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Solving the Lane–Emden–Fowler Type Equations of Higher Orders by the Adomian Decomposition Method

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Abstract: In this paper, we construct the Lane–Emden–Fowler type equations of higher orders. We study the linear and the nonlinear Lane–Emden–Fowler type equations of the third and fourth orders, where other forms can be treated in a similar manner. We use the systematic Adomian decomposition method to handle these types of equations with specified initial conditions. We confirm that the Adomian decomposition method provides an efficient algorithm for exact and approximate analytic solutions of these equations. We corroborate this study by investigating several numerical examples that emphasize initial value problems.

Keywords: Initial value problems, Singularities, Lane–Emden–Fowler equation, Adomian decomposition method, Adomian polynomials.

1 Introduction

Many problems in the literature of mathematical physics can be distinctively formulated as equations of the Lane–Emden–Fowler type defined in the form

$$y'' + \frac{k}{x}y' + f(x)g(y) = 0,\ y(0) = y_0, y'(0) = 0,\ k > 0, \tag{1}$$

where $x > 0$, $f(x)$ and $g(y)$ are some given functions of x and y, respectively. The Emden–Fowler equation (1) arises in the study of fluid mechanics, relativistic mechanics, pattern formation, population evolution and in the study of chemical reactor systems.

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For $f(x) = 1$ and $g(y) = y^m$, Eq. (1) becomes the standard Lane–Emden equation of the first kind and index m, that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Moreover, the Lane–Emden equation of the first kind is a useful equation in astrophysics for computing the interior structure of polytropic stars. However, for $f(x) = 1$, $g(y) = e^y$ and $y(0) = 0$, Eq. (1) becomes the standard Lane–Emden equation of the second kind that models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere [Richardson (1921); Chandrasekhar (1967); Davis (1962); Wazwaz (2009)].

We note that (1) was derived by using the equation

$$x^{-k}\frac{d}{dx}\left(x^k\frac{d}{dx}\right)y + f(x)g(y) = 0, y(0) = y_0, y'(0) = 0, \tag{2}$$

where $x > 0$, and $k > 0$ is called the shape factor.

The singular behavior that occurs at $x = 0$ constitutes the main difficulty in solving the Lane–Emden and Emden–Fowler equations. A substantial amount of work has been devoted to understanding these type of problems for various structures represented by $g(y)$. Several analytic and numeric methods, which have been used in similar investigations, include the Hermite functions collocation method [Parand, Dehghan, Rezaeia, and Ghaderi (2010)], the Legendre pseudospectral method [Parand, Shahini, and Dehghan (2009)], the Adomian decomposition method [Wazwaz (2009); Adomian, Rach, and Shawagfeh (1995); Wazwaz (2005a,b, 2001, 2002)], the variational iteration method [Kanth and Aruna (2010); Wazwaz (2011b); Yildirim and Özis (2009); Dehghan and Shakeri (2008a)], the homotopy analysis method [Liao (2003, 2012)], the homotopy perturbation method [Yildirim and Özis (2007)], the hybrid functions method [Tabrizidooz, Marzban, and Razzaghi (2009)], the symmetry method [Khanlique, Mahomed, and Muatjetjeja (2008); Muatjetjeja and Khanlique (2011)], etc. Implicit solutions were also considered in [Momoniat and Harley (2011)].

The Adomian decomposition method [Adomian and Rach (1983); Adomian (1983, 1986, 1994); Wazwaz (2009, 2011a); Duan, Rach, Baleanu, and Wazwaz (2012); Rach (2012)] is a systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations [Dehghan and Tatari (2010); Dehghan and Shakeri (2009)], partial differential equations [Dehghan and Salehi (2011); Dehghan (2004a,b)], integral equations [Wazwaz (2011a); Fu, Wang, and Duan (2013)], integro-differential equations [Wazwaz (2011a)], delay differential equation [Dehghan and Shakeri (2008b); Shakeri and Dehghan (2010); Dehghan and Salehi (2010)], etc. The Adomian decomposition method provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering.

The main goal of this work is to extend the second-order Emden–Fowler equation to Emden–Fowler type equations of higher order. To achieve this goal, we follow the sense of (2), where the first differential operator will be replaced by other differential operators of higher orders. Then we use the systematic Adomian decomposition method to handle these types of equations with specified initial conditions.

