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A Fully Discrete SCNFVE Formulation for the Non-stationary Navier-Stokes Equations

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Abstract: A semi-discrete Crank-Nicolson (CN) formulation about time and a fully discrete stabilized CN finite volume element (SCNFVE) formulation based on two local Gauss integrals and parameter-free with the second-order time accuracy are established for the non-stationary Navier-Stokes equations. The error estimates of the semi-discrete and fully discrete SCNFVE solutions are derived. Some numerical experiments are presented to illustrate that the fully discrete SCNFVE formulation possesses more advantages than its stabilized finite volume element formulation with the first-order time accuracy, thus validating that the fully discrete SCNFVE formulation is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations.

Keywords: Non-stationary Navier-Stokes equations, stabilized Crank-Nicolson finite volume element formulation, local Gauss integral and parameter-free, error estimate.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected polygonal domain. We consider the following system of incompressible non-stationary Navier-Stokes equations.

Problem I. Find $\boldsymbol{u} = (u_1, u_2)$ and p such that, for $T > 0$,

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, & (x,y,t)\in\Omega\times(0,T), \\ \mathrm{div}\boldsymbol{u} = 0, & (x,y,t)\in\Omega\times(0,T), \\ \boldsymbol{u}(x,y,t) = \boldsymbol{\varphi}(x,y,t), & (x,y,t)\in\partial\Omega\times(0,T], \\ \boldsymbol{u}(x,y,0) = \boldsymbol{u}^0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2)$ represents the fluid velocity vector, p the pressure, T the total time, $\nu = 1/Re$, Re the Reynolds number, $\boldsymbol{f}(x,y,t)$ the given body force vector,

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$\boldsymbol{\varphi}(x,y,t)$ the boundary value function, and $\boldsymbol{u}^0(x,y)$ the initial value function. For the sake of convenience and without loss of generality, we may as well suppose that $\boldsymbol{\varphi}(x,y,t) = \mathbf{0}$.

Problem I constitutes a system of nonlinear equations in fluid dynamics. It has been successfully and extensively applied in many fields of practical engineering [Temam (1984); Girault and Raviart (1986); Heywood and Rannacher (1982)]. It is a difficult task to find the analytical solutions for the non-stationary Navier-Stokes equations due to its nonlinearity. Especially, it is a more difficult task when its computational field is an irregular geometrical shape in actual applications. One has to rely on numerical solutions.

The finite volume element (FVE) method [Cannon and Lin (1990); Cai and McCormick (1990); Süli (1991); Quarteroni and Ruiz-Baier (2011)] is regarded as one of the most valid numerical methods thanks to its following virtues. First, it retains the integral conservation of gross energy and that of mass. Second, it has higher accuracy and is more suitable for cases involving complicated computational field than the finite deference (FD) scheme. Third, it has the same accuracy as the finite element (FE) method but is simpler and more convenient to apply than the FE method (In fact, a FVE formulation can be changed into a FD scheme in numerical computation by means of FVE so that its algorithm implementation is very simple and convenient, however its numerical analysis is provided by FE technique). It is also known as a box method [Jones and Menziest (2000)] or a generalized difference method [Li, Chen, and Wu (2002); Li, Luo, and Li (2007)]. It has been widely used to solve various types of partial differential equations, but it focused on stationary partial differential equations and linear equations, for example, elliptic problems, viscoelastic problems, parabolic equations, Stokes equations, and the stationary Navier-Stokes equations [Cai and McCormick (1990); Süli (1991); Quarteroni and Ruiz-Baier (2011); Jones and Menziest (2000); Li, Chen, and Wu (2002); Li, Luo, and Li (2007); Blanc, Eymerd, and Herbin (2004); Li and Chen (2009); Ye (2001); Chou and Kwak (1998); Shen, Li, and Chen (2009); Yang and Song (2009); Ammara and Masson (2004); Li, Shen, and Chen (2010); He and He (2007)].

