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A Universal Model-Independent Algorithm for Structural Damage Localization

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Abstract: Although the model-independent damage localization algorithms have been extensively developed in recent years, the theoretical relationship between these damage indicators and the definition of damage is not clear. Moreover the existing damage localization methods are usually dependent on the boundary conditions and the type of structure. In view of this, the paper presents a universal model-independent algorithm for structural damage localization. To this end, the explicit relationship between the damage and damage-induced displacement variation is firstly clarified by using the well-known Sherman-Morrison and Woodbury formulas. A theorem is then presented for structural damage localization. According to the theorem, the universal model-independent damage localization algorithm has been concluded and verified in some common structures. Presented in this article also can be seen as a theoretical proof of the existing non model methods for structural damage localization. It has been shown that the presented algorithm may be useful in the long term health monitoring and the damage localization.

Keywords: damage localization, structural displacement variation, Sherman-Mo rrison-Woodbury formulas, stiffness matrix, spectral decomposition.

1 Introduction

The detection of structural damage is a vital part of the monitoring and servicing of structural systems during their lifetime. Structural damage in normal service may include corrosion, fatigue, and aging, or it may be caused by impact loads, earthquakes, and wind. The presence of damages may reduce the performance of a structure, such as decreasing the service life, or even progressing to catastrophic failure. They may also increase the cost of maintenance or repairs. It is very beneficial if the damage can be detected before some critical conditions occur. During

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the last decades, a significant amount of research has been conducted on structural damage detection based on the changes in measured static/dynamic response. The basic idea of this technique is that structural response parameters are functions of structural physical properties (mass, damping, and stiffness). Therefore, changes in the physical properties will cause changes in the response parameters. Recent surveys on the technical literature show that extensive efforts have been developed to find reliable and efficient numerical and experimental models to identify damage in structure [Mottershead and Friswell (1993, 1995); Zou, Tong and Steven (2000); Chang (1997); Salawu (1997); Modak, Kundra and Nakra (2002); Alvandi and Cremona (2006); Prinaris, Alampalli and Ettouney (2008); Kanev, Weber and Verhaegen (2007); Yang and Liu (2007); Rainieri, Fabbrocino and Cosenza (2011); Lam and Yin (2011)].

Based on the extent of prior knowledge required, two main types of damage detection techniques can be discriminated: model-dependent and model-independent, also called parametric and nonparametric, respectively [Araujo dos Santos, Mota Soares and Mota Soares (2005)]. The model-dependent methods [Yu, Cheng and Yam (2007); Jiang, Tang and Wang (2007); Ashokkumar and Lyengar (2011); Yang and Wang (2010); Zhu, Li and He (2011); Perera and Ruiz (2008); Yang and Liu (2009); Weber and Paultre (2010); Adewuyi and Wu (2011); Limongelli (2010); Lima, Faria and Rade (2010); Lu and Law (2010); Lu, Huang and Liu (2011); Wong, Huang and Xiong (2011); Yang (2011); Yang and Sun (2011); Yang, Liu and Li (2013a,2013b)] identify damage by correlating an analytical model, which is usually based on the finite element theory, with test modal data of the damaged structure. Comparisons of the updated model to the original one provide an indication of damage and further information on the damage location and its severity. It is well known that model-based methods are computationally intensive and require a high quality finite element model (FEM) that could accurately predict the behavior of the intact structure. However, the precise FEM is often difficult to achieve in engineering practice for the simplified assumptions in the construction of the FEM, which makes that the correct results might be missed.

Model-independent damage detection methods are relatively simple and straightforward since these methods do not need a detailed model of the structure. The changes of response parameters between the intact and damaged states of the structure are directly used, or correlated with other relevant information, to develop the damage indicators for localizing damage in the structure. Pandey A.K. and Biswas M. (1994, 1995) propose to use the column-wise maximum absolute difference of flexibility matrices as an indicator for damage localization. Simulated and measured flexibilities for steel beams with different support conditions illustrate that an ad hoc interpretation of this indicator is necessary for different types of structures and different types of boundary conditions. Toksoy T. and Aktan A.E. (1994) used dynamically measured flexibilities for damage localization of a reinforced concrete bridge which is subject to progressive damage tests. They observe that flexibility is a very sensitive damage indicator as opposed to eigenfrequencies and damping ratios. Aktan A.E., Lee K.L., and Chuntavan C. (1994) find a very good match between the measured deflection of a steel stringer bridge under truck loading and the corresponding deflection calculated using dynamically measured flexibility. Furthermore, damage localization on a steel truss highway that was subjected to a progressive damage test by comparing the combination of the dynamically measured flexibility with a uniform load and a point load, was reported to be successful. Zhang Z. and Aktan A.E. (1998) studied the modal flexibility and its derivative ULS (uniform load surface) in structural identification. Wu D. and Law S.S. (2004, 2005) applied the ULS curvature to plate structures for damage localization. It is found that the ULS curvature is sensitive to the presence of local damages, even with truncated, incomplete, and noisy measurements. Wang J. and Qiao P.Z. (2007) used the ULS method to localize damage in a composite beam. A simplified gapped-smoothing method using a fourth order polynomial is considered to fit the ULS curve of a damaged beam, and the difference between the polynomial and ULS curves is squared to obtain the damage index. Choi I.Y., Lee J.S., Choi E. (2004) proposed an elastic damage load theorem for damage localization in statically determinate beams. They observe that the shapes of displacements variation before and after damage are consistent with the influence line of the moment of conjugate beam at a point where damage occurs. A comparative study of the damage localization methods can be found in references [Fan and Qiao (2011); Hamze, Gueguen and Roux (2012)].

It is well known that the coefficients of the *i*th column of the flexibility matrix represent the displacement shape of the structure with a unit load applied at the *i*th degree of freedom (DOF), and that the ULS obtained by the sum of all columns of the flexibility matrix represents the displacement shape of the structure if a unit load is applied at each DOF. Thus, all the above-mentioned methods can be attributed to the type of displacement-based damage localization techniques. Although the displacement-based localization algorithms have been extensively developed, the theoretical relationship between these damage indicators and the definition of damage is not clear. In other words, these methods lack a solid theoretical and physical base. The first objective of this paper is to clarify the explicit relationship between the damage and damage-induced displacement variation by using the well-known Sherman-Morrison and Woodbury formulas. The second objective of this paper is to propose a universal model-independent algorithm for structural damage localization. Presented in this article also can be seen as a theoretical proof of the existing damage localization methods. The paper begins by introducing the SMW formulas and relating them to the damage localization problem. A theorem is then presented for structural damage localization. According to the theorem, the universal damage localization algorithm has been concluded and verified in some common structures. It has been founded that the proposed method has a solid theoretical and physical base and can be used in damage localization for any structural type.