2 Formulation of the Emden–Fowler type equations

In this section, we follow the sense of (2) to formulate the singular Emden–Fowler type equations of higher orders and for a variety of shape factors. We will first set the generalized formula that can be used for any order, but we will focus our work on the Emden–Fowler type equations of the third and fourth orders only. Other higher-order equations can be derived in a similar manner and can be investigated by an identical approach. For the construction of these equations, we therefore introduce the following generalized formula, which subsumes Eq. (1), as

$$\begin{aligned} &x^{-k}\frac{d^n}{dx^n}\left(x^k\frac{d}{dx}\right)y+f(x)g(y)=0,\\ &y(0)=y_0,y'(0)=y''(0)=\cdots=y^{(n)}(0)=0,\ n\geq 1. \end{aligned} \tag{3}$$

2.1 Formulation of the Emden–Fowler type equations of the third order

To derive the Lane-Emden-Fowler type equations of the third order, we set $n=2$ in (3) to find

$$x^{-k}\frac{d^2}{dx^2}\left(x^k\frac{d}{dx}\right)y+f(x)g(y)=0,\ y(0)=y_0,y'(0)=y''(0)=0. \tag{4}$$

This in turn gives the Emden–Fowler type equations of the third order in the form

$$y'''+\frac{2k}{x}y''+\frac{k(k-1)}{x^2}y'+f(x)g(y)=0,y(0)=y_0,y'(0)=y''(0)=0. \tag{5}$$

Notice that the singular point $x=0$ appears twice as x and x^2 with shape factors $2k$ and $k(k-1)$, respectively. Moreover, the third term vanishes for $k=1$ and the shape factor in this case reduces to 2.

2.2 Formulation of the Emden–Fowler type equations of the fourth order

To derive the Lane-Emden-Fowler type equations of the fourth order, we set $n = 3$ in (3) to find

$$x^{-k}\frac{d^3}{dx^3}\left(x^k\frac{d}{dx}\right)y + f(x)g(y) = 0. \tag{6}$$

This in turn gives the Emden-Fowler type equations of the fourth order in the form

$$\begin{aligned} &y^{(iv)} + \frac{3k}{x}y''' + \frac{3k(k-1)}{x^2}y'' + \frac{k(k-1)(k-2)}{x^3}y' + f(x)g(y) = 0,\\ &y(0) = y_0,\ y'(0) = y''(0) = y'''(0) = 0. \end{aligned} \tag{7}$$

Notice that the singular point $x = 0$ appears three times as x, x^2 and x^3 with shape factors $3k, 3k(k-1)$ and $k(k-1)(k-2)$, respectively. Moreover, the third and the fourth terms vanish for $k = 1$ and the shape factor in this case is 3, whereas the fourth term vanishes for $k = 2$, and the shape factors reduce to 6 and 6.

For $f(x) = 1$, Eqs. (5) and (7) are the Lane–Emden type equations of the third and the fourth order, respectively.

2.3 Formulation of the Emden–Fowler type equations of higher orders

To derive the generalized Emden–Fowler type equations for higher orders, we first set

$$x^{-k}\frac{d^n}{dx^n}\left(x^k\frac{d}{dx}\right)y + f(x)g(y) = 0. \tag{8}$$

This in turn gives the generalized Emden–Fowler type equations of higher orders in the form

$$\begin{aligned} &y^{(n+1)} + \sum_{r=1}^{n}\binom{n}{r}\left(\prod_{j=1}^{r}(k-j+1)\right)\frac{1}{x^r}y^{(n-r+1)} + f(x)g(y) = 0,\\ &y(0) = y_0, y'(0) = y''(0) = \cdots = y^{(n)}(0) = 0, \end{aligned} \tag{9}$$

Notice that the singular point at $x = 0$ appears n time as $x, x^2, \cdots, x^n$ with the distinct shape factors $k_{n,r} = \binom{n}{r}\prod_{j=1}^{r}(k-j+1)$. For $f(x) = 1$, Eq. (9) reduces to the generalized Lane-Emden type equations of higher orders.

3 Analysis of the proposed method

As stated before, the Adomian decomposition method will be used for analytic treatment of the Emden–Fowler type equations. The Adomian decomposition method usually starts by defining the equation in an operator form. Although our treatment will be identical for all cases, we will present the analysis for the third order case in detail. However, we will also summarize the main steps for the fourth order case.

3.1 Framework for the Emden–Fowler type equations of third order

We begin our analysis by first studying the Emden–Fowler type equation of the third order

$$x^{-k}\frac{d^2}{dx^2}\left(x^k\frac{d}{dx}\right)y+f(x)g(y)=0, y(0)=\alpha, y'(0)=y''(0)=0. \tag{10}$$