Although a semi-discrete stabilized FVE (SFVE) formulation with respect to spatial variables based on the jump operator in pressure for the non-stationary Navier-Stokes equations is proposed [He, He, and Feng (2007), without mentioning fully discrete SFVE method], it is well known that it is yet unable to carry out directly actual numerical computations. Though the SFVE method based on two local Gauss integrals and parameter-free is used to treat the stationary Navier-Stokes equations [Li, Shen, and Chen (2010)] and the semi-discrete FVE formulation with respect to space variables for the transient Stokes equations [Shen, Li, and Chen (2009)], to the best of our knowledge, there are no published results where the fully discrete stabilized Crank-Nicolson (CN) FVE (SCNFVE) formulation based on two local Gauss integrals and parameter-free with second-order time accuracy for the non-stationary Navier-Stokes equations is established or the error estimates of the fully discrete SCNFVE solutions for the non-stationary Navier-Stokes equations are provided. Therefore, in this paper, a fully discrete SCNFVE formulation based on two local Gauss integrals and parameter-free with the second-order time accuracy is directly established from the time semi-discrete CN formulation for the non-stationary Navier-Stokes equations so that it could avoid the semi-discrete SCNFVE formulation with respect to spatial variables, namely it is unnecessary to discuss the semi-discrete SCNFVE formulation with respect to spatial variables such that theoretical analysis here becomes simpler than the existing other methods [see, e.g., Shen, Li, and Chen (2009); He, He, and Feng (2007)]. Consequently, it is the improvement and innovation for the existing other methods and a new type of study attempt for the non-stationary Navier-Stokes equations.

The plan of this paper is organized as follows. In Section 2, the time semi-discrete CN formulation with the second-order time accuracy for the non-stationary Navier-Stokes equations and the error estimates are derived. In Section 3, the fully discrete SCNFVE formulation based on two local Gauss integrals and parameter-free with the second-order time accuracy is directly established from the time semi-discrete CN formulation. In Section 4, the existence and uniqueness and the error estimates of the fully discrete SCNFVE solutions are proved by means of the stabilized CN FE (SCNFE) method. In Section 5, some numerical experiments are used to validate that the fully discrete SCNFVE formulation with the second-order time accuracy is more effective than the SFVE formulation with the first-order time accuracy, thus meaning that the fully discrete SCNFVE formulation is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations. Section 6 provides main conclusions and some discussions.

2 Time semi-discrete CN formulation with the second-order time accuracy and error estimates

The Sobolev spaces and norms used in this paper are standard [Adams (1975)]. Let $U = H_0^1(\Omega)^2$, $M = L_0^2(\Omega) = \left\{q \in L^2(\Omega):\ \int_\Omega q \mathrm{d}x\mathrm{d}y = 0\right\}$. Then the variational formulation for Problem I is read as follows.

Problem II. Find $(\boldsymbol{u}(t), p(t)) : [0,T] \to U \times M$ such that

$$\begin{cases} (\boldsymbol{u}_t, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in U, \\ b(\boldsymbol{u}, q) = 0, \quad \forall q \in M, \\ \boldsymbol{u}(x,y,0) = \boldsymbol{u}^0(x,y), \quad (x,y) \in \Omega, \end{cases} \tag{1}$$

where $a(\boldsymbol{u},\boldsymbol{v}) = \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})$, $a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = [(\boldsymbol{u}\nabla\boldsymbol{v},\boldsymbol{w}) - (\boldsymbol{u}\nabla\boldsymbol{w},\boldsymbol{v})]/2$, $b(\boldsymbol{v},q) = (\mathrm{div}\boldsymbol{v}, q)$, and $(\cdot,\cdot)$ denotes inner product in $L^2(\Omega)$ or $L^2(\Omega)^2$ or $L^2(\Omega)^{2\times 2}$.