2 Theoretical development

2.1 Sherman-Morrison-Woodbury formulas

The SMW formulas have existed for over 60 years. An excellent historical survey of the origin of the SMW formulas is available in the literature (Hager W.W., 1989). The Sherman-Morrison formula gives the change in the inverse of a matrix K due to a change ΔK, which is of rank one and therefore may be written as $\Delta K = xy^T$. The formula may be stated as

$$(K+xy^T)^{-1} = K^{-1} - K^{-1}x(1+y^T K^{-1}x)^{-1}y^T K^{-1} \tag{1}$$

The Woodbury formula, which gives the change in the inverse of a matrix K due to a rank-m change, ΔK, is

$$(K+XY^T)^{-1} = K^{-1} - K^{-1}X(I+Y^T K^{-1}X)^{-1}Y^T K^{-1} \tag{2}$$

In recent years, the SMW formulas have been successfully applied in many areas, such as electrical networks, least squares, asymptotic analysis, sensitivity in linear programming, quasi-Newton methods, structural reanalysis, etc. In the following discussion, we will use the SMW formulas to solve the problem of structural damage localization.

2.2 Implementation for structural damage localization

In applications to structural problems, the SMW formulas are typically applied to update the solution of a system of equations for the displacement field u obtained from a finite element model rather than to update an inverse. For an intact structure, the analytical static model can be expressed as

$$Ku = l \tag{3}$$

where K is the $(n \times n)$ global stiffness matrix of the undamaged structure, and u is the displacement vector under the applied static load vector l. Rewriting equation (3), one has

$$u = K^{-1}l, \tag{4a}$$

$$\text{or} \quad u = Fl \tag{4b}$$

in which F is the flexibility matrix of the undamaged structure, i.e., $F = K^{-1}$. As is well known, structural damage reduces the stiffness and increases the flexibility of structures. Let ΔF and ΔK be the exact perturbation matrices that reflect the nature of the structural damage. Then the undamaged model matrices and the damaged model matrices are related as follows:

$$F_d = F + \Delta F \tag{5}$$

$$K_d = K - \Delta K \tag{6}$$

Then the displacement vector u_d for the damaged structure can be obtained by

$$u_d = K_d^{-1} l, \tag{7a}$$

$$\text{or} \quad u_d = F_d l \tag{7b}$$

Therefore the displacement variation (DV) Δu can be obtained as

$$\Delta u = u_d - u = [(K - \Delta K)^{-1} - K^{-1}] l, \tag{8a}$$

$$\text{or} \quad \Delta u = \Delta F \cdot l \tag{8b}$$

In structural finite element model, the global stiffness matrix K of the intact structure is a sum of the elemental stiffness matrices, i.e.

$$K = \sum_{i=1}^{N} K_i = \sum_{i=1}^{N} T_i K_i^e T_i^T \tag{9}$$

where K_i is the ith $(n \times n)$ elemental stiffness matrix in global co-ordinates, K_i^e is the ith $(n_e \times n_e)$ elemental stiffness matrix in local co-ordinates (n_e is the number of elemental DOFs), T_i is the transformation matrix from elemental degree of freedom (DOF) to global DOF, N is the total number of elements. Note that the elemental stiffness matrix K_i^e is commonly rank deficient and can be decomposed [Doebling, Peterson and Alvin (1998); Bathe (1996)], using its non-zero eigenvalues and the corresponding eigenvectors, as

$$K_i^e = [\eta]_i [p]_i [\eta]_i^T \tag{10a}$$

$$[p]_i = diag(p_i^1, \cdots, p_i^r) \tag{10b}$$

where $[p]_i$ is the diagonal matrix consist of the non-zero eigenvalues $(p_i^1, \cdots p_i^r)$ of K_i^e (r is the rank of K_i^e), and $[\eta]_i$ is the corresponding eigenvector matrix in local co-ordinates. Substituting equation (10) into (9), one has

$$K = \sum_{i=1}^{N} (T_i [\eta]_i) [p]_i ([\eta]_i^T T_i^T) \tag{11}$$

Equation (11) can be rewritten as

$$K = APA^T \tag{12a}$$

$$A = [T_1 [\eta]_1, T_2 [\eta]_2, \cdots, T_N [\eta]_N] \tag{12b}$$

$$P = diag([p]_1, [p]_2, \cdots, [p]_N) \tag{12c}$$

where the sparse matrix A is called the stiffness topology matrix, and $T_i [\eta]_i$ is essentially the eigenvector matrix of the elemental stiffness matrix K_i in global co-ordinates, that is

$$K_i = (T_i [\eta]_i) [p]_i (T_i [\eta]_i)^T, \quad (i = 1 \sim N) \tag{13}$$

Physically, the columns of $T_i [\eta]_i$ reflect the connectivity between DOFs, while the diagonal matrix $[p]_i$ is a function purely of the material properties such as the elastic modulus, the cross-sectional area, the moment of inertia, etc. Therefore the matrix A is independent of P and unchanged as damage occurs. As a result, the stiffness matrix perturbation ΔK can also be obtained in the form of equation (12) as

$$\Delta K = A \Delta P A^T \tag{14a}$$

$$P = diag([p]_1, [p]_2, \cdots, [p]_N) \tag{14b}$$

$$[\Delta p]_i = diag((\alpha_i^1 p_i^1), \cdots, (\alpha_i^r p_i^r)) \tag{14c}$$

where the coefficient $\alpha_i^j (0 \le \alpha_i^j \le 1, i = 1 \sim N, j = 1 \sim r)$ is defined as the elemental damage parameter. The value of α_i^j is 0 if the ith element is undamaged and α_i^j is 1 or less than 1 if the corresponding element is completely or partially damaged.

The following discussion will be divided into two cases. For the first case of rank-1 perturbation ΔK, equation (14a) reduces to

$$\Delta K = a_i (\alpha_i p_i) a_i^T \tag{15}$$

where $\alpha_i (\alpha_i \neq 0), p_i$, and a_i are the damage parameter, the eigenvalue, and the eigenvector of the single damaged element, respectively. Substituting equation (15) into equation (8a), one has

$$\Delta u = [(K - a_i (\alpha_i p_i) a_i^T)^{-1} - K^{-1}] l \tag{16}$$

Let $x = a_i$ and $y = a_i(\alpha_i p_i)$. Then equation (16) can be expanded by the Sherman-Morrison formula (equation (1)) to get the following equation for the displacement change:

$$\Delta u = (K^{-1}a_i)(1 + (\alpha_i p_i)a_i^T K^{-1} a_i)^{-1}(\alpha_i p_i)a_i^T K^{-1} l \tag{17}$$

Equation (17) can be rewritten as

$$\Delta u = \delta d_i \tag{18a}$$

$$d_i = K^{-1}a_i \tag{18b}$$

$$\delta = (1 + (\alpha_i p_i)a_i^T K^{-1} a_i)^{-1}(\alpha_i p_i)a_i^T K^{-1} l \tag{18c}$$

The implication of equation (18) is very important. In equation (18b), we define the vector d_i as the characteristic displacement (CD) by considering a_i as a static load vector (a_i can be renamed as the characteristic force (CF) associated with the damaged element). That is to say, the characteristic displacement (d_i) is obtained by applying the corresponding characteristic force (a_i) to the undamaged structure. It has been shown from equation (18a) that structural displacement variation (Δu) is proportional to the characteristic displacement (d_i). In other words, the shape of displacement variation in a structure due to damage equals to the shape of characteristic displacement associated with the unique damaged element.