The proposed framework rests mainly on defining the differential operator L in terms of the first three derivatives in the form

$$L(y)=-f(x)g(y), \tag{11}$$

where the differential operator L contains the first three derivatives as

$$L=x^{-k}\frac{d^2}{dx^2}\left(x^k\frac{d}{dx}\right), \tag{12}$$

in order to overcome the singular behavior at $x=0$. Based on (12), the optimal definition of L^{-1} is a threefold definite integration operator defined as

$$L^{-1}(.)=\int_0^x x^{-k}\int_0^x\int_0^x x^k(.)\,dx\,dx\,dx. \tag{13}$$

Applying L^{-1} of (13) to the first term of (10), which is equivalent to the first three terms of (5), we find that

$$\begin{aligned}
&L^{-1}\left(y'''+\frac{2k}{x}y''+\frac{k(k-1)}{x^2}y'\right)\\
&=\int_0^x x^{-k}\int_0^x\int_0^x x^k\left(y'''+\frac{2k}{x}y''+\frac{k(k-1)}{x^2}y'\right)dx\,dx\,dx\\
&=\int_0^x x^{-k}\int_0^x\left[x^k y''+kx^{k-1}y'\right]dx\,dx\\
&=\int_0^x y'\,dx=y(x)-y(0),
\end{aligned} \tag{14}$$

where integration by parts was used more than once in the evaluation of this triple integral. We note that for the case of $0 < k < 1$, the limit calculated by L'Hospital's rule,

$$\lim_{x\to 0} x^{k-1}y' = \lim_{x\to 0} \frac{y'}{x^{1-k}} = \lim_{x\to 0} \frac{x^k y''}{1-k} = 0,$$

is used in the derivation of (14). Combining (11) and (14) yields

$$y(x) = \alpha - L^{-1}\left(f(x)g(y)\right). \tag{15}$$

3.2 Framework for the Emden–Fowler type equations of the fourth order

We continue our analysis by studying the Emden–Fowler type equation of the fourth order

$$\frac{d^4y}{dx^4} + \frac{3k}{x}\frac{d^3y}{dx^3} + \frac{3k(k-1)}{x^2}\frac{d^2y}{dx^2} + \frac{k(k-1)(k-2)}{x^3}\frac{dy}{dx} + f(x)g(y) = 0,$$
$$y(0) = \alpha, y'(0) = y''(0) = y'''(0) = 0, \tag{16}$$

The proposed framework rests mainly on defining the differential operator L in terms of the first four derivatives in the form

$$L(y) = -f(x)g(y), \tag{17}$$

where the differential operator L contains the first four derivatives of (16) as

$$L = x^{-k}\frac{d^3}{dx^3}\left(x^k\frac{d}{dx}\right), \tag{18}$$

in order to overcome the singular behavior at $x = 0$. Based on (18), the optimal definition of L^{-1} is a fourfold definite integration operator defined as

$$L^{-1}(.) = \int_0^x x^{-k}\int_0^x\int_0^x\int_0^x x^k(.)\,dx\,dx\,dx\,dx. \tag{19}$$

Applying L^{-1} of (19) to the first four terms of (16) gives

$$\begin{aligned} &L^{-1}\left(y^{(iv)} + \frac{3k}{x}y''' + \frac{3k(k-1)}{x^2}y'' + \frac{k(k-1)(k-2)}{x^3}y'\right)\\ &= \int_0^x x^{-k}\int_0^x\int_0^x\int_0^x x^k\left(y^{(iv)} + \frac{3k}{x}y''' + \frac{3k(k-1)}{x^2}y'' + \frac{k(k-1)(k-2)}{x^3}y'\right)dxdxdxdx\\ &= y(x) - y(0), \end{aligned} \tag{20}$$

where integration by parts was used more than once in the evaluation of this fourfold integral. Combining (17) and (20) gives

$$y(x) = \alpha - L^{-1}\left(f(x)g(y)\right). \tag{21}$$

3.3 The Adomian decomposition method

In what follows, we give a brief presentation of the Adomian decomposition method (ADM). The details of this method are now well known and widely applied in the literature; see, for example, [Adomian (1983, 1986, 1994); Adomian and Rach (1983); Rach (2008); Wazwaz (2009, 2011a); Duan, Rach, Baleanu, and Wazwaz (2012); Rach (2012)].

The ADM admits the use of the infinite decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{22}$$

for the solution $y(x)$, and the infinite series of polynomials

$$g(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \cdots, y_n), \tag{23}$$

for the nonlinear term $g(y)$, where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and the A_n are the Adomian polynomials.