The following properties of trilinear form $a_1(\cdot,\cdot,\cdot)$ are known [Heywood and Rannacher (1990) or Luo (2006)]:

$$a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -a_1(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}),\ \ a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0,\quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in U. \tag{2}$$

The following properties of bilinear form $a(\cdot,\cdot)$ are also known [also see Heywood and Rannacher (1990) or Luo (2006)]:

$$a(\boldsymbol{v},\boldsymbol{v}) = \nu\|\nabla\boldsymbol{v}\|_0^2,\ \ \forall \boldsymbol{v}\in U;\ |a(\boldsymbol{u},\boldsymbol{v})| \leqslant \nu|\boldsymbol{u}|_1|\boldsymbol{v}|_1,\ \ \forall \boldsymbol{u},\ \boldsymbol{v}\in U. \tag{3}$$

The bilinear form $b(\cdot,\cdot)$ satisfies the following B-B (Brezzi-Babuška) condition [Brezzi and Fortin (1991) or Luo (2006)]:

$$\sup_{\boldsymbol{v}\in U}\frac{b(q,\boldsymbol{v})}{|\boldsymbol{v}|_1} \geqslant \beta\|q\|_0,\quad \forall q\in M, \tag{4}$$

where β is a constant independent of $\boldsymbol{v}$ and q. Put

$$N_0 = \sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in U}\frac{a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})}{|\boldsymbol{u}|_1\cdot|\boldsymbol{v}|_1\cdot|\boldsymbol{w}|_1}. \tag{5}$$

Thanks to (2)–(5), if $N_0\nu^{-1}\|\boldsymbol{f}\|_{L^2(H^{-1})} \leqslant 1$, Problem II has a unique solution such that [Heywood and Rannacher (1990) or Luo (2006)]:

$$\begin{aligned}&\|\boldsymbol{u}\|_0 + \|\boldsymbol{u}_t\|_{L^2(L^2)} + \|\nabla\boldsymbol{u}\|_{L^2(L^2)} + \|p\|_{L^2(L^2)}\\ &\leqslant C(\|\boldsymbol{u}^0\|_1 + \|\boldsymbol{f}\|_{L^2(L^2)} + N_0\|\boldsymbol{f}\|^2_{L^2(L^2)} + \|\boldsymbol{f}\|_{L^2(H^{-1})}),\end{aligned} \tag{6}$$

where $\|\cdot\|_{H^m(H^l)}$ are the norm of $H^m(0,T;H^l(\Omega))$ or $H^m(0,T;H^l(\Omega)^2)^2 (m\geqslant 0$ and $l\geqslant -1)$ and C is a constant.

For given positive integer N, let $k = T/N$ denote time step, $t_n = nk$, $\boldsymbol{u}^n$ be the time semi-discrete CN approximation of $\boldsymbol{u}$ at $t_n \equiv nk$ $(n = 0,1,\cdots,N)$. Let $\bar{\partial}_t\boldsymbol{u}^n = (\boldsymbol{u}^n - \boldsymbol{u}^{n-1})/k$ denote the approximation of $\boldsymbol{u}_t$, $\bar{\boldsymbol{u}}^n = (\boldsymbol{u}^n + \boldsymbol{u}^{n-1})/2$, then the time semi-discrete CN scheme with the second-order time accuracy for Problem II is read as follows.

Problem III. Find $(\boldsymbol{u}^n, p^n)\in U\times M$ $(1\leqslant n\leqslant N)$ such that

$$\begin{cases}(\bar{\partial}_t\boldsymbol{u}^n,\boldsymbol{v}) + a(\bar{\boldsymbol{u}}^n,\boldsymbol{v}) + a_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\boldsymbol{v}) - b(\boldsymbol{v},p^n) = (\boldsymbol{f}^{n-\frac{1}{2}},\boldsymbol{v}),\ \forall\boldsymbol{v}\in U,\\ b(\boldsymbol{u}^n,q) = 0,\quad \forall q\in M,\\ \boldsymbol{u}^0 = \boldsymbol{u}^0(x,y),\quad (x,y)\in\Omega,\end{cases} \tag{7}$$

where $\boldsymbol{f}^{n-\frac{1}{2}} = \boldsymbol{f}(t_{n-\frac{1}{2}})$.

There is the following theorem for the time semi-discrete CN formulation, i.e., Problem III.