For the second case of rank>1 perturbation ΔK, equation (14a) reduces to

$$\Delta K = A^* \Delta P^* (A^*)^T \tag{19}$$

where A^* and ΔP^* are associated with the damaged elements. Substituting equation (19) into equation (8a) yields

$$\Delta u = [(K - A^* \Delta P^* (A^*)^T)^{-1} - K^{-1}]l \tag{20}$$

Let $X = A^*$ and $Y = A^* \Delta P^*$. Then equation (20) can be expanded by the Woodbury formula (equation (2)) to get the following equation for the displacement change:

$$\Delta u = (K^{-1}A^*)(I + \Delta P^* (A^*)^T K^{-1} A^*)^{-1} \Delta P^* (A^*)^T K^{-1} l \tag{21}$$

Equation (21) can be rewritten as

$$\Delta u = D\zeta \tag{22a}$$

$$D = K^{-1}A^* \tag{22b}$$

$$\zeta = (I + \Delta P^* (A^*)^T K^{-1} A^*)^{-1} \Delta P^* (A^*)^T K^{-1} l \tag{22c}$$

As before, considering A^* as a set of load vectors, the shape of each column vector in D represents the characteristic displacement corresponding to the damaged element. According to the theory of linear algebra [Herstein and Winter (1988); Datta (1995)], equation (22a) is valid only if the vector Δu is a linear combination of the columns of D. That is to say, the displacement variation in a structure due to damage is a linear combination of the characteristic displacements associated with the damaged elements.

According to the above discussion, we have proved the following two propositions using the SMW formulas: (1) For the rank-1 damage, the displacement variation in a structure under an arbitrary load is proportional to the characteristic displacement corresponding to the unique damaged element; (2) For the rank>1 damage, the displacement variation in a structure under an arbitrary load is a linear combination of the characteristic displacements associated with the damaged elements. It is noted that the rank-1 damage case can be seen as a special case of the rank>1 damage case. In engineering practice, the displacement variation Δu can be obtained by a static load-deflection test or the dynamically flexibility test (i.e., the measured flexibility change is multiplied with a virtual force vector using equation (10b)). As a result, by comparing the geometric features of the displacement variation with that of the characteristic displacement, structural damage localization is possible. In addition, the measured data are usually incomplete in practice because of the limited number of sensors, and the rotational DOFs are difficult to measure. Therefore only the characteristic displacement associated with the partial transnational DOFs will be studied in the following discussion.

2.3 The physical features of the characteristic force and the corresponding characteristic displacement

As stated previously, the characteristic displacement is obtained by applying the corresponding characteristic force to the undamaged structure, and the characteristic forces of each element are essentially the eigenvectors of each elemental stiffness matrix. Without loss of generality, the beam element is used in the following to investigate the physical features of the characteristic force and the corresponding characteristic displacement.

Consider a two-node Bernoulli-Euler plane beam element with four DOFs (shown in Fig.1), the node displacement vector and the elemental stiffness matrix in local co-ordinates are given as

$$u^e = [v_1, \theta_1, v_2, \theta_2]^T \tag{23}$$

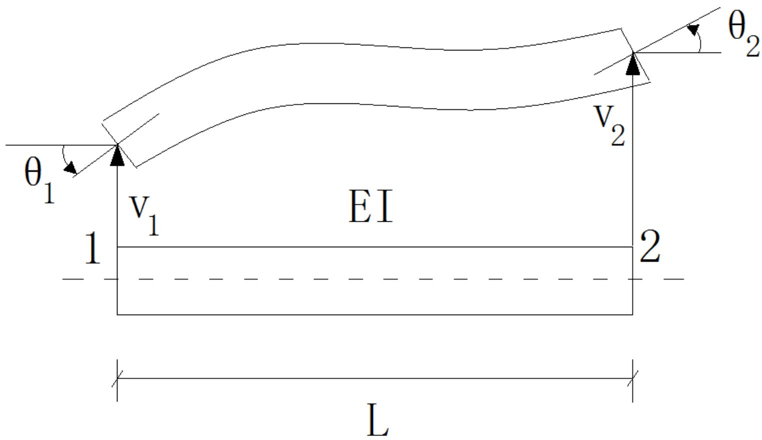


Figure 1: The two-node Bernoulli-Euler plane beam element with four DOFs.

$$K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{24}$$

where E is the modulus of elasticity, I is the moment of inertia, and L is the length of the beam element. Then the spectral decomposition of element stiffness K^e yields the following non-zero eigenvalues and the corresponding eigenvectors as

$$[p] = \begin{bmatrix} \frac{6EI(L^2+4)}{L^3} & \\ & \frac{2EI}{L} \end{bmatrix} \tag{25a}$$

$$[\eta] = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{L^2+4}} & 0 \\ \frac{L}{\sqrt{2}\sqrt{L^2+4}} & \frac{-1}{\sqrt{2}} \\ \frac{-\sqrt{2}}{\sqrt{L^2+4}} & 0 \\ \frac{L}{\sqrt{2\sqrt{L^2+4}}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{25b}$$

Apparently, the diagonal matrix $[p]$ in (25a) is a function purely of the material properties (E and I). Thus the matrix $[p]$ is changed, while the matrix $[\eta]$ is unchanged as structural damage occurs. In equation (25b), the two column vectors in $[\eta]$ are the characteristic forces of the beam element in local co-ordinates. For convenience, the two load configurations associated with the characteristic forces in local co-ordinates are shown in Fig.2. It is very important to note that both load configurations shown in Fig.2 are the self-equilibrating force systems. Using other types of finite elements, we will get the same conclusion. When this self-equilibrating characteristic force was applied to the structure (see Fig.3), the internal force in most parts of the structure will be zero except the element associated with this characteristic force. In a word, it can be concluded from figures 2 and 3 that: (1) the characteristic force is a self-equilibrating force; (2) the characteristic force acts only on its own element not on the rest of the structure.