The definitional formula of the Adomian polynomials for the nonlinearity $F(y)$ is [Adomian and Rach (1983)]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots. \tag{24}$$

We list the formulas of the first several Adomian polynomials for the one-variable, simple analytic nonlinearity $Nu = F(y(x))$ from A_0 through A_5, inclusively, for convenient reference as

$$\begin{aligned}
A_0 &= F(y_0),\\
A_1 &= y_1F'(y_0),\\
A_2 &= y_2F'(y_0) + \frac{1}{2!}y_1^2F''(y_0),\\
A_3 &= y_3F'(y_0) + y_1y_2F''(y_0) + \frac{1}{3!}y_1^3F^{(3)}(y_0),\\
A_4 &= u_4F'(y_0) + (y_1y_3 + \frac{1}{2!}y_2^2)F''(y_0) + \frac{1}{2!}y_1^2y_2F^{(3)}(y_0) + \frac{1}{4!}y_1^4F^{(4)}(y_0).\\
A_5 &= y_5F'(y_0) + (y_2y_3 + y_1y_4)F''(y_0) + (\frac{1}{2!}y_1y_2^2 + \frac{1}{2!}y_1^2y_3)F^{(3)}(y_0)\\
&\quad + \frac{1}{3!}y_1^3y_2F^{(4)}(y_0) + \frac{1}{5!}y_1^5F^{(5)}(y_0).
\end{aligned}$$

The Adomian polynomials can be generated by using different algorithms such as in [Abbaoui, Cherruault, and Seng (1995); Abdelwahid (2003); Adomian and Rach (1983); Azreg-Aïnou (2009); Rach (1984, 2008); Wazwaz (2000, 2009); D-uan and Guo (2010); Duan (2010a,b, 2011)]. Duan (2010a,b, 2011) has recently developed several new, more efficient algorithms for fast generation of the one-variable and multivariable Adomian polynomials. For the case of the one-variable Adomian polynomials, we favor Duan's Corollary 3 algorithm [Duan (2011)], s-ince it does not involve the differentiation operator in the recurrence procedure but only requires the operations of addition and multiplication, which is eminently con-venient for computer algebra systems such as MATHEMATICA®, MAPLE® or MATLAB®,

$$\begin{aligned} &C_n^1 = y_n,\ n \geq 1, \\ &C_n^i = \frac{1}{n}\sum_{j=0}^{n-i}(j+1)y_{j+1}C_{n-1-j}^{i-1},\ 2 \leq i \leq n, \end{aligned} \tag{25}$$

from which we can quickly and easily calculate Adomian polynomials as

$$A_n = \sum_{i=1}^{n} C_n^i F^{(i)}(y_0),\ n \geq 1. \tag{26}$$

We list the corresponding MATHEMATICA® code AP[f, M] for Duan's Corollary 3 algorithm in the Appendix, which generates the first 30 Adomian polynomials within 0.90 seconds; the benchmark computer has an Intel Core$^{\text{TM}}$ i5-650 proces-sor with a clock frequency of 3.20 GHz and a RAM capacity of 4.00 GB using Microsoft's Windows® 7 OS. We remark that it has been timed in speed tests to be one of the fastest on record [Duan (2011)], including the earliest computer al-gorithm for fast generation of the Adomian polynomials as published by Adomian and Rach (1983).

Substituting (22) and (23) into (15) or (21) yields

$$\sum_{n=0}^{\infty} y_n(x) = \alpha - L^{-1}\left(f(x)\sum_{n=0}^{\infty} A_n(y_0, y_1, \cdots, y_n)\right), \tag{27}$$

where the inverse operator L^{-1} is defined as a threefold or fourfold integral operator as defined in (13) or (19), respectively.

The classic ADM admits the use of the recursive relation

$$\begin{aligned} &y_0(x) = \alpha, \\ &y_{j+1}(x) = -L^{-1}\left(f(x)A_j\right),\ j \geq 0, \end{aligned} \tag{28}$$

that will lead to the complete determination of the components $y_n(x)$, for $n \geq 0$, of $y(x)$. The series solution of $y(x)$ follows immediately and converges to the closed form solution if such a solution exists. In practice, we denote the n-term approximant of the series solution as $\phi_n(x;k) = \sum_{j=0}^{n-1} y_j(x)$.

Several investigators [Abbaoui and Cherruault (1994); Abdelrazec and Pelinovsky (2011); Cherruault and Adomian (1993)] have previously proved convergence of the Adomian decomposition series and the series of the Adomian polynomials. For example, Abdelrazec and Pelinovsky (2011) have published a rigorous proof of convergence for the ADM under the aegis of the Cauchy-Kovalevskaya theorem for initial value problems.

We remark that the domain of convergence for the decomposition solution may not always be sufficiently large for engineering purposes. But we can readily solve this problem by means of solution continuation or convergence acceleration techniques, such as analytic continuation [Adomian, Rach, and Meyers (1997)], the diagonal Padé approximants [Adomian (1994); Wazwaz (2009); Dehghan, Hamidi, and Shakourifar (2007); Dehghan, Shakourifar, and Hamidi (2009)], the iterated Shanks transform [Adomian (1994)], Adomian's asymptotic decomposition method [Adomian (1986, 1994); Rach and Duan (2011)], the parametrized recursion scheme [Duan (2010b); Duan and Rach (2011a); Duan, Rach, and Wang (2013); Lu and Duan (2014)], the higher-order discretized one-step subroutines based on the ADM and its modifications [Adomian, Rach, and Meyers (1997); Duan and Rach (2011b, 2012); Duan, Rach, and Wazwaz (2013)], and so on. For recent developments and a comprehensive bibliography of the ADM, see [Duan, Rach, Baleanu, and Wazwaz (2012); Rach (2012)].