Theorem 1 *If $\boldsymbol{u}^0 \in H^1(\Omega)^2$ and $\boldsymbol{f} \in L^2(0,T;L^2(\Omega)^2)^2$, then Problem III has a unique series of solutions $(\boldsymbol{u}^n, p^n) \in U \times M$ ($n = 1,2,...,N$) satisfying the following stability*

$$\|\boldsymbol{u}^n\|_0^2 + \nu k \sum_{i=1}^{n} \|\nabla \bar{\boldsymbol{u}}^i\|_0^2 \leqslant \|\boldsymbol{u}^0\|_0^2 + \nu^{-1} k \sum_{i=0}^{n} \|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2, \tag{8}$$

$$\|\nabla \boldsymbol{u}^n\|_0^2 \leqslant \|\nabla \boldsymbol{u}^0\|_0^2 + k\nu^{-1} \sum_{i=1}^{n} \|\boldsymbol{f}^{i-\frac{1}{2}}\|_0^2, \tag{9}$$

$$\begin{aligned} k\sum_{i=1}^{n} \|p^i\|_0 &\leqslant \beta^{-1}(2+\sqrt{kn\nu})\|\boldsymbol{u}^0\|_0 + \beta^{-1} N_0 \nu^{-1} \|\boldsymbol{u}^0\|_0^2 \\ &+\beta^{-1}(1+2\sqrt{kn\nu}) \left(\nu^{-1} k \sum_{i=0}^{n} \|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right)^{1/2} + kN_0\nu^{-2}\beta^{-1} \sum_{i=1}^{n} \|\boldsymbol{f}^{i-\frac{1}{2}}\|_0^2. \end{aligned} \tag{10}$$

And if $\boldsymbol{u}^0 \in H^2(\Omega)^2$, $\boldsymbol{f} \in W^{2,\infty}(0,T;L^2(\Omega)^2)^2$, and $N_0\nu^{-1}\|\nabla \boldsymbol{u}(t)\|_0 \leqslant 1/4$, there hold the following error estimates:

$$\begin{aligned} &\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|_0 + (k\nu)^{1/2} \|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}^n)\|_0 \leqslant C(\boldsymbol{f})k^2, \\ &\|p(t_n) - p^n\|_0 \leqslant \tilde{C}(\boldsymbol{f})k, \quad n = 1,2,...,N, \end{aligned} \tag{11}$$

where $\tilde{C}(\boldsymbol{f}) = \beta^{-1}\Big[(\nu k)^{1/2} C(\boldsymbol{f}) + \frac{(\nu k)^{1/2}}{4} C(\boldsymbol{f}) + \frac{k}{24}\|\boldsymbol{u}\|_{W^{3,\infty}(0,T;L^2)} + \frac{1}{2}\|p\|_{W^{1,\infty}(L^2)} + 2C(\boldsymbol{f}) + \frac{9\nu k}{64}\|\boldsymbol{u}\|_{W^{2,\infty}(H^1)} + \Big(C(\boldsymbol{f}) + \frac{k^{1/2}}{16\nu^{1/2}}\|\boldsymbol{u}\|_{W^{2,\infty}(H^1)}\Big)(\|\boldsymbol{u}^0\|_0 + \nu^{-1/2}\|\boldsymbol{f}\|_{L^\infty(H^{-1})})\Big]$ *and* $C^2(\boldsymbol{f}) = TN_0^2\|\boldsymbol{u}\|_{W^{2,\infty}(H^1)}^4 + N_0^2\nu^{-1}\|\boldsymbol{u}\|_{W^{1,\infty}(H^1)}^2(\|\boldsymbol{u}^0\|_2 + \nu^{-1}\|\boldsymbol{f}\|_{L^\infty(H^{-1})}^2) + T\nu^{-1}\|\boldsymbol{u}\|_{W^{3,\infty}(H^{-1})}^2 + 16\nu T\|\boldsymbol{u}\|_{W^{2,\infty}(H^1)}^2$.