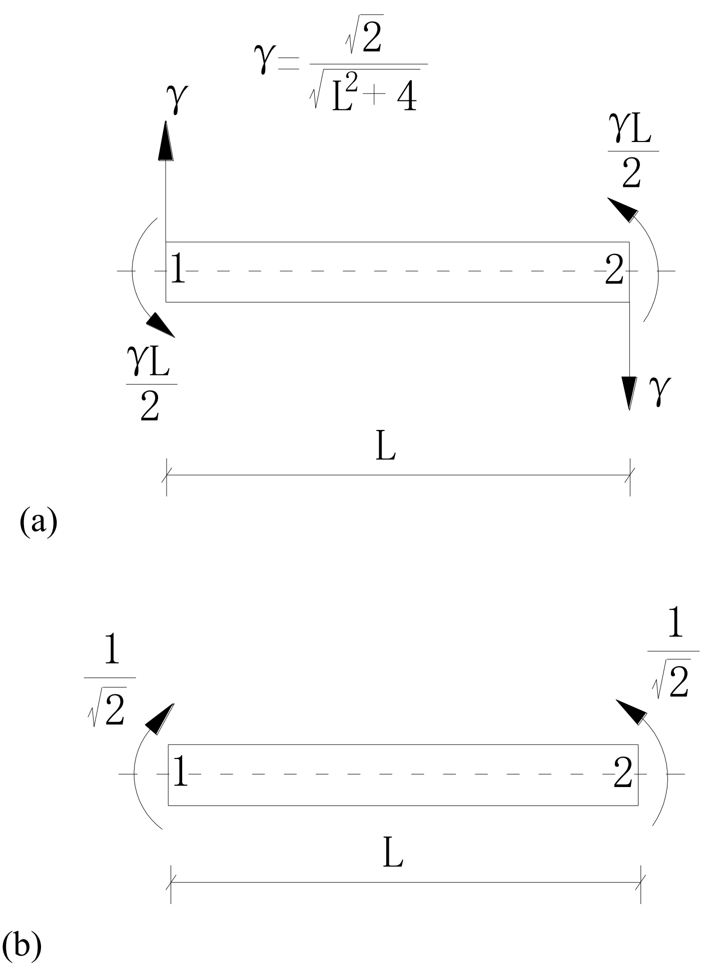


Figure 2: (a) Load configuration of the characteristic force 1 for the plane beam element; (b) Load configuration of the characteristic force 2 for the plane beam element.

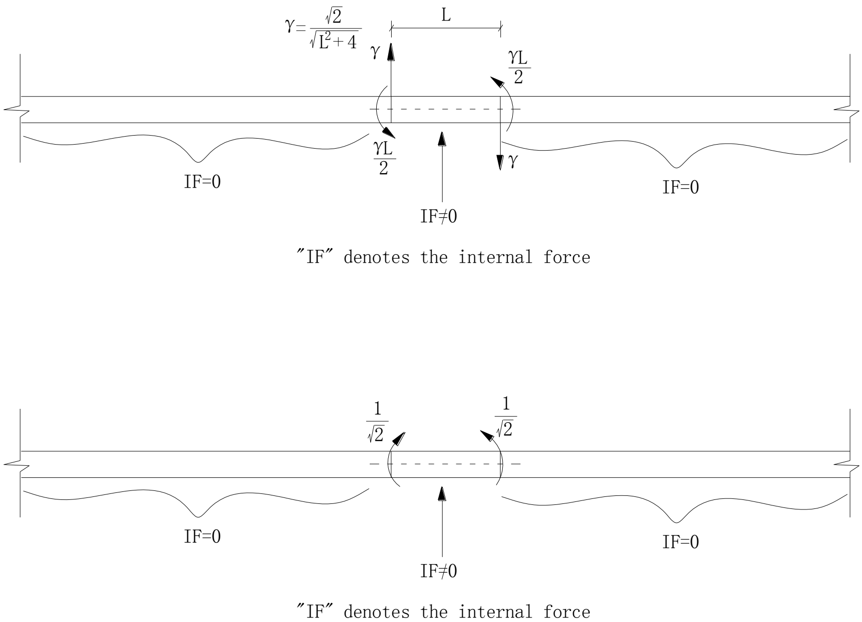


Figure 3: The internal force distribution in the structure under the action of the characteristic force.

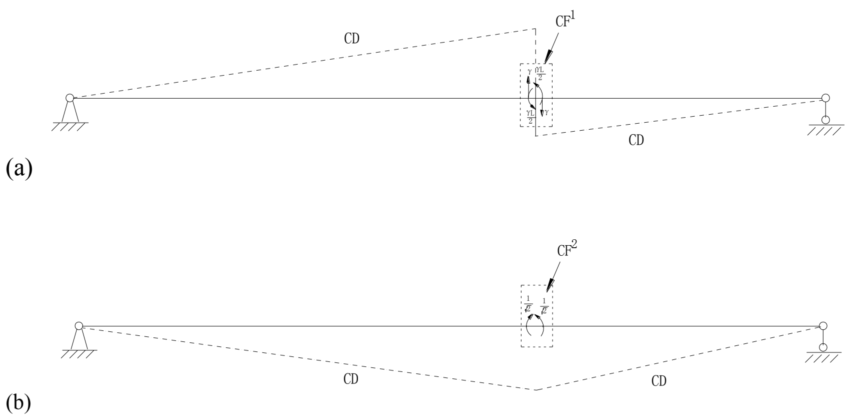


Figure 4: (a) The characteristic displacement (CD) for the simple supported beam under the characteristic force 1 (CF^1); (b) The characteristic displacement (CD) for the simple supported beam under the characteristic force 2 (CF^2).

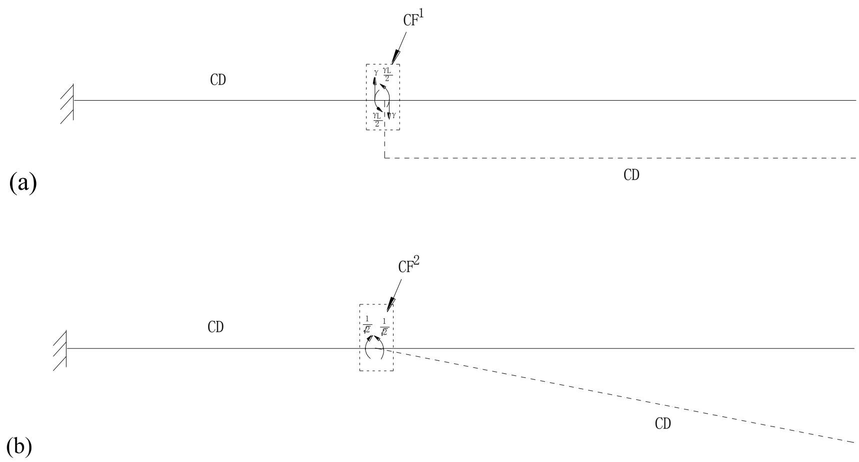


Figure 5: (a) The characteristic displacement (CD) for the cantilever beam under the characteristic force 1 (CF^1); (b) The characteristic displacement (CD) for the cantilever beam under the characteristic force 2 (CF^2).

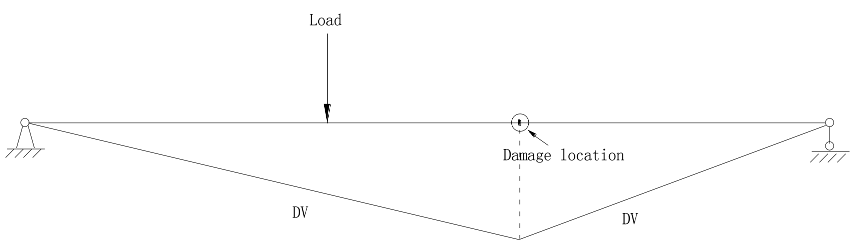


Figure 6: The displacement variation (DV) due to damage for the simple supported beam under a point load.