In what follows, we examine several numerical examples that represent the Emden–Fowler type and Lane–Emden type equations of the third and fourth orders.

4 Numerical examples for the Emden–Fowler type equations of the third order

In this section, we study several numerical examples for the Emden–Fowler type and Lane–Emden type equations of the third order in the form

$$y''' + \frac{2k}{x}y'' + \frac{k(k-1)}{x^2}y' + f(x)g(y) = 0, y(0) = \alpha, y'(0) = y''(0) = 0. \tag{29}$$

We will study this equation for a variety of values for the shape factor k and for the given functions $f(x)g(y)$.

Example 1. We first consider the Emden–Fowler type equation

$$y''' + \frac{2}{x}y'' - \frac{9}{8}(x^6+8)y^{-5} = 0, y(0) = 1, y'(0) = y''(0) = 0, \tag{30}$$

obtained by substituting $k = 1$ in (29) and by setting $f(x)g(y) = -\frac{9}{8}(x^6+8)y^{-5}$. The Adomian polynomials for the nonlinear term y^{-5} are given by

$$\begin{aligned}
&A_0 = y_0^{-5},\\
&A_1 = -5y_0^{-6}y_1,\\
&A_2 = -5y_0^{-6}y_2 + \frac{30}{2!}y_0^{-7}y_1^2,\\
&A_3 = -5y_0^{-6}y_3 + 30y_0^{-7}y_1y_2 - \frac{210}{3!}y_0^{-8}y_1^3,\\
&\cdots.
\end{aligned}$$

Using (28), the recursive relation becomes

$$\begin{aligned}
&y_0(x) = 1,\\
&y_{j+1}(x) = -L^{-1}\left(f(x)A_j\right),\ j \geq 0,
\end{aligned} \tag{31}$$

where L^{-1} is defined in (13). We list the first several calculated solution components

$$\begin{aligned}
&y_0(x) = 1,\\
&y_1(x) = \frac{1}{2}x^3 + \frac{1}{576}x^9,\\
&y_2(x) = -\frac{1}{8}x^6 - \frac{185}{101376}x^{12} - \frac{5}{2820096}x^{18},\\
&y_3(x) = \frac{35}{576}x^9 + \frac{11801}{7096320}x^{15} + \cdots,\\
&\cdots.
\end{aligned}$$

This in turn gives the series solution

$$y(x) = 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + \frac{1}{16}x^9 - \frac{5}{128}x^{12} + \cdots, \tag{32}$$

that converges to the exact solution

$$y(x) = \sqrt{1+x^3}. \tag{33}$$

Example 2. We next consider the linear Emden–Fowler type equation

$$y''' + \frac{4}{x}y'' + \frac{2}{x^2}y' - 9(4+10x^3+3x^6)y = 0, y(0) = 1, y'(0) = y''(0) = 0, \quad (34)$$

obtained by substituting $k = 2$ in (29) and by setting $f(x)g(y) = -9(4+10x^3+3x^6)y$.

Using the recursive relation (28), we obtain the first several solution components

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= x^3 + \frac{5}{14}x^6 + \frac{1}{30}x^9,\\
y_2(x) &= \frac{1}{7}x^6 + \frac{8}{63}x^9 + \frac{44}{1365}x^{12} + \cdots,\\
y_3(x) &= \frac{2}{315}x^9 + \frac{61}{6552}x^{12} + \cdots,\\
&\cdots.
\end{aligned}$$

This in turn gives the series solution

$$y(x) = 1 + x^3 + \frac{1}{2!}x^6 + \frac{1}{3!}x^9 + \frac{1}{4!}x^{12} + \cdots, \quad (35)$$

that converges to the exact solution

$$y(x) = e^{x^3}. \quad (36)$$

Example 3. We now consider the Emden–Fowler type equation

$$y''' + \frac{6}{x}y'' + \frac{6}{x^2}y' - 6(10+2x^3+x^6)e^{-3y} = 0, y(0) = 0, y'(0) = y''(0) = 0, \quad (37)$$

obtained by substituting $k = 3$ in (29) and by setting $f(x)g(y) = -6(10+2x^3+x^6)e^{-3y}$.

