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An Artificial Boundary Method for Burgers' Equation in the Unbounded Domain

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Abstract: In this paper, we construct a numerical method for one-dimensional Burgers' equation in the unbounded domain by using artificial boundary conditions. The original problem is converted by Hopf-Cole transformation to the heat equation in the unbounded domain, the latter is reduced to an equivalent problem in a bounded computational domain by using two artificial integral boundary conditions, a finite difference method with discrete artificial boundary conditions is established by using the method of reduction of order for the last problem, and thereupon the numerical solution of Burgers' equation is obtained. This artificial boundary method is proved and verified to be uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time for solving Burgers' equation on the computational domain.

Keywords: Burgers' equation, Unbounded domain, Hopf-Cole transformation, Artificial boundary method, Finite difference method.

1 Introduction

Burgers' equation is an important and basic parabolic PDE in fluid mechanics [Burgers (1948)]. It is a model equation of numerous nonlinear problems in aerodynamics, traffic dynamics and so on. It can be used as a simplified form of Navier-Stokes equations in fluid dynamics. Burgers' equation was first given by H. Bateman (1915), then J.M. Burgers introduced the equation to simulate turbulence problems. When an analytic solution is not available, or the analytic one is not suitable to be used, a numerical method is necessary. Since Burgers' equation possesses complexity and universality of applications, its numerical solution is crucial in theory and practice. Burgers' equation can be converted to heat equation by Hopf-Cole transformation. Even for Burgers' equation with homogeneous

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Dirichlet boundary condition, the topic of the stability and the convergence is still valuable [Kadalbajoo and Awasthi (2006); Liao (2008)].

In fact, many numerical approaches have been applied to solve evolution equations which include Burgers' equation. They are listed as finite difference method, finite element method, finite volume method, collocation method, spectral element method, etc. Recently, a Local Radial Basis Function Meshless Method is applied for solution of the Burgers' equation with different initial and boundary conditions of various complexities [Hosseini and Hashemi (2011)]. Moreover, a Meshless Local Petrov-Galerkin Mixed Collocation Method is developed to solve the Cauchy inverse problems of heat transfer [Zhang, He, Dong, Li, Alotaibi and Atluri (2014)], and a Radial Basis Function Collocation Method is constructed for solving ill-posed time domain inverse problems in systems of nonlinear ODEs [Elgohary, Dong, Junkins and Atluri (2014)]. Although tremendous efforts have been devoted to solve the so-called direct problems and inverse problems, the finite difference method among them is early and fundamental in applications and can be combined with other methods [Dhawan, Kapoor, Kumar and Rawat (2012)]. The combination scheme of the finite difference method with the discretization of artificial integral boundary conditions, as an important algorithm for solving the PDEs on unbounded domain, needs to be elaborately established and theoretically analyzed.

As well-known, several kinds of problems in the areas of heat transfer, fluid dynamics and other applications are on unbounded domains and are solved numerically by using artificial boundary conditions [Feng (1983); Givoli (1992); Yu (2002); Liu and Yu (2008); Yu and Huang (2008); Zheng, Wang and Li (2011)]. The artificial boundary methods were obtained for various problems of heat equation on unbounded domains and the feasibility and effectiveness of the methods were shown by the numerical examples [Han and Huang (2002A); Han and Huang (2002B)]. Moreover, for the heat equation in a semi-unbounded domain $[-1,\infty)\times[0,\infty)$, Sun and Wu (2004) firstly proved that the finite difference scheme with an artificial boundary condition is uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time under an energy norm. Wu and Zhang (2011) also obtained the high-order artificial boundary conditions for the heat equation in unbounded domains, but only proved that the reduced initial-boundary-value problems were stable.

Furthermore, Han, Wu and Xu (2006) started to consider the nonlinear Burgers' equation in the unbounded domain as follows:

$$w_t + ww_x - \nu w_{xx} = F(x,t), \quad -\infty < x < +\infty, 0 < t \leq T, \tag{1}$$

$$w(x,0) = f(x), \quad -\infty < x < +\infty, \tag{2}$$

$$w(x,t) \to 0, \quad \text{when } |x| \to +\infty, \quad 0 \leq t \leq T, \tag{3}$$

where $\nu = \frac{1}{Re}$, Re is the Reynolds number, and the given functions F and f are sufficiently smooth with compact supports $\text{supp}\{F(x,t)\} \subset [x_l, x_r] \times [0,T]$ and supp $\{f(x)\} \subset [x_l, x_r]$. They obtained nonlinear artificial boundary conditions, constructed a nonlinear difference method with no theoretical convergence analysis, and supported it by numerical examples. Recently, Sun and Wu (2009) introduced a function transformation to reduce nonlinear Burgers' equation to a linear initial boundary value problem, deduced a linear finite difference scheme, and also proved that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and 3/2 in time.