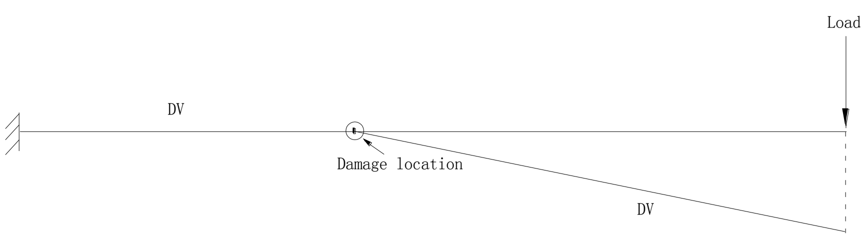
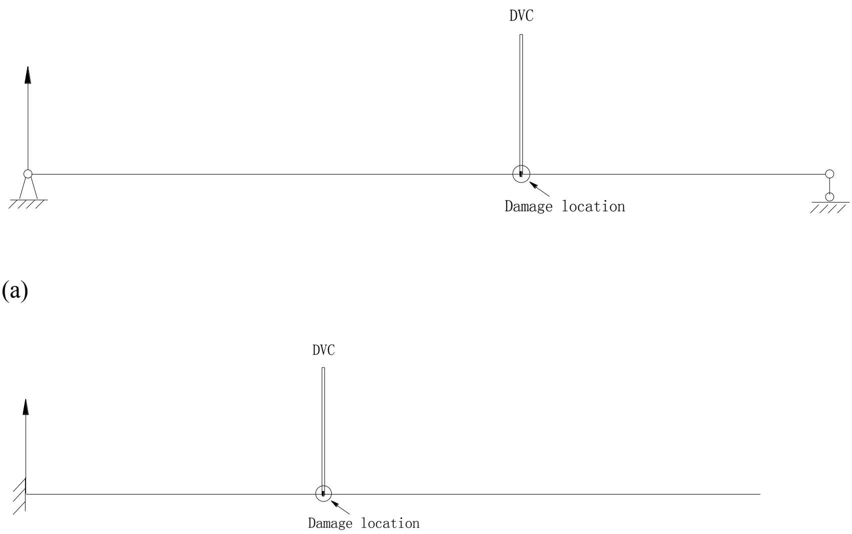


Figure 7: The displacement variation (DV) due to damage for the cantilever beam under a point load.

As a result, for the statically determinate structure, only the element associated with the characteristic force is deformed, whereas the remaining part of the structure is not deformed and only has rigid body motion. In other words, the displacement of the statically determinate structure under the characteristic force will consist of several parts of the rigid displacement except the segment associated with this characteristic force. For example, figures 4 and 5 presented the characteristic displacement shapes for the simple supported beam and the cantilever beam, respectively. One can see from figures 4 and 5 that the characteristic displacements for this type of statically determine beam consist of two line segments. As discussed previously, structural displacement variation due to damage is a linear combination of the characteristic displacements associated with the damaged elements. Therefore, the displacement variation of the statically determinate structure under an arbitrary load will consist of several parts of the rigid displacement except the damaged segments. For example, the displacement variations due to damage for the simple supported beam and the cantilever beam under a point load are shown in figures 6 and 7, respectively. From figure 6, the displacement variation of the simple supported beam is zero at the supports and increases linearly as one move towards the mid-span of the beam. For each damage location, the change in the displacement reaches its maximum at the damaged element. Hence, for the simply supported beam, the region in which the displacement variation is maximum is the damaged region. From figure 7, the displacement variation of the cantilever beam is zero between the clamped end and the damaged element. As one moves away from the damage location towards the free end of the cantilever, the change in displacement increases linearly. Thus, for a cantilever beam, the point from which the displacement variation starts increasing linearly is the location of damage. Similar results have been observed in references [32], [33], and [40]. Additionally, it is obvious that the curvature of structural displacement variation, that is, the second derivatives of the displacement variation, should be zero except the damaged element for these statically determinate beams (shown in figure 8). In other words, the variation of the rotational angle due to damage is changed at the damage location and is constant along the remaining part of the structure. Hence, for these statically determinate beams, the region in which the curvature of structural displacement variation is peak value is the damaged region. Similar results have been observed in references [37-39], and [41-42].

For the statically indeterminate structure, the internal force in some parts of the structure under the characteristic force will be slightly greater than zero due to the limitation of redundant constraints. As a result, the characteristic displacement of the statically indeterminate structure may be consisting of the deformational displacements and the rigid displacements. Accordingly, the displacement variation

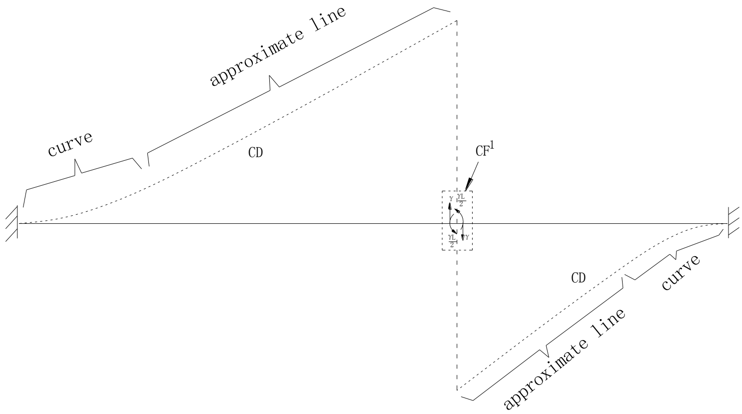


(a)

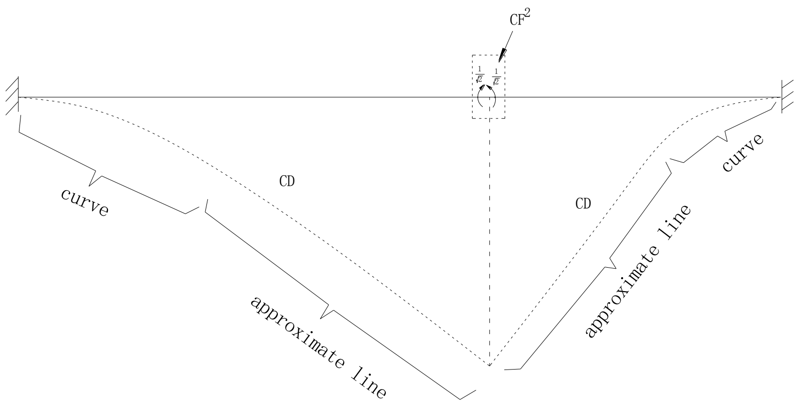
(b)

Figure 8: (a) The displacement variation curvature (DVC) due to damage for the simple supported beam; (b) The displacement variation curvature (DVC) due to damage for the cantilever beam.

