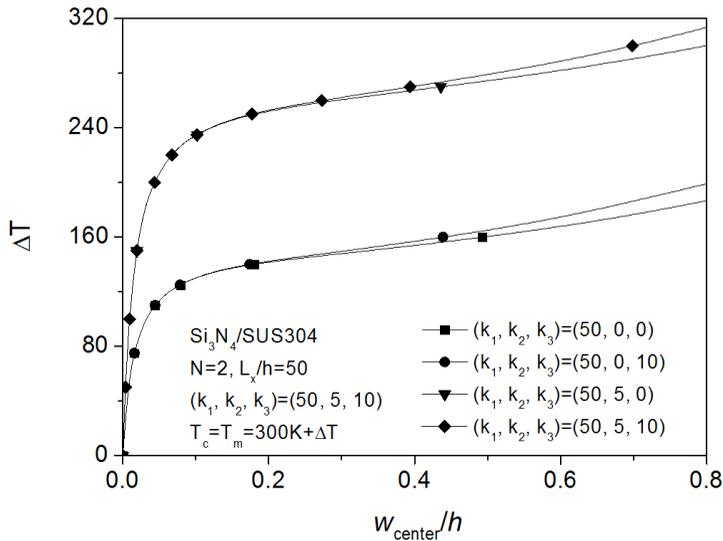


Figure 14: Effect of elastic foundations on thermal post-buckling of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable simply supported ends subjected to uniform temperature rise.

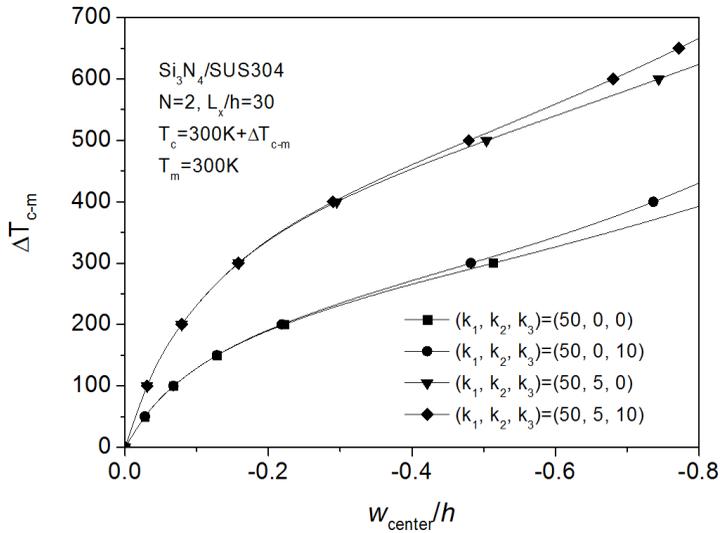


Figure 15: Effect of elastic foundations on thermal post-buckling of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable simply supported ends subjected to heat conduction.

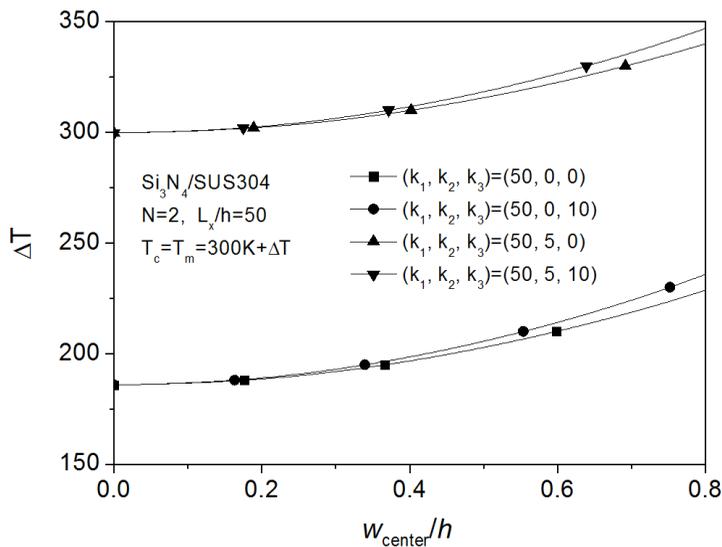


Figure 16: Effect of elastic foundations on thermal post-buckling of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable clamped ends subjected to uniform temperature rise.