In this paper, we consider the problem (1)-(3) with $F \equiv 0$ and convert it into an initial value problem of heat equation by using Hopf-Cole transformation in the following. Let

$$\omega(x,t) = -\int_x^{\infty} w(y,t)dy,$$

we obtain

$$\omega_t + \frac{1}{2}\omega_x^2 - \nu\omega_{xx} = 0, \;\; \omega(x,0) = -\int_x^{\infty} f(y)dy, \text{ and } \omega(x,t) \to 0 \text{ when } |x| \to +\infty.$$

Let $u = \exp(-\omega/2\nu) - 1$, then we have the initial value problem of heat equation:

$$u_t - \nu u_{xx} = 0, \quad -\infty < x < +\infty, 0 < t \le T, \tag{4}$$

$$u(x,0) = \phi(x) := \exp\left(\frac{1}{2\nu}\int_x^{\infty} f(y)dy\right) - 1, \quad -\infty < x < +\infty \tag{5}$$

$$u(x,t) \to 0, \quad \text{when } |x| \to +\infty, \quad 0 \le t \le T, \tag{6}$$

where the sufficiently smooth given function $\phi(x)$ has compact support $\text{supp}\{\phi(x)\} \subset [x_l, x_r]$.

In section 2, we derive artificial boundary conditions for the problem (4)-(6). In section 3, we construct a finite difference scheme for solving the problem on bounded computational domain with the artificial boundary conditions. Then a new solution of Burgers' equation is obtained and the difficulty for solving the nonlinear problem is avoided. In section 4, we prove that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and 3/2 in time. In section 5, a numerical example confirms the stability and convergence of the finite difference method.

2 The problem with artificial boundary conditions

Let us consider firstly the restriction of u for the problem (4)-(6) on $[x_r, +\infty) \times [0, T]$ as follows:

$u_t - \nu u_{xx} = 0, \quad x_r \leq x < +\infty, 0 < t \leq T,$

$u(x,0) = 0, \quad x_r \leq x < +\infty,$

$u(x,t) \to 0, \quad \text{when } |x| \to +\infty, \quad 0 \leq t \leq T.$

We can get the solution $u(x,t)$ by given $u(x_r,t)$:

$$u(x,t) = \frac{x - x_r}{2\sqrt{\pi\nu}} \int_0^t u(x_r, \lambda)(t-\lambda)^{-3/2} e^{-(x-x_r)^2/4(t-\lambda)} d\lambda.$$

Let $\mu = (x - x_r)/2\sqrt{t-\lambda}$, we obtain

$$u(x,t) = \frac{2}{\sqrt{\pi\nu}} \int_{(x-x_r)/2\sqrt{t}}^{\infty} u(x_r, t - \frac{(x-x_r)^2}{4\mu^2}) e^{-\mu^2} d\mu,$$

and

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= -\frac{2}{\sqrt{\pi\nu}} u(x_r, 0) e^{-(x-x_r)^2/4t} \frac{1}{2\sqrt{t}} \\ &\quad + \frac{2}{\sqrt{\pi\nu}} \int_{(x-x_r)/2\sqrt{t}}^{\infty} u_t(x_r, t - \frac{(x-x_r)^2}{4\mu^2})(-\frac{2(x-x_r)}{4\mu^2}) e^{-\mu^2} d\mu. \end{aligned}$$

Returning to the variable λ we get

$$\frac{\partial u(x,t)}{\partial x} = -\frac{1}{\sqrt{\pi\nu}} \int_0^t \frac{\partial u(x_r,\lambda)}{\partial \lambda} \frac{1}{\sqrt{t-\lambda}} e^{-(x-x_r)^2/4(t-\lambda)} d\lambda,$$

and taking the limit $x \to +x_r$ we obtain a relation between $u_t(x_r,t)$ and $u_x(x_r,t)$:

$$u_x(x_r,t) = -\frac{1}{\sqrt{\pi\nu}} \int_0^t \frac{u_\lambda(x_r,\lambda)}{\sqrt{t-\lambda}} d\lambda.$$

Similarly, we can obtain the artificial boundary condition on $x = x_l$.