of the statically indeterminate structure under an arbitrary load also might consist of the deformational displacements and the rigid displacements. For example, figure 9 presented the characteristic displacement shapes for the fixed-fixed beam. Then figure 10 showed the displacement variations due to damage for the fixed-fixed beam under a point load. It can be found from figure 9 that the characteristic displacement of the fixed-fixed beam consists of several curve segments. Correspondingly, the displacement variation due to damage for the fixed-fixed beam in figure 10 also consists of several curve segments. The curve segment in the characteristic displacement is attributed to the limitation of the fixed end in this statically indeterminate beam. It is noteworthy from figures 9 and 10 that the curvature of the curve segment gradually decreases as one move away from the fixed end towards the damage location. It can be explained by the decrease of the constraint effect of the fixed end as one move away from the fixed end towards the damage location. That is to say, the curve segments in figures 9 and 10 can be approximated as the line segments except the region near the restrained end. The curvature distribution of the displacement variations for the fixed-fixed beam is presented in figure 11. Therefore, most parts of the displacement variation for the statically indeterminate



(a)



(b)

Figure 9: (a) The characteristic displacement (CD) for the fixed-fixed beam under the characteristic force 1 (CF^1); (b) The characteristic displacement (CD) for the fixed-fixed beam under the characteristic force 2 (CF^2).

structure can be approximately seen as the rigid displacements except the damaged locations and the region near the restrained end.

According to the above discussion, the following important theorem for displacement-based damage localization has been derived as:

Theorem. (1) For the statically determinate structure under an arbitrary load, the displacement variation in a structure due to damage will consist of several parts of the rigid displacement except the damaged locations. (2) For the statically indeter-

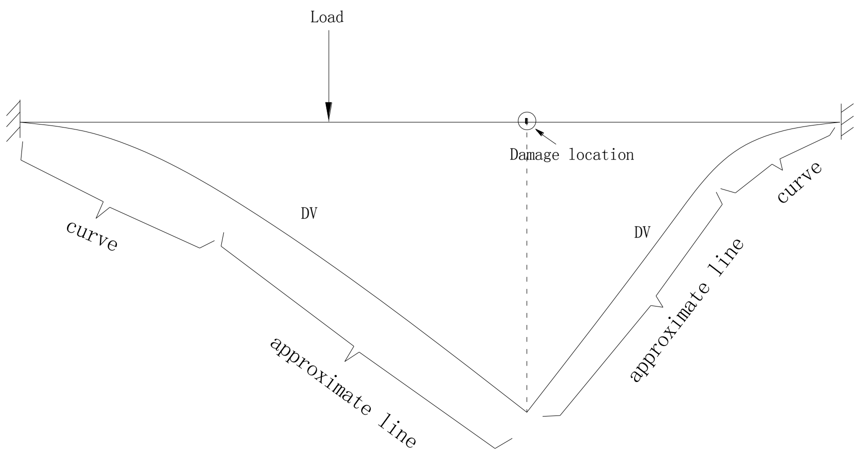


Figure 10: The displacement variation (DV) due to damage for the fixed-fixed beam under a point load.

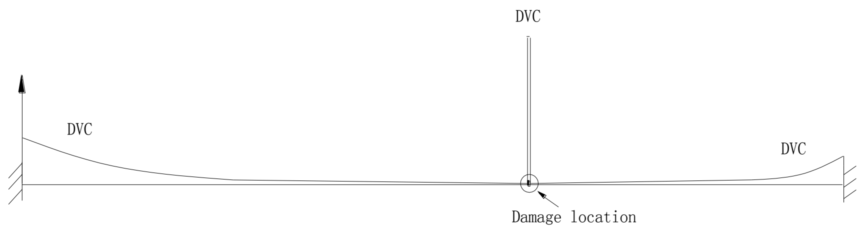


Figure 11: The displacement variation curvature (DVC) due to damage for the fixed-fixed beam.

minate structure under an arbitrary load, the displacement variation in a structure due to damage may be consisting of the deformational displacements and the rigid displacements. Moreover, most parts of the deformational displacement variation can be approximately seen as the rigid displacements except the damaged locations and the region near the restrained end.

3 Applications of the Theorem: a universal damage localization algorithm

The above theorem is in principle applicable to any structural type. From the theorem, the following universal algorithm for displacement-based damage localization can be concluded that:

(1) For the statically determinate structure, the turning points between each segment of the rigid displacement in the shape of the displacement variation are the locations

of damage;

(2) For the statically indeterminate structure, the turning points between each segment of the approximate rigid displacement (or the rigid displacement) in the shape of the displacement variation are the locations of damage;

(3) From the point of view of the curvature, the region in which the curvature of structural displacement variation is peak value is the damaged region.

As stated previously, figures 6,7,8,10 and 11 support the above conclusions. In this section, a single storey frame and a three-span continuous beam are used to further illustrate the applications of the presented universal method in structural damage localization.

3.1 A single storey frame

The first example is a single storey frame as shown in figure 12. It is noted that this example is a statically determinate structure. Figure 13 showed the displacement variations due to damage for the frame under a point load. It can be found from figure 13(a) that the shape of displacement variations due to damage at beam BC consists of two segments of the rigid displacement ($AB'E$ and $EC'D'$) and the turning point E is the damage location. Similarly, figure 13(b) shows that the shape of displacement variations due to damage at column AB consists of two segments of the rigid displacement (AE and $EB'C'D'$) and the turning point E is the damage location.

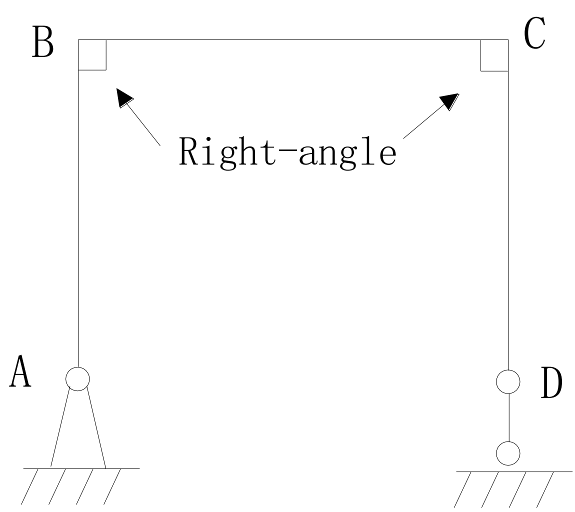


Figure 12: A single storey frame.