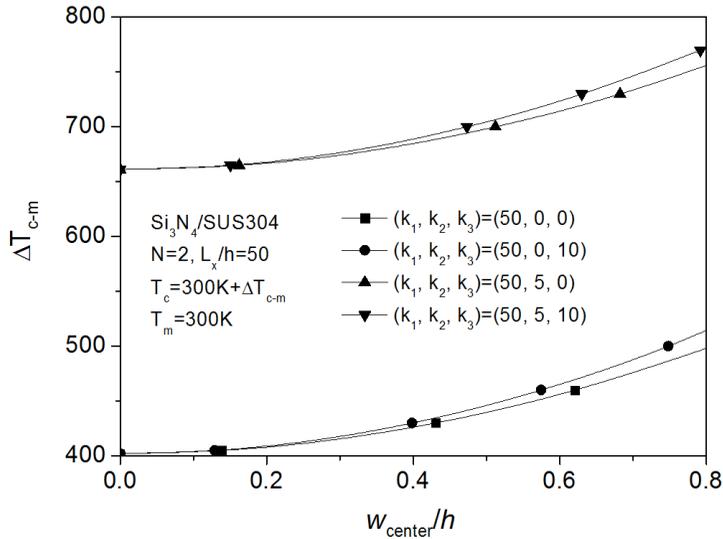


Figure 17: Effect of elastic foundations on thermal post-buckling of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with immovable clamped ends subjected to heat conduction.

Thermal post-buckling behaviors for  $\text{Si}_3\text{N}_4/\text{SUS304}$  FGM beams with different supported ends resting on nonlinear elastic foundations subjected to uniform temperature rise and heat conduction are calculated, see. Figs. 10-13. Bifurcation of buckling can occur for FGM beams with simply supported ends due to effect of uniform temperature rise or heat conduction, while bifurcation of buckling can not occur for FGM beams with clamped ends due to effect of uniform temperature rise and heat conduction. The effect of elastic foundations on thermal post-buckling of  $\text{Si}_3\text{N}_4/\text{SUS304}$  beams with different supported ends and volume fraction index  $N = 2$  resting on nonlinear elastic foundations subjected to uniform temperature rise and heat conduction are calculated, see. Figs. 14-17, and it can be observed that the effect of nonlinear elastic foundation increase with increasing deflection.

## 6 Conclusions

A model of the FGM beams resting on nonlinear elastic foundations is successfully established by physical neutral surface and high-order shear deformation theory. In nonlinear bending and post-buckling analysis, influences played by different supported boundaries, thermal environmental conditions, different elastic foundation and volume fraction index are discussed in detail. In nonlinear bending analysis, downward initial deflections under uniform temperature rise fields and the upward

initial deflections under heat conduction fields can be observed for the FGM beams with immovable simply supported ends, while no initial deflections under both uniform temperature rise and heat conduction fields can be observed for the FGM beams with immovable clamped ends. In thermal post-buckling analysis, bifurcation of buckling can occur for FGM beams with simply supported ends due to effect of uniform temperature rise or heat conduction, while bifurcation of buckling can not occur for FGM beams with clamped ends due to effect of uniform temperature rise and heat conduction. It is worth noting that the effect of nonlinear elastic foundation is significant with increasing deflection.