So, we reduce the problem (4)-(6) to a problem in the bounded computational domain:

$$u_t - \nu u_{xx} = 0, \quad x_l \leq x \leq x_r, 0 < t \leq T, \tag{7}$$

$$u(x,0) = \phi(x), \quad x_l \leq x \leq x_r, \tag{8}$$

$$u_x(x_l,t) = \frac{1}{\sqrt{\pi\nu}} \int_0^t \frac{u_\lambda(x_l,\lambda)}{\sqrt{t-\lambda}} d\lambda, \quad 0 \leq t \leq T, \tag{9}$$

$$u_x(x_r,t) = -\frac{1}{\sqrt{\pi\nu}} \int_0^t \frac{u_\lambda(x_r,\lambda)}{\sqrt{t-\lambda}} d\lambda, \quad 0 \leq t \leq T. \tag{10}$$

3 The derivation of the difference scheme

In order to derive the finite difference method, the bounded computational domain is divided into an $M \times N$ uniform mesh. Let $h = (x_r - x_l)/M$, $x_i = x_l + ih$ for $0 \le i \le M$, $\tau = T/N$, $t_n = n\tau$ for $0 \le n \le N$, $r = \frac{\nu\tau}{h^2}$, and u_i^n be the numerical solution of $u(x,t)$ at (x_i, t_n). Introduce the notations:

$$u_{i-\frac{1}{2}}^n = \frac{1}{2}(u_i^n + u_{i-1}^n), \quad \delta_x u_{i-\frac{1}{2}}^n = \frac{1}{h}(u_i^n - u_{i-1}^n), \quad u_i^{n-\frac{1}{2}} = \frac{1}{2}(u_i^n + u_i^{n-1}),$$

$$\delta_t u_i^{n-\frac{1}{2}} = \frac{1}{\tau}(u_i^n - u_i^{n-1}), \quad \delta_x^2 u_i^n = \frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n),$$

$$\|u^n\|_A = \sqrt{h\sum_{i=1}^{M}(u_{i-\frac{1}{2}}^n)^2}, \quad \|\delta_x u^n\| = \sqrt{h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}}^n)^2}.$$

Lemma 1 *(see [Han and Wu (2012)]) Suppose $f(t) \in C^2[0,t_n]$, then*

$$\left|\int_0^{t_n} f'(t)\frac{dt}{\sqrt{t_n - t}} - \sum_{k=1}^{n}\frac{f(t_k) - f(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{dt}{\sqrt{t_n - t}}\right| \le \frac{1}{12}(20\sqrt{2} - 23)\max_{0\le t\le t_n}|f''(t)|\tau^{\frac{3}{2}}.$$

By introducing a new variable $v = \frac{\partial u}{\partial x}$ to reduce the order of heat equation, the problem (7)-(10) is equivalent to the problem of first-order differential equations:

$$\frac{\partial u}{\partial x} = \nu\frac{\partial v}{\partial x}, \quad \forall(x,t) \in [x_l, x_r] \times [0,T], \tag{11}$$

$$v - \frac{\partial u}{\partial x} = 0, \quad \forall(x,t) \in [x_l, x_r] \times [0,T], \tag{12}$$

$$u(x,0) = \phi(x), \quad x_l \le x \le x_r, \tag{13}$$

$$v(x_l,t) = \frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{\partial u(x_l,\lambda)}{\partial\lambda}\frac{1}{\sqrt{t-\lambda}}d\lambda, \tag{14}$$

$$v(x_r,t) = -\frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{\partial u(x_r,\lambda)}{\partial\lambda}\frac{1}{\sqrt{t-\lambda}}d\lambda. \tag{15}$$

Define the grid functions:

$$U_i^n = u(x_i,t_n), \quad V_i^n = v(x_i,t_n), \quad 0 \le i \le M, \quad n \ge 0.$$

Using Lemma 1, it follows from (15) that

$$\begin{aligned}
V_M^n &= -\frac{1}{\sqrt{\pi\nu}}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{\partial u(x_r,\lambda)}{\partial\lambda}\frac{d\lambda}{\sqrt{t_n-\lambda}} \\
&= -\frac{1}{\sqrt{\pi\nu}}\sum_{k=1}^{n}\frac{U_M^k-U_M^{k-1}}{\tau}\int_{t_{k-1}}^{t_k}\frac{d\lambda}{\sqrt{t_n-\lambda}}+O(\tau^{\frac{3}{2}}) \\
&= -\frac{2}{\sqrt{\pi\nu}}\sum_{k=1}^{n}(U_M^k-U_M^{k-1})a_{n-k}+O(\tau^{\frac{3}{2}}) \\
&= -\frac{2}{\sqrt{\pi\nu}}[a_0U_M^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})U_M^k-a_{n-1}U_M^0]+O(\tau^{\frac{3}{2}}),\ n=1,2,\cdots.
\end{aligned}$$

Therefore, we have

$$V_M^{n-\frac{1}{2}}=\frac{1}{2}(V_M^{n-1}+V_M^n)=-\frac{2}{\sqrt{\pi\nu}}[a_0U_M^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})U_M^{k-\frac{1}{2}}-a_{n-1}U_M^0]+O(\tau^{\frac{3}{2}}),$$

and similarly,

$$V_0^{n-\frac{1}{2}}=\frac{1}{2}(V_0^{n-1}+V_0^n)=\frac{2}{\sqrt{\pi\nu}}[a_0U_0^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})U_0^{k-\frac{1}{2}}-a_{n-1}U_0^0]+O(\tau^{\frac{3}{2}}).$$