Proof Let $A(\boldsymbol{u}^n, \boldsymbol{v}) = 4(\boldsymbol{u}^n, \boldsymbol{v}) + 2ka(\boldsymbol{u}^n, \boldsymbol{v}) + ka_1(\boldsymbol{u}^n, \boldsymbol{u}^{n-1}, \boldsymbol{v}) + ka_1(\boldsymbol{u}^{n-1}, \boldsymbol{u}^n, \boldsymbol{v})$, $F(\boldsymbol{v}) = 4k(\boldsymbol{f}^{n-\frac{1}{2}}, \boldsymbol{v}) + 4(\boldsymbol{u}^{n-1}, \boldsymbol{v}) - 2ka(\boldsymbol{u}^{n-1}, \boldsymbol{v}) - ka_1(\boldsymbol{u}^{n-1}, \boldsymbol{u}^{n-1}, \boldsymbol{v})$. Then, for given $\boldsymbol{u}^{n-1}$ satisfying $N_0\nu^{-1}\|\nabla \boldsymbol{u}^{n-1}\|_0 \leqslant 1$ and fixed n and k, $A(\cdot,\cdot)$ is bounded bilinear form and satisfies the following coerciveness

$$A(\boldsymbol{u}^n, \boldsymbol{u}^n) = 4\|\boldsymbol{u}^n\|_0^2 + 2k\nu\|\nabla \boldsymbol{u}^n\|_0^2 - kN_0\|\nabla \boldsymbol{u}^n\|_0^2\|\nabla \boldsymbol{u}^{n-1}\|_0 \geqslant \alpha_0\|\boldsymbol{u}^n\|_1^2, \tag{12}$$

where $\alpha_0 = \min\{k\nu, 4\}$ is a constant independent of $\boldsymbol{u}^n$. For given $\boldsymbol{u}^{n-1}$ and $\boldsymbol{f}$, $F(\cdot)$ is a continuous linear function on U, we know from (4) that $kb(\boldsymbol{v}, q)$ also satisfies B-B condition and $ka_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v})$ is a continuous trilinear form too. Then Problem III, i.e., $A(\boldsymbol{u}^n, \boldsymbol{v}) + ka_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}) - kb(\boldsymbol{v}, p^n) + kb(\boldsymbol{u}^n, q) = F(\boldsymbol{v})$ $(\forall(\boldsymbol{v}, q) \in U \times M)$

has a unique series of solutions $(\boldsymbol{u}^n, p^n) \in U \times M$ $(n = 1, 2, ..., N)$ from the existence and uniqueness of solution for the stationary Navier-Stokes equations [Luo (2006) or Brezzi and Fortin (1991)].

Let $\boldsymbol{v} = \boldsymbol{u}^n + \boldsymbol{u}^{n-1}$ in Problem III. With (2), Hölder inequality, and Cauchy inequality, we obtain that

$$\begin{aligned}&2(\|\boldsymbol{u}^n\|_0^2 - \|\boldsymbol{u}^{n-1}\|_0^2) + k\nu\|\nabla(\boldsymbol{u}^n + \boldsymbol{u}^{n-1})\|_0^2 = 2k(\boldsymbol{f}^{n-\frac{1}{2}}, \boldsymbol{u}^n + \boldsymbol{u}^{n-1})\\ &\leqslant 2k\|\boldsymbol{f}^{n-\frac{1}{2}}\|_{-1}\|\nabla(\boldsymbol{u}^n + \boldsymbol{u}^{n-1})\|_0\\ &\leqslant 2k\nu^{-1}\|\boldsymbol{f}^{n-\frac{1}{2}}\|_{-1}^2 + \frac{k\nu}{2}\|\nabla(\boldsymbol{u}^n + \boldsymbol{u}^{n-1})\|_0^2.\end{aligned} \tag{13}$$

It follows from (13) that

$$4(\|\boldsymbol{u}^n\|_0^2 - \|\boldsymbol{u}^{n-1}\|_0^2) + k\nu\|\nabla(\boldsymbol{u}^n + \boldsymbol{u}^{n-1})\|_0^2 \leqslant 4k\nu^{-1}\|\boldsymbol{f}^{n-\frac{1}{2}}\|_{-1}^2. \tag{14}$$

Summing (14) from 1 to n yields (8).