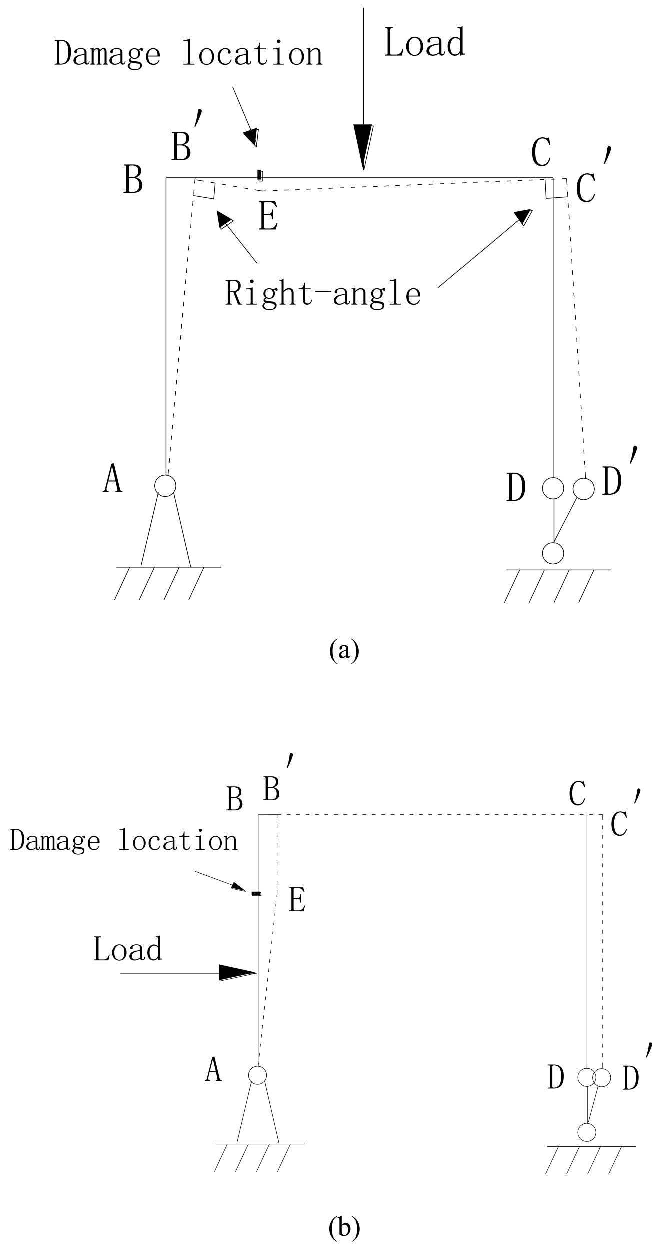


Figure 13: (a) The displacement variation (DV: —-) due to damage at beam BC for the frame under a point load; (b) The displacement variation (DV: —-) due to damage at column AB for the frame under a point load.

3.2 A three-span continuous beam

The second example is a three-span continuous beam as shown in figure 14. Note that this example is a statically indeterminate structure. The basic parameters of the structure are as follows: Young's modulus $E = 200GPa$, density $\rho = 7.8 \times 10^3 Kg/m^3$, moment of inertia $I = 1.0416 \times 10^{-6} m^4$, and cross-sectional area $A = 0.0025m^2$. The beam is modeled using 36 elements giving 70 DOFs (33 translational, 37 rotational) and the length of each element is $L = 0.1m$. Three load cases are given in figures 14(a)-(c), respectively. Damage in the beam was simulated as a reduction in the Young's modulus of individual elements. Three damage cases are studied in the example. Case 1: element 5 is damaged with a stiffness loss of 20%. Case 2: element 17 is damaged with a stiffness loss of 20%. Case 3: element 32 is damaged with a stiffness loss of 20%. Figures 15(a)-(c) showed the displacement variation due to damage at element 5 under the three load cases, respectively. Figures 16 and 17 presented the results of damage at element 17 and 32, respectively. From figures 15-17, the following observations are made: (1) the variation of the displacement has the largest value at the damage location independent of the location of the load; (2) the left and right sides of the damage location in figures 15-17 are line segments or approximate line segments. These observations supported the universal technique again.

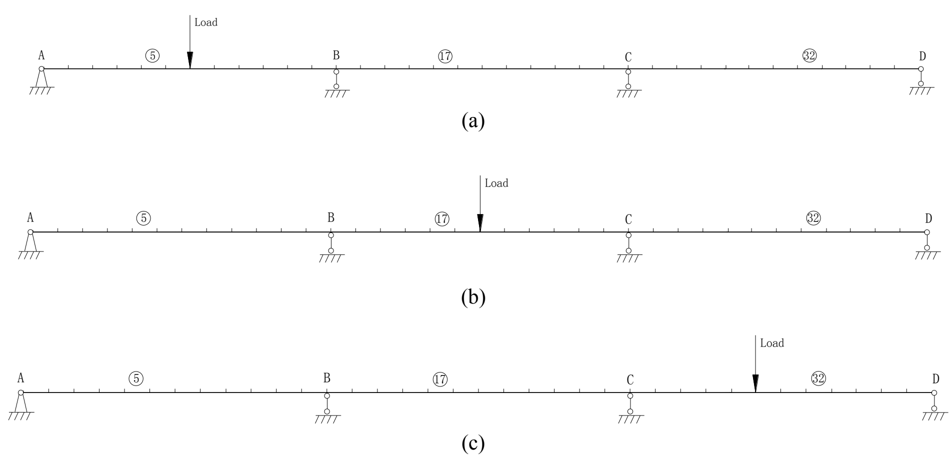
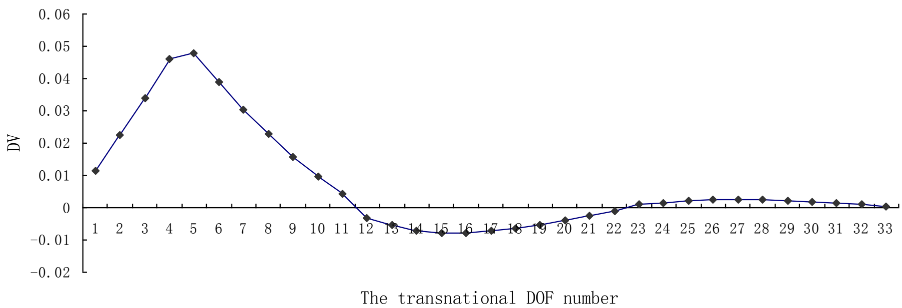
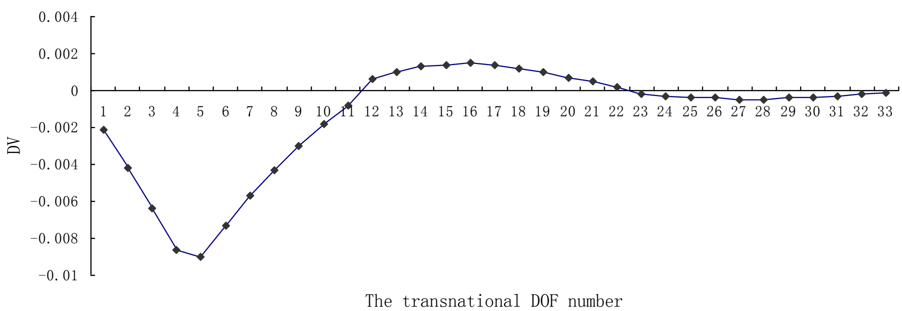


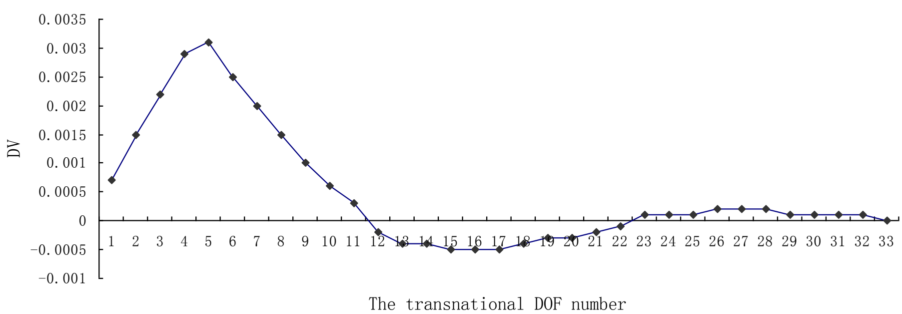
Figure 14: A three-span continuous beam.