Using Taylor expansion, we have

$$\delta_t U_{i-\frac{1}{2}}^{n-\frac{1}{2}}-\nu\delta_x V_{i-\frac{1}{2}}^{n-\frac{1}{2}}=p_{i-\frac{1}{2}}^{n-\frac{1}{2}},\quad 1\le i\le M,\quad n\ge 1,$$

$$V_{i-\frac{1}{2}}^{n-\frac{1}{2}}-\delta_x U_{i-\frac{1}{2}}^{n-\frac{1}{2}}=q_{i-\frac{1}{2}}^{n-\frac{1}{2}},\quad 1\le i\le M,\quad n\ge 1,$$

$$U_i^0=\phi(x_i),\quad 0\le i\le M,$$

$$V_0^{n-\frac{1}{2}}=\frac{2}{\sqrt{\pi\nu}}[a_0U_0^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})U_0^{k-\frac{1}{2}}-a_{n-1}U_0^0]+s^{n-\frac{1}{2}},\quad n\ge 1,$$

$$V_M^{n-\frac{1}{2}}=-\frac{2}{\sqrt{\pi\nu}}[a_0U_M^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})U_M^{k-\frac{1}{2}}-a_{n-1}U_M^0]+t^{n-\frac{1}{2}},\quad n\ge 1,$$

where

$$|p_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2+h^2), \quad |q_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2+h^2), \quad 1\le i\le M, \quad n\ge 1,$$

$$|t^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}, \quad |s^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}, \quad n\ge 1,$$

and c is a constant.

Thus, we construct a difference scheme for (11)-(15) in the following:

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu\delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1\le i\le M, \quad n\ge 1, \tag{16}$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1\le i\le M, \quad n\ge 1, \tag{17}$$

$$u_i^0 = \phi(x_i), \quad 0\le i\le M, \tag{18}$$

$$v_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0], \quad n\ge 1, \tag{19}$$

$$v_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0]. \quad n\ge 1. \tag{20}$$

Theorem 1 *The difference scheme (16)-(20) is equivalent to the following (21)-(25):*

$$u_i^0 = \phi(x_i), \quad 0\le i\le M, \tag{21}$$

$$\frac{1}{2}(\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}) - \nu\delta_x^2 u_i^{n-\frac{1}{2}} = 0, \quad 1\le i\le M-1, \quad n\ge 1, \tag{22}$$

$$\delta_t u_{\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\nu}{h}[\frac{2}{\sqrt{\pi\nu}}(a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0) - \delta_x u_{\frac{1}{2}}^{n-\frac{1}{2}}] = 0, \quad n\ge 1, \tag{23}$$

$$\delta_t u_{M-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\nu}{h}[\frac{2}{\sqrt{\pi\nu}}(a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_M^0) + \delta_x u_{M-\frac{1}{2}}^{n-\frac{1}{2}}] = 0, \quad n\ge 1, \tag{24}$$

where

$$a_m = \frac{1}{\sqrt{t_{m+1}}+\sqrt{t_m}} = \frac{1}{\sqrt{\tau}(\sqrt{m+1}+\sqrt{m})}, \quad m = 0,1,2,\cdots. \tag{25}$$

Proof Multiplying (16) by $\frac{1}{2}h$ and using (17) we obtain

$$v_i^{n-\frac{1}{2}} = \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2\nu}\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{26}$$

$$v_i^{n-\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2\nu}\delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}, \quad 0 \le i \le M-1, \quad n \ge 1. \tag{27}$$

From (26) and (27) for i from 1 to $M-1$ we obtain

$$\delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2\nu}\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2\nu}\delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M-1, \quad n \ge 1,$$

or

$$\frac{1}{2}(\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}) - \nu\delta_x^2 u_i^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M-1, \quad n \ge 1,$$

which is (22).

When $i = 0$, from (19) and (27), we know that

$$\frac{2\sqrt{\nu}}{\sqrt{\pi}}[a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0] = \nu\delta_x u_{\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2}\delta_t u_{\frac{1}{2}}^{n-\frac{1}{2}}.$$

Dividing by $h/2$ on the both sides we obtain (23).

Similarly, when $i = M$, from (20) and (26), we know that

$$-\frac{2\sqrt{\nu}}{\sqrt{\pi}}[a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0] = \nu\delta_x u_{M-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2}\delta_t u_{M-\frac{1}{2}}^{n-\frac{1}{2}}.$$

Dividing by $h/2$ on the both sides we obtain (24).