The Adomian polynomials for the nonlinear term e^{-3y} are given by

$$\begin{aligned}
A_0 &= e^{-3y_0},\\
A_1 &= -3y_1e^{-3y_0},\\
A_2 &= (-3y_2 + \frac{9}{2}y_1^2)e^{-3y_0},\\
A_3 &= (-3y_3 + 9y_1y_2 - \frac{27}{3!}y_1^3)e^{-3y_0},\\
&\cdots.
\end{aligned}$$

Using the recursive relation (28), we calculate the first several solution components

$$
\begin{aligned}
y_0(x) &= 0,\\
y_1(x) &= x^3 + \frac{1}{28}x^6 + \frac{1}{165}x^9,\\
y_2(x) &= -\frac{15}{28}x^6 - \frac{3}{70}x^9 - \frac{523}{56056}x^{12} + \cdots,\\
y_3(x) &= \frac{57}{154}x^9 + \frac{9}{196}x^{12} + \cdots,\\
&\cdots.
\end{aligned}
$$

This in turn gives the series solution

$$y(x) = x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 - \frac{1}{4}x^{12} + \cdots, \tag{38}$$

that converges to the exact solution

$$y(x) = \ln(1 + x^3). \tag{39}$$

Example 4. We conclude this section by considering the Lane–Emden type equation

$$y''' + \frac{8}{x}y'' + \frac{12}{x^2}y' + y^m = 0,\ y(0) = 1, y'(0) = y''(0) = 0, \tag{40}$$

obtained by substituting $k = 4$ in (29) and by setting $g(y) = y^m, f(x) = 1$.
The Adomian polynomials for the nonlinear term y^m are given by

$$
\begin{aligned}
A_0 &= y_0^m,\\
A_1 &= my_0^{m-1}y_1,\\
A_2 &= my_0^{m-1}y_2 + \frac{1}{2!}m(m-1)y_0^{m-2}y_1^2,\\
A_3 &= my_0^{m-1}y_3 + m(m-1)y_0^{m-2}y_1y_2 + \frac{1}{3!}m(m-1)(m-2)y_0^{m-3}y_1^3,\\
&\cdots.
\end{aligned} \tag{41}
$$

Using the recursive relation (28), we calculate the first several solution components

$$
\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= -\frac{1}{90}x^3,\\
y_2(x) &= \frac{m}{38880}x^6,\\
y_3(x) &= -\frac{m(17m-12)}{230947200}x^9,\\
y_4(x) &= \frac{m(679m^2-1182m+528)}{2909934720000}x^{12},\\
&\cdots.
\end{aligned}
$$

This in turn gives the series solution

$$
y(x) = 1 - \frac{1}{90}x^3 + \frac{m}{38880}x^6 - \frac{m(17m-12)}{230947200}x^9 + \frac{m(679m^2-1182m+528)}{2909934720000}x^{12} + \cdots, \tag{42}
$$

where the n-term approximant is $\phi_n(x) = \sum_{j=0}^{n-1} y_j(x)$.

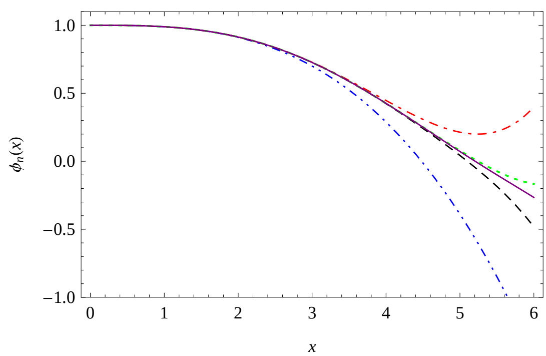

Figure 1: The curves of the approximate solutions $\phi_n(x)$ versus x for $m = 1.5$ and for $n = 2$ (dot-dot-dash line), $n = 3$ (dot-dash line), $n = 4$ (dash line), $n = 5$ (dot line) and $n = 6$ (solid line).

We note that when $m = 0$ Eq. (40) has the exact solution

$$
y^*(x) = 1 - \frac{1}{90}x^3. \tag{43}
$$

In Fig. 1, we plot the curves of the n-term approximate solutions $\phi_n(x)$ versus x for $m = 1.5$ and for $n = 2$ through 6, where a fast rate of convergence is displayed. In

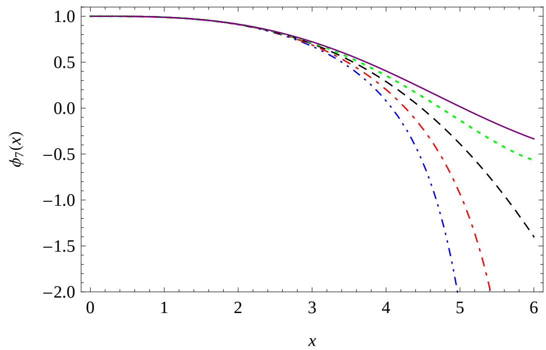

Figure 2: The curves of the approximate solutions $\phi_7(x)$ versus x for $m=-1.2$ (dot-dot-dash line), $m=-0.6$ (dot-dash line), $m=0$ (dash line), $m=0.6$ (dot line) and $m=1.2$ (solid).