(a)

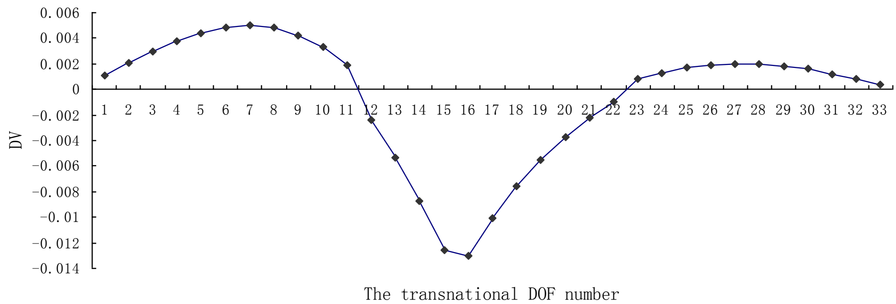


(b)

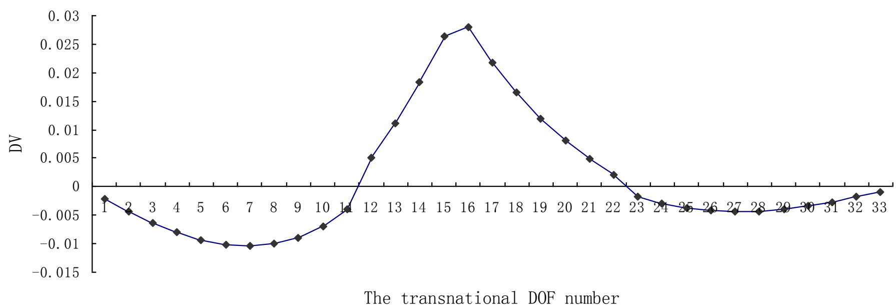


(c)

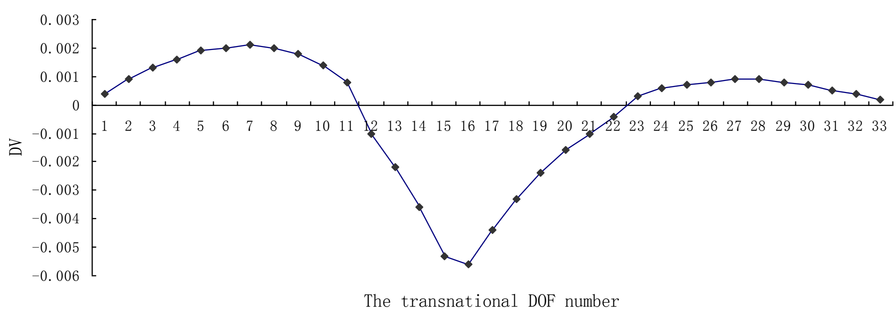
Figure 15: The displacement variation (DV) due to damage at element 5 for the three-span continuous beam: (a) load case 1; (b) load case 2; (c) load case 3.



(a)

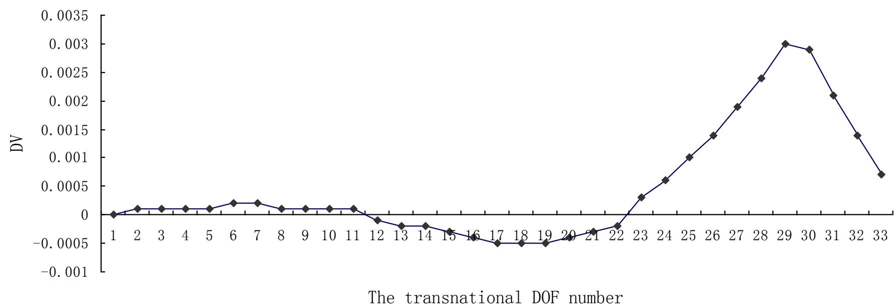


(b)

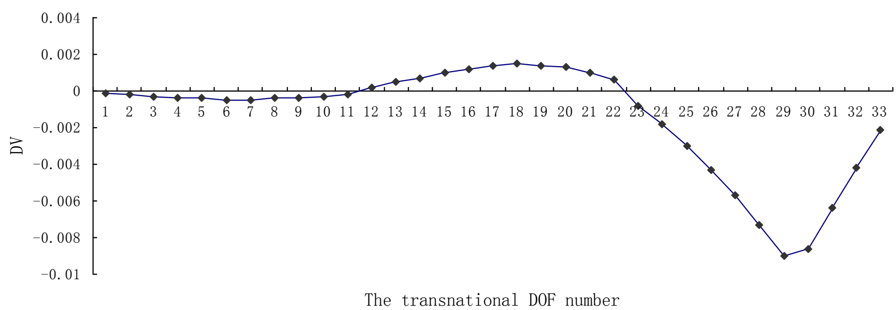


(c)

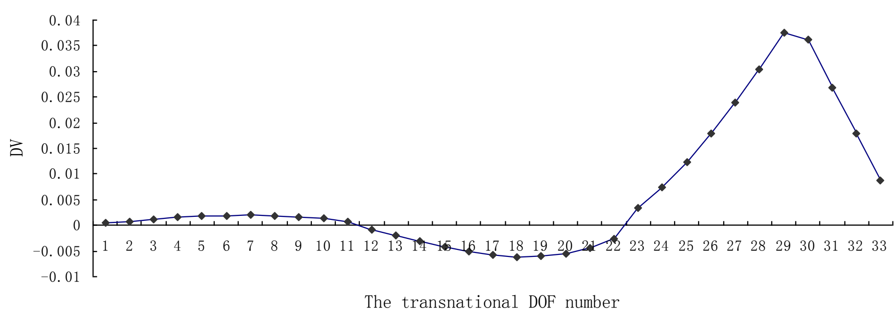
Figure 16: The displacement variation (DV) due to damage at element 17 for the three-span continuous beam: (a) load case 1; (b) load case 2; (c) load case 3.



(a)



(b)



(c)

Figure 17: The displacement variation (DV) due to damage at element 32 for the three-span continuous beam: (a) load case 1; (b) load case 2; (c) load case 3.

4 Conclusions

Using the well-known SMW formulas, a universal model-independent algorithm is presented in this paper for structural damage localization. The proposed method has a solid theoretical and physical base and can be used in damage localization for any structural type. Some common structures are used to illustrate the applications of the presented technique in structural damage localization. It has been shown that the presented algorithm is in principle generally applicable and useful for structural damage localization.

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