The difference scheme (21)-(24) can be sorted as the following:

$$\begin{aligned}&(\frac{1}{2} - r)u_{i+1}^n + (1+2r)u_i^n + (\frac{1}{2} - r)u_{i-1}^n \\ &= (\frac{1}{2} + r)u_{i+1}^{n-1} + (1-2r)u_i^{n-1} + (\frac{1}{2} + r)u_{i-1}^{n-1}, \quad 1 \le i \le M-1,\end{aligned} \tag{28}$$

$$\begin{aligned}&(1+2r+\frac{4\sqrt{r}}{\sqrt{\pi}})u_0^n + (1-2r)u_1^n = (1-2r-\frac{4\sqrt{r}}{\sqrt{\pi}})u_0^{n-1} + (1+2r)u_1^{n-1} \\ &+ \frac{4\sqrt{r\tau}}{\sqrt{\pi}}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(u_0^k + u_0^{k-1}) + \frac{8\sqrt{r\tau}}{\sqrt{\pi}}a_{n-1}u_0^0,\end{aligned} \tag{29}$$

$$\begin{aligned}&(1+2r+\frac{4\sqrt{r}}{\sqrt{\pi}})u_M^n + (1-2r)u_{M-1}^n = (1-2r-\frac{4\sqrt{r}}{\sqrt{\pi}})u_M^{n-1} + (1+2r)u_{M-1}^{n-1} \\ &+ \frac{4\sqrt{r\tau}}{\sqrt{\pi}}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(u_M^k + u_M^{k-1}) + \frac{8\sqrt{r\tau}}{\sqrt{\pi}}a_{n-1}u_M^0.\end{aligned} \tag{30}$$

4 Analysis of the difference scheme

Lemma 2 *For any $F=\{F_1,F_2,F_3,\cdots\}$, we have*

$$\sum_{l=1}^{n}[a_0F_l-\sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})F_k]F_l \geq \frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}F_l^2, \quad n=1,2,\cdots,$$

where a_m is defined in (25).

Proof Let $b_m=a_{m-1}-a_m=\frac{1}{\sqrt{\tau}}(\frac{1}{\sqrt{m}+\sqrt{m-1}}-\frac{1}{\sqrt{m+1}+\sqrt{m}})$, $m\geq 1$, then $b_m>0$, and

$$\begin{aligned}
&\sum_{l=1}^{n}[a_0F_l-\sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})F_k]F_l\\
=\;&\sum_{l=1}^{n}a_0F_l^2-\sum_{l=1}^{n}\sum_{m=1}^{l-1}(a_{m-1}-a_m)F_{l-m}F_l\\
\geq\;&\sum_{l=1}^{n}a_0F_l^2-\frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_m(F_{l-m}^2+F_l^2)\\
=\;&\sum_{l=1}^{n}a_0F_l^2-\frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_{l-m}F_m^2-\frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_mF_l^2\\
=\;&\sum_{l=1}^{n}a_0F_l^2-\frac{1}{2}\sum_{m=1}^{n}\sum_{l=m+1}^{n}b_{l-m}F_m^2-\frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_mF_l^2\\
\geq\;&\sum_{l=1}^{n}a_0F_l^2-(\sum_{m=1}^{n-1}b_m)\sum_{l=1}^{n}F_l^2\\
=\;&[\frac{1}{\sqrt{\tau}}-\frac{1}{\sqrt{\tau}}(1-\frac{1}{\sqrt{n}+\sqrt{n-1}})]\sum_{l=1}^{n}F_l^2\\
\geq\;&\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}F_l^2. \quad \square
\end{aligned}$$

Lemma 3 *Suppose $\{u_i^n\}$ be the solution of*

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}}-\nu\delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}}=P_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1\leq i\leq M, \quad n\geq 1, \tag{31}$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}}-\delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}}=Q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1\leq i\leq M, \quad n\geq 1, \tag{32}$$

$$u_i^0=\phi(x_i), \quad 0\leq i\leq M, \tag{33}$$

$$v_0^{n-\frac{1}{2}}=\frac{2}{\sqrt{\pi\nu}}[a_0u_0^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_0^{k-\frac{1}{2}}-a_{n-1}u_0^0]+S^{n-\frac{1}{2}}, \quad n\geq 1, \tag{34}$$

$$v_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0] + T^{n-\frac{1}{2}}. \quad n \geq 1, \quad (35)$$

where $Supp\{\phi(x)\} \subset [x_0, x_M]$, *then*

$$\|u^n\|_A^2 \leq \exp(\frac{2T}{4-\tau}) \cdot \frac{1}{1-\frac{\tau}{4}}\{\|u^0\|_A^2 + \frac{\sqrt{\pi\nu t_n}}{2}\tau\sum_{l=1}^{n}[(T^{l-\frac{1}{2}})^2 + (S^{l-\frac{1}{2}})^2] + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2)\}, \quad n = 1, 2, \cdots. \quad (36)$$