Fig. 2, we plot the curves of the 7-term approximate solutions $\phi_7(x)$ versus x for $m=-1.2,-0.6,0,0.6,1.2$, where in the case of $m=0$, $\phi_7(x)=y^*(x)$.

5 Numerical examples for the Emden–Fowler type equations of the fourth order

We follow our previous analysis to study several numerical examples for the Emden–Fowler type and Lane–Emden type equations of the fourth order in the form

$$\begin{aligned} &y^{(iv)}+\frac{3k}{x}y'''+\frac{3k(k-1)}{x^2}y''+\frac{k(k-1)(k-2)}{x^3}y'+f(x)g(y)=0,\\ &y(0)=y_0, y'(0)=y''(0)=y'''(0)=0. \end{aligned} \tag{44}$$

We will study this equation for a variety of values for the shape factor k and for the given functions $f(x)g(y)$.

Example 5. We first consider the Emden–Fowler type equation

$$y^{(iv)}+\frac{3}{x}y'''-96(1-10x^4+5x^8)e^{-4y}=0, y(0)=y'(0)=y''(0)=y'''(0)=0, \tag{45}$$

obtained by substituting $k=1$ in (44) and by setting $f(x)g(y)=-96(1-10x^4+5x^8)e^{-4y}$.

The Adomian polynomials for the nonlinear term e^{-4y} are given by

$$
\begin{aligned}
&A_0 = e^{-4y_0},\\
&A_1 = -4y_1 e^{-4y_0},\\
&A_2 = (-4y_2 + 8y_1^2)e^{-4y_0},\\
&A_3 = (-4y_3 + 16y_1y_2 - \frac{64}{3!}y_1^3)e^{-4y_0},\\
&\cdots.
\end{aligned}
$$

Using (28), the recursive relation becomes

$$
\begin{aligned}
&y_0(x) = 0,\\
&y_{j+1}(x) = -L^{-1}\left(f(x)A_j\right),\ j \geq 0,
\end{aligned}
\tag{46}
$$

where L^{-1} is a fourfold integral operator defined in (19). We list the first several calculated solution components

$$
\begin{aligned}
&y_0(x) = 0,\\
&y_1(x) = x^4 - \frac{5}{14}x^8 + \frac{1}{33}x^{12},\\
&y_2(x) = -\frac{1}{7}x^8 + \frac{58}{231}x^{12} + \cdots,\\
&y_3(x) = \frac{4}{77}x^{12} + \cdots,\\
&\cdots.
\end{aligned}
$$

This in turn gives the series solution

$$
y(x) = x^4 - \frac{1}{2}x^8 + \frac{1}{3}x^{12} - \frac{1}{4}x^{16} + \cdots, \tag{47}
$$

that converges to the exact solution

$$
y(x) = \ln(1 + x^4). \tag{48}
$$

Example 6. We next consider a linear version of the Emden–Fowler type equation

$$
\begin{aligned}
&y^{(iv)} + \frac{6}{x}y''' + \frac{6}{x^2}y'' - 16(-15 + 111x^4 - 96x^8 + 16x^{12})y = 0,\\
&y(0) = 1, y'(0) = y''(0) = y'''(0) = 0,
\end{aligned}
\tag{49}
$$

obtained by substituting $k=2$ in (44) and by setting $f(x)g(y)=-16(-15+111x^4-96x^8+16x^{12})y$.
Using the recursive relation (28), we calculate the first several solution components

$$
\begin{aligned}
&y_0(x)=1,\\
&y_1(x)=-x^4+\frac{37}{84}x^8-\frac{32}{429}x^{12}+\cdots,\\
&y_2(x)=\frac{5}{84}x^8-\frac{3293}{36036}x^{12}+\cdots,\\
&y_3(x)=-\frac{25}{36036}x^{12}+\cdots,\\
&\cdots.
\end{aligned}
$$

This in turn gives the series solution

$$y(x)=1-x^4+\frac{1}{2!}x^8-\frac{1}{3!}x^{12}+\cdots, \tag{50}$$

that converges to the exact solution

$$y(x)=e^{-x^4}. \tag{51}$$

Example 7. We consider the Emden–Fowler type equation

$$
\begin{aligned}
&y^{(iv)}+\frac{9}{x}y'''+\frac{18}{x^2}y''+\frac{6}{x^3}y'-96(-5+35x^4-23x^8+x^{12})y^5=0,\\
&y(0)=1, y'(0)=y''(0)=y'''(0)=0,
\end{aligned}
\tag{52}
$$

obtained by substituting $k=3$ in (44) and by setting $f(x)g(y)=-96(-5+35x^4-23x^8+x^{12})y^5$.
The Adomian polynomials for the nonlinear term y^5 are given by

$$
\begin{aligned}
&A_0=y_0^5,\\
&A_1=5y_1y_0^4,\\
&A_2=5y_2y_0^4+10y_1^2y_0^3,\\
&A_3=5y_3y_0^4+20y_1y_2y_0^3+10y_1^3y_0^2,\\
&\cdots.
\end{aligned}
$$