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**Appendix A**

The displacement fields

$$u = u_0 + (z - z_0)\psi_x - \frac{4}{3h^2} (z^3 - c_0) \left( \frac{dw}{dx} + \psi_x \right) \tag{A.1a}$$

$$w = w(x) \tag{A.1b}$$

in which  $z_0$  and  $c_0$  are defined by

$$z_0 = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} zE(z, T) dz}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z, T) dz}, \quad c_0 = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} z^3 E(z, T) dz}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z, T) dz} \tag{A.2}$$

Considering nonlinear von Kármán strain-displacement relationships, the strains can be expressed by

$$\epsilon_x = \epsilon_x^{(0)} + (z - z_0)\epsilon_x^{(1)} + (z^3 - c_0)\epsilon_x^{(3)}, \quad \gamma_{xz} = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)} \tag{A.3}$$

in which

$$\epsilon_x^{(0)} = \frac{du_0}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2, \quad \epsilon_x^{(1)} = \frac{d\psi_x}{dx}, \quad \epsilon_x^{(3)} = -\frac{4}{3h^2} \left( \frac{d\psi_x}{dx} + \frac{d^2w}{dx^2} \right) \tag{A.4a}$$

$$\gamma_{xz}^{(0)} = \psi_x + \frac{dw}{dx}, \quad \gamma_{xz}^{(2)} = -\frac{4}{h^2} \left( \psi_x + \frac{dw}{dx} \right) \tag{A.4b}$$

According to Hooke’s law, the stresses can be determined as

$$\sigma_x = E(z, T) [\epsilon_x - \alpha(z, T)\Delta T], \quad \tau_{xz} = \frac{E(z, T)}{2[1 + \nu(z)]} \gamma_{xz} \tag{A.5}$$

The constitutive equations can be deduced by proper integration.

$$\begin{bmatrix} N_x \\ M_x \\ P_x \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & D_{11} & F_{11} \\ 0 & F_{11} & H_{11} \end{bmatrix} \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_x^{(1)} \\ \epsilon_x^{(3)} \end{bmatrix} - \begin{bmatrix} N_T \\ M_T \\ P_T \end{bmatrix},$$

$$\begin{bmatrix} Q_x \\ R_x \end{bmatrix} = \begin{bmatrix} A_{44} & D_{44} \\ D_{44} & F_{44} \end{bmatrix} \begin{bmatrix} \gamma_{xz}^{(0)} \\ \gamma_{xz}^{(2)} \end{bmatrix} \quad (\text{A.6})$$

In which the beam stiffnesses are defined by

$$(A_{11}, D_{11}, F_{11}, H_{11}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z, T) \left( 1, (z - z_0)^2, (z - z_0)(z^3 - c_0), (z^3 - c_0)^2 \right) dz \quad (\text{A.7a})$$

$$(A_{44}, D_{44}, F_{44}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z, T)}{2[1 + \nu(z)]} (1, z^2, z^4) dz \quad (\text{A.7b})$$

The axial forces, bending moments and higher order bending moment caused by elevated temperature are defined by

$$(N_T, M_T, P_T) = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z, T) \alpha(z, T) \Delta T (1, z - z_0, z^3 - c_0) dz \quad (\text{A.8})$$

where  $\Delta T = T - T_0$  is temperature rise from some reference temperature  $T_0$  at which there are no thermal strains.

The symbols used in the governing equations are defined by

$$\begin{aligned} \bar{D}_{11} &= D_{11} - \frac{4F_{11}}{3h^2}, & \bar{F}_{11} &= F_{11} - \frac{4H_{11}}{3h^2}, & \tilde{D}_{11} &= \bar{D}_{11} - \frac{4\bar{F}_{11}}{3h^2}, \\ \bar{A}_{44} &= A_{44} - \frac{4D_{44}}{h^2}, & \bar{D}_{44} &= D_{44} - \frac{4F_{44}}{h^2}, & \tilde{A}_{44} &= \bar{A}_{44} - \frac{4\bar{D}_{44}}{h^2}, & \bar{M}_T &= M_T - \frac{4}{3h^2} P_T, \end{aligned} \quad (\text{A.9})$$