Let $\boldsymbol{v} = \boldsymbol{u}^n - \boldsymbol{u}^{n-1}$ in Problem III. With (2), Hölder inequality, and Cauchy inequality, we get that

$$\begin{aligned}&2\|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|_0^2 + k\nu[\|\nabla\boldsymbol{u}^n\|_0^2 - \|\nabla\boldsymbol{u}^{n-1}\|_0^2] = 2k(\boldsymbol{f}^{n-\frac{1}{2}}, \boldsymbol{u}^n - \boldsymbol{u}^{n-1})\\ &\leqslant 2k\|\boldsymbol{f}^{n-\frac{1}{2}}\|_0\|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|_0 \leqslant k^2\|\boldsymbol{f}^{n-\frac{1}{2}}\|_0 + \|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|_0^2.\end{aligned} \tag{15}$$

It is obtained from (15) that

$$\|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|_0^2 + k\nu[\|\nabla\boldsymbol{u}^n\|_0^2 - \|\nabla\boldsymbol{u}^{n-1}\|_0^2] \leqslant k^2\|\boldsymbol{f}^{n-\frac{1}{2}}\|_0^2. \tag{16}$$

It is gotten by summing (16) from 1 to n that

$$(k\nu)^{-1}\sum_{i=1}^{n}\|\boldsymbol{u}^i - \boldsymbol{u}^{i-1}\|_0^2 + \|\nabla\boldsymbol{u}^n\|_0^2 \leqslant \|\nabla\boldsymbol{u}_0\|_0^2 + k\nu^{-1}\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_0^2, \tag{17}$$

which yields (9). With (4) and Problem III, we have

$$\begin{aligned}&k\beta\|p^n\|_0 \leqslant \sup_{\boldsymbol{v}\in U}\frac{kb(\boldsymbol{v}, p^n)}{\|\nabla v\|_0}\\ &= \sup_{\boldsymbol{v}\in U}\frac{(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \boldsymbol{v}) + ka(\bar{\boldsymbol{u}}^n, \boldsymbol{v}) + ka_1(\bar{\boldsymbol{u}}^n, \bar{\boldsymbol{u}}^n, \boldsymbol{v}) - k(\boldsymbol{f}^{n-\frac{1}{2}}, \boldsymbol{v})}{\|\nabla v\|_0}.\end{aligned} \tag{18}$$

By summing (18) from 1 to n and using Hölder inequality, (9), and (17), we get that

$$
\begin{aligned}
& k\beta\sum_{i=1}^{n}\|p^i\|_0 \\
& \leqslant \|\boldsymbol{u}^n\|_{-1}+\|\boldsymbol{u}^0\|_{-1}+k\nu\sum_{i=1}^{n}\|\nabla\bar{\boldsymbol{u}}^i\|_0+kN_0\sum_{i=1}^{n}\|\nabla\bar{\boldsymbol{u}}^i\|_0^2+k\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1} \\
& \leqslant \|\boldsymbol{u}^n\|_0+\|\boldsymbol{u}^0\|_0+\sqrt{kn\nu}\left(k\nu\sum_{i=1}^{n}\|\nabla\bar{\boldsymbol{u}}^i\|_0^2\right)^{1/2} \\
& +kN_0\sum_{i=1}^{n}\|\nabla\bar{\boldsymbol{u}}^i\|_0^2+k\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1} \\
& \leqslant \left(\|\boldsymbol{u}^0\|_0^2+\nu^{-1}k\sum_{i=0}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right)^{1/2} \\
& +\|\boldsymbol{u}^0\|_0+\sqrt{kn\nu}\left(k\nu^{-1}\sum_{i=0}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right)^{1/2} \\
& +N_0\nu^{-1}\left(\|\boldsymbol{u}^0\|_0^2+\nu^{-1}k\sum_{i=0}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right) \\
& +\sqrt{kn\nu}\left(\|\boldsymbol{u}^0\|_0^2+\nu^{-1}k\sum_{i=0}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right)^{1/2} \\
& \leqslant (2+\sqrt{kn\nu})\|\boldsymbol{u}^0\|_0+N_0\nu^{-1}\|\boldsymbol{u}^0\|_0^2 \\
& +(1+2\sqrt{kn\nu})\left(\nu^{-1}k\sum_{i=0}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right)^{1/2}+kN_0\nu^{-2}\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_0^2,
\end{aligned}
\tag{19}
$$

which yields (10).