Proof Multiplying (31) by $2u_{i-\frac{1}{2}}^{n-\frac{1}{2}}$ and multiplying (32) by $2v_{i-\frac{1}{2}}^{n-\frac{1}{2}}$, then adding the results, we have

$$\begin{aligned} &\frac{1}{\tau}[(u_{i-\frac{1}{2}}^n)^2 - (u_{i-\frac{1}{2}}^{n-1})^2] + 2(v_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 \\ &= \frac{2}{h}(u_i^{n-\frac{1}{2}}v_i^{n-\frac{1}{2}} - u_{i-1}^{n-\frac{1}{2}}v_{i-1}^{n-\frac{1}{2}}) + 2u_{i-\frac{1}{2}}^{n-\frac{1}{2}}P_{i-\frac{1}{2}}^{n-\frac{1}{2}} + 2v_{i-\frac{1}{2}}^{n-\frac{1}{2}}Q_{i-\frac{1}{2}}^{n-\frac{1}{2}} \\ &\leq \frac{2}{h}(u_i^{n-\frac{1}{2}}v_i^{n-\frac{1}{2}} - u_{i-1}^{n-\frac{1}{2}}v_{i-1}^{n-\frac{1}{2}}) + \frac{1}{2}(u_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + 2(P_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + \frac{1}{2}(v_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + 2(Q_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2, \\ &1 \leq i \leq M, n \geq 1. \end{aligned} \quad (37)$$

Multiplying the above inequality by τh and summing up for i from 1 to M, we obtain

$$(\|u^n\|_A^2 - \|u^{n-1}\|_A^2) + 2\tau\|v^{n-\frac{1}{2}}\|_A^2 \leq 2\tau(u_M^{n-\frac{1}{2}}v_M^{n-\frac{1}{2}} - u_0^{n-\frac{1}{2}}v_0^{n-\frac{1}{2}}) + \frac{\tau}{2}\|u^{n-\frac{1}{2}}\|_A^2 + \frac{\tau}{2}\|v^{n-\frac{1}{2}}\|_A^2 + 2\tau\|P^{n-\frac{1}{2}}\|_A^2 + 2\tau\|Q^{n-\frac{1}{2}}\|_A^2 \quad n \geq 1. \quad (38)$$

Noticing $\frac{\tau}{2}\|u^{n-\frac{1}{2}}\|_A^2 \leq \frac{\tau}{4}(\|u^n\|_A^2 + \|u^{n-1}\|_A^2)$, thus

$$\|u^l\|_A^2 - \|u^{l-1}\|_A^2 \leq 2\tau(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) + \frac{\tau}{4}(\|u^l\|_A^2 + \|u^{l-1}\|_A^2) + 2\tau\|P^{l-\frac{1}{2}}\|_A^2 + 2\tau\|Q^{l-\frac{1}{2}}\|_A^2, \quad l = 1, 2, \ldots, n.$$

Summing up for l from 1 to n, we have

$$\|u^n\|_A^2 \leq \|u^0\|_A^2 + 2\tau\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) + \frac{\tau}{4}\|u^n\|_A^2 + \frac{\tau}{2}\sum_{l=0}^{n-1}\|u^l\|_A^2 + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2).$$

Substituting (34) and (35) into the above inequality, and using Lemma 2, we have

$$
\begin{aligned}
\|u^n\|_A^2 \le & \frac{1}{1-\frac{\tau}{4}}[\|u^0\|_A^2 + 2\tau\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) \\
& + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{\tau}{2}\sum_{l=0}^{n-1}\|u^l\|_A^2] \\
= & \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 + \frac{2\tau}{1-\frac{\tau}{4}}\cdot(-\frac{2}{\sqrt{\pi\nu}})\sum_{l=1}^{n}[a_0 u_M^{l-\frac{1}{2}} - \sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})u_M^{k-\frac{1}{2}}]u_M^{l-\frac{1}{2}} \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}u_M^{l-\frac{1}{2}}T^{l-\frac{1}{2}} - \frac{2\tau}{1-\frac{\tau}{4}}\cdot(\frac{2}{\sqrt{\pi\nu}})\sum_{l=1}^{n}[a_0 u_0^{l-\frac{1}{2}} - \sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})u_0^{k-\frac{1}{2}}]u_0^{l-\frac{1}{2}} \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}u_0^{l-\frac{1}{2}}S^{l-\frac{1}{2}} + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2 \\
\le & \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 - \frac{2\tau}{1-\frac{\tau}{4}}\cdot\frac{2}{\sqrt{\pi\nu}}\cdot\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}})^2 + \frac{\tau}{1-\frac{\tau}{4}}(\frac{2}{\sqrt{\pi\nu t_n}}\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}})^2 \\
& + \frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}(T^{l-\frac{1}{2}})^2) - \frac{2\tau}{1-\frac{\tau}{4}}\cdot\frac{2}{\sqrt{\pi\nu}}\cdot\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}(u_0^{l-\frac{1}{2}})^2 \\
& + \frac{\tau}{1-\frac{\tau}{4}}(\frac{2}{\sqrt{\pi\nu t_n}}\sum_{l=1}^{n}(u_0^{l-\frac{1}{2}})^2 + \frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}(S^{l-\frac{1}{2}})^2) \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2 \\
\le & \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 + \frac{\tau}{1-\frac{\tau}{4}}\frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}[(T^{l-\frac{1}{2}})^2 + (S^{l-\frac{1}{2}})^2] \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2, \quad n = 1, 2, \cdots.
\end{aligned}
$$