Using the recursive relation (28), we calculate the first several solution components

$$
\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= -x^4 + \frac{7}{12}x^8 - \frac{23}{273}x^{12} + \cdots,\\
y_2(x) &= \frac{5}{12}x^8 - \frac{25}{36}x^{12} + \cdots,\\
y_3(x) &= -\frac{725}{3276}x^{12} + \cdots,\\
&\cdots.
\end{aligned}
$$

This gives the series solution

$$y(x) = 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots, \tag{53}$$

that converges to the exact solution

$$y(x) = \frac{1}{1+x^4}. \tag{54}$$

Example 8. We conclude this section by considering the Emden–Fowler type equation

$$
\begin{aligned}
&y^{(iv)} + \frac{12}{x}y''' + \frac{36}{x^2}y'' + \frac{24}{x^3}y' + 48(35 - 25x^4 + 5x^8 + x^{12})e^{2y} = 0,\\
&y(0) = 0, y'(0) = y''(0) = y'''(0) = 0,
\end{aligned} \tag{55}
$$

obtained by substituting $k = 4$ in (44) and by setting $f(x)g(y) = 48(35 - 25x^4 + 5x^8 + x^{12})e^{2y}$.
The Adomian polynomials for the nonlinear term e^{2y} are given by

$$
\begin{aligned}
A_0 &= e^{2y_0},\\
A_1 &= 2y_1 e^{2y_0},\\
A_2 &= (2y_2 + 2y_1^2)e^{2y_0},\\
A_3 &= (2y_3 + 4y_1y_2 + \frac{8}{3!}y_1^3)e^{2y_0},\\
&\cdots.
\end{aligned}
$$

Using the recursive relation (28), we calculate the first several solution components

$$
\begin{aligned}
y_0(x) &= 0,\\
y_1(x) &= -2x^4 + \frac{5}{33}x^8 - \frac{2}{273}x^{12} - \frac{1}{1938}x^{16},\\
y_2(x) &= \frac{28}{33}x^8 - \frac{1460}{9009}x^{12} + \frac{6025}{415701}x^{16} + \cdots,\\
y_3(x) &= -\frac{640}{1287}x^{12} + \frac{190600}{1247103}x^{16} + \cdots,\\
y_4(x) &= \frac{415520}{1247103}x^{16} + \cdots,\\
&\cdots.
\end{aligned}
$$

This in turn gives the series solution

$$y(x) = -2(x^4 - \frac{1}{2}x^8 + \frac{1}{3}x^{12} - \frac{1}{4}x^{16} + \cdots), \tag{56}$$

that converges to the exact solution

$$y(x) = -2\ln(1+x^4). \tag{57}$$

6 Conclusion

In this work, we have presented a new framework to establish a canonical form of the Lane–Emden–Fowler type equations of order greater than or equal to two. We introduced a generalized formula for the solution scheme. Unlike the standard Lane–Emden–Fowler equations where the shape factor is unique, we have demonstrated that more than one shape factor exists for equations of order greater than or equal to 3. Similarly, the singular point appears once in the standard form, whereas in higher-order cases it appears more than once. We used the ADM with the Adomian polynomials to easily and rapidly solve these nonlinear problems to illustrate our analysis. The calculated results from the recursion scheme are effective for all values of the shape factor $k > 0$. The obtained results validate the reliability and rapid convergence of the ADM for solving strongly nonlinear differential equations including higher-order Emden-Fowler type and Lane-Emden type equations.

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Appendix: MATHEMATICA code for the one-variable Adomian polynomials based on Duan's Corollary 3 algorithm [Duan (2011)]

```
AP[f_,M_]:=Module[{c,n,k,j,der},
Table[c[n,k],{n,1,M},{k,1,n}];
der=Table[D[f[Subscript[u,0]],{Subscript[u,0],k}],{k,1,M}];
A[0]=f[Subscript[u,0]];
For[n=1,n<=M,n++,c[n,1]=Subscript[u,n];
For[k=2,k<=n,k++,
c[n,k]=Expand[1/n*Sum[(j+1)*Subscript[u,j+1]*c[n-1-j,k-1],
{j,0,n-k}]]];
A[n]=Take[der,n].Table[c[n,k],{k,1,n}]];
Table[A[n],{n,0,M}]]
```