Put $\boldsymbol{e}^n=\boldsymbol{u}(t_n)-\boldsymbol{u}^n$ and $\eta^n=p(t_n)-p^n$. Subtracting Problem III from Problem II taking $t=t_{n-\frac{1}{2}}$, $\boldsymbol{v}=\boldsymbol{e}^n+\boldsymbol{e}^{n-1}$, and $q=\eta^n$, using Taylor's formula, we obtain that

$$
\begin{aligned}
& \|\boldsymbol{e}^n\|_0^2-\|\boldsymbol{e}^{n-1}\|_0^2+\frac{k\nu}{2}\|\nabla(\boldsymbol{e}^n+\boldsymbol{e}^{n-1})\|_0^2 \\
& =\frac{k^3}{24}(\boldsymbol{u}_{ttt}(\xi_{1n}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1})+\frac{k^3\nu}{4}(\nabla\boldsymbol{u}_{tt}(\xi_{2n}),\nabla(\boldsymbol{e}^n+\boldsymbol{e}^{n-1}))+\Phi,
\end{aligned}
\tag{20}
$$

where $\Phi=ka_1(\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1})-ka_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\boldsymbol{e}^n+\boldsymbol{e}^{n-1})$ $(t_{n-1}\leqslant\xi_{1n},\ \xi_{2n}\leqslant t_n)$. If $N_0\nu^{-1}\|\nabla\boldsymbol{u}(t)\|_0\leqslant 1/4$, by using Taylor's formula, Hölder inequality, and Cauchy inequality, there are $\xi_{in} \in [t_{n-1}, t_n]$ $(i = 3,4,5,6)$ such that

$$\begin{aligned}
\Phi &= ka_1(\boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) - ka_1(\bar{\boldsymbol{u}}^n, \bar{\boldsymbol{u}}^n, \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) \\
&= ka_1(\boldsymbol{u}(t_{n-\frac{1}{2}}) - \bar{\boldsymbol{u}}^n, \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) + ka_1(\bar{\boldsymbol{u}}^n, \boldsymbol{u}(t_{n-\frac{1}{2}}) - \bar{\boldsymbol{u}}^n, \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) \\
&= ka_1(\boldsymbol{e}^n + \boldsymbol{e}^{n-1}, \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) + \frac{k^3}{16} a_1(\boldsymbol{u}_{tt}(\xi_{3n}), \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) \\
&\quad + \frac{k^3}{16} a_1(\boldsymbol{u}_{tt}(\xi_{4n}), \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) + \frac{k^3}{16} a_1(\bar{\boldsymbol{u}}^n, \boldsymbol{u}_{tt}(\xi_{5n}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) \\
&\quad + \frac{k^3}{16} a_1(\bar{\boldsymbol{u}}^n, \boldsymbol{u}_{tt}(\xi_{6n}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) \\
&\leqslant \frac{k\nu}{4} \|\nabla(\boldsymbol{e}^n + \boldsymbol{e}^{n-1})\|_0^2 + \frac{k\nu}{16} \|\nabla(\boldsymbol{e}^n + \boldsymbol{e}^{n-1})\|_0^2 \\
&\quad + \frac{k^5 N_0^2}{16} \|\nabla \boldsymbol{u}(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)} (\|\nabla \boldsymbol{u}(t)\|^2_{L^{2,\infty}(t_{n-1},t_n;L^2)} + \|\nabla \bar{\boldsymbol{u}}^n\|_0^2).
\end{aligned} \tag{21}$$