Using Gronwall's lemma,

$$
\begin{aligned}
\|u^n\|_A^2 \quad = \quad & \exp(\frac{2T}{4-\tau})\cdot\frac{1}{1-\frac{\tau}{4}}\cdot\{\|u^0\|_A^2 + \frac{\sqrt{\pi\nu t_n}}{2}\tau\sum_{l=1}^{n}[(T^{l-\frac{1}{2}})^2 + (S^{l-\frac{1}{2}})^2] \\
& + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2)\}, \quad n = 1, 2, \cdots.
\end{aligned}
$$

Theorem 2 *The difference scheme (21)-(25) is uniquely solvable.*

Proof From Theorem 1, it suffices to prove that the difference scheme (16)-(20) is uniquely solvable. When initial value is homogeneous, using Lemma 3, we have

$$\|u^n\|_A^2 = 0, \quad n = 1, 2, \cdots.$$

Theorem 3 *Let* $\{u_i^n | 0 \le i \le M, n \ge 1\}$ *be the solution of (21)-(25), then*

$$\|u^n\|_A^2 \le \frac{\exp(\frac{2T}{4-\tau})}{1-\frac{\tau}{4}} \|u^0\|_A^2, \quad n = 1, 2, \cdots. \tag{39}$$

Proof From Theorem 2.2, it suffices to prove that (39) holds for the difference scheme (16)-(20). Therefore, (39) follows directly from Lemma 3.2.

Theorem 4 *Suppose that the problem (4)-(6) has solution* $u(x,t) \in C_{x,t}^{4,3}(\mathbb{R} \times [0,T])$. *Let* $\{u_i^n\}$ *be the solution of (21)-(25), and let* $\tilde{u}_i^n = U_i^n - u_i^n$, *then*

$$\|\tilde{u}^n\|_A^2 \le \frac{CT}{4-\tau}(\sqrt{\pi\nu T} + 4)\exp(\frac{2T}{4-\tau})(\tau^{\frac{3}{2}} + h^2)^2, \quad n = 1, 2, \cdots, [T/\tau], \tag{40}$$

where C *is a constant independent of* τ *and* h.

Proof We obtain the error equations:

$$\delta_t \tilde{u}_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu\delta_x \tilde{v}_{i-\frac{1}{2}}^{n-\frac{1}{2}} = p_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1,$$

$$\tilde{v}_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x \tilde{u}_{i-\frac{1}{2}}^{n-\frac{1}{2}} = q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1,$$

$$\tilde{u}_i^0 = 0, \quad 0 \le i \le M,$$

$$\tilde{v}_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0\tilde{u}_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\tilde{u}_0^{k-\frac{1}{2}} - a_{n-1}\tilde{u}_0^0] + s^{n-\frac{1}{2}}, \quad n \ge 1,$$

$$\tilde{v}_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0\tilde{u}_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\tilde{u}_M^{k-\frac{1}{2}} - a_{n-1}\tilde{u}_M^0] + t^{n-\frac{1}{2}}, \quad n \ge 1.$$

By using Lemma 3 and noticing $|p_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2 + h^2), |q_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2 + h^2), |t^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}$ and $|s^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}$, we obtain

$$\begin{aligned}
\|\tilde{u}^n\|_A^2 &= \exp(\frac{2T}{4-\tau}) \cdot \frac{1}{1-\frac{\tau}{4}} \cdot \{\|\tilde{u}^0\|_A^2 + \frac{\sqrt{\pi\nu t_n}}{2}\tau\sum_{l=1}^{n}[(t^{l-\frac{1}{2}})^2 + (s^{l-\frac{1}{2}})^2] \\
&\quad + 2\tau\sum_{l=1}^{n}(\|p^{l-\frac{1}{2}}\|^2 + \|q^{l-\frac{1}{2}}\|^2)\} \\
&\le \frac{CT}{4-\tau}(\sqrt{\pi\nu T} + 4)\exp(\frac{2T}{4-\tau})(\tau^{\frac{3}{2}} + h^2)^2, \quad n = 1, 2, \cdots, [T/\tau].
\end{aligned}$$

Theorem 4 shows that the convergence order of (21)-(24) is 2 in space and 3/2 in time for the problem (7)-(10) of the heat equation with artificial boundary conditions. Finally, the numerical solution of Burgers' equation is obtained by using central difference w.r.t. x as the following:

$$w_i^n = -\frac{\nu}{h}\frac{u_{i+1}^n - u_{i-1}^n}{1+u_i^n}, \tag{41}$$

which keeps the corresponding unique solvability, unconditional stability and convergence in space and in time.

By the way, the artificial integral boundary method in [Sun and Wu (2009)] is suitable for the inhomogenous Burgers' equation. For the homogenous problem (4)-(6), it gives

$$u(x_l,t) = \frac{1}{\sqrt{\pi\nu}}\int_0^t [\nu u_x(x_l,\lambda) - \frac{1+u(x_l,\lambda)}{2(x_r-x_l)}\int_{x_l}^{x_r} f(x)dx]\frac{d\lambda}{\sqrt{t-\lambda}}, \quad 0 \le t \le T,$$

which deduces a discrete boundary condition different from (23), and finally the numerical solution of Burgers' equation is derived from

$$w_{i-\frac{1}{2}}^n = -\frac{2\nu\delta_x u_{i-\frac{1}{2}}^n}{1+u_{i-\frac{1}{2}}^n},$$

which is slightly different from (41).

5 Numerical examples

We test the stability and accuracy of the proposed method by solving Burgers' equation with an initial condition $f(x) = \frac{4\nu x}{e^{x^2}+1}$. The support of f is approximately compact since $|f(x)|$ is small enough outside the computational domain $[x_l, x_r] = [-5,5]$. The exact solution is

$$w(x,t) = -2\nu\frac{\frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{\infty}\frac{\xi-x}{2\nu t}\exp(-\xi^2 - \frac{(x-\xi)^2}{4\nu t})d\xi}{1+\frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{\infty}\exp(-\xi^2 - \frac{(x-\xi)^2}{4\nu t})d\xi}.$$

The error of numerical solutions and the convergence order w.r.t τ are shown in table 1. The error of numerical solutions and the convergence order w.r.t h are shown in table 2.

Table 1: Convergence w.r.t. τ of examples for $\nu = 0.5$, $T = 1$, $h = 0.002$ and $h = \tau^{3/4}$.

N	M	L^∞-error	order	L^2-error	order	M	L^∞-error	order	L^2-error	order
10	5000	3.4349e-04	—	5.0342e-04	—	56	3.6538e-03	—	5.3702e-03	—
20	5000	8.6462e-05	1.9901	1.2597e-04	1.9987	95	1.2531e-03	1.5439	1.8421e-03	1.5436
40	5000	3.6936e-05	1.2270	3.2692e-05	1.9461	159	4.4110e-04	1.5063	6.4865e-04	1.5058
80	5000	3.1874e-05	0.2126	1.2036e-05	1.4416	267	1.5473e-04	1.5114	2.2819e-04	1.5072

Table 2: Convergence w.r.t. h of examples for $\nu = 0.5$, $T = 1$, $\tau = 0.005$ and $\tau = h^{4/3}$.

M	N	L^∞-error	order	L^2-error	order	N	L^∞-error	order	L^2-error	order
25	200	1.5826e-02	—	2.3957e-02	—	3	1.7558e-02	—	2.8237e-02	—
50	200	4.1785e-03	1.9212	6.2111e-03	1.9475	9	4.5211e-03	1.9574	6.7089e-03	2.0734
100	200	1.0618e-03	1.9765	1.5793e-03	1.9756	22	1.1260e-03	2.0055	1.6559e-03	2.0185
200	200	2.6775e-04	1.9876	3.9703e-04	1.9920	54	2.7728e-04	2.0218	4.0895e-04	2.0176

6 Concluding Remarks

In this study, motivated by works of Han, Wu, Sun and their co-authors, an artificial boundary method for Burgers' equation in the unbounded domain is presented by (21), (28)-(30) and (41) succinctly. The inequality in Lemma 2 is slightly stronger than that in [Wu and Sun (2004)] and [Han and Wu (2012)]. Lemma 3 is proved by using Gronwall's lemma, a similar Lemma 4 in [Wu and Sun (2004)] for heat equation on the semi-infinite domain, i.e. Lemma 3.2.4 in [Han and Wu (2012)], was incorrectly proved by not using Gronwall's lemma, and could be modified and proved by the way of Lemma 3. Finally, the suggested method is clearly proved and verified to be uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time under an energy norm to solve Burgers' equation in the unbounded domain.

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