By using Hölder inequality and Cauchy inequality, we have that

$$\begin{aligned}
&\frac{k^3}{24}(\boldsymbol{u}_{ttt}(\xi_{1n}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}) + \frac{k^3\nu}{4}(\nabla \boldsymbol{u}_{tt}(\xi_{2n}), \nabla(\boldsymbol{e}^n + \boldsymbol{e}^{n-1})) \\
&\leqslant \frac{k\nu}{16}\|\nabla(\boldsymbol{e}^n + \boldsymbol{e}^{n-1})\|_0^2 + \frac{k^5}{64\nu}\|\boldsymbol{u}\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})} + 2\nu k^5 \|\nabla \boldsymbol{u}\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}.
\end{aligned} \tag{22}$$

Combining (21) and (22) with (20) yields that

$$\begin{aligned}
&\|\boldsymbol{e}^n\|_0^2 - \|\boldsymbol{e}^{n-1}\|_0^2 + \frac{k\nu}{8}\|\nabla(\boldsymbol{e}^n + \boldsymbol{e}^{n-1})\|_0^2 \\
&\leqslant \frac{k^5}{64\nu}\|\boldsymbol{u}\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})} + 2\nu k^5 \|\nabla \boldsymbol{u}\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)} \\
&\quad + \frac{k^5 N_0^2}{16}\|\nabla \boldsymbol{u}(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)} (\|\nabla \boldsymbol{u}(t)\|^2_{L^{2,\infty}(t_{n-1},t_n;L^2)} + \|\nabla \bar{\boldsymbol{u}}^n\|_0^2).
\end{aligned} \tag{23}$$

Summing (23) from 1 to n yields that

$$\begin{aligned}
&\|\boldsymbol{e}^n\|_0^2 + \frac{k\nu}{8}\sum_{i=1}^{n}\|\nabla(\boldsymbol{e}^i + \boldsymbol{e}^{i-1})\|_0^2 \\
&\leqslant \frac{Tk^4}{64\nu}\|\boldsymbol{u}\|^2_{W^{3,\infty}(H^{-1})} + 2T\nu k^4\|\nabla \boldsymbol{u}\|^2_{W^{2,\infty}(L^2)} \\
&+ \frac{k^4 N_0^2}{16}\|\nabla \boldsymbol{u}(t)\|^2_{W^{2,\infty}(L^2)} \left(T\|\nabla \boldsymbol{u}(t)\|^2_{L^{2,\infty}(L^2)} + k\sum_{i=1}^{n}\|\nabla \bar{\boldsymbol{u}}^i\|_0^2 \right).
\end{aligned} \tag{24}$$

It is obtained from (24) and (9) that

$$\|\boldsymbol{e}^n\|_0 + (k\nu)^{1/2}\|\nabla \boldsymbol{e}^n\|_0 \leqslant C(\boldsymbol{f})k^2, \quad 1 \leqslant n \leqslant N, \tag{25}$$

where $C^2(\boldsymbol{f}) = N_0^2\nu^{-1}\|\boldsymbol{u}\|^2_{W^{1,\infty}(H^1)}(\|\boldsymbol{u}^0\|_2 + \nu^{-1}\|\boldsymbol{f}\|^2_{L^\infty(H^{-1})}) + T\nu^{-1}\|\boldsymbol{u}\|^2_{W^{3,\infty}(H^{-1})}$ $+16\nu T\|\boldsymbol{u}\|^2_{W^{2,\infty}(H^1)} + TN_0^2\|\boldsymbol{u}\|^4_{W^{2,\infty}(H^1)}$.

By using Taylor's formula, there are $\xi_{in} \in [t_{n-1}, t_n]$ $(i = 7, 8, 9, 10, 11)$ such that

$$\begin{aligned}
&\frac{1}{k}(\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \boldsymbol{v}) + \frac{\nu}{2}(\nabla(\boldsymbol{e}^n + \boldsymbol{e}^{n-1}), \boldsymbol{v}) + \frac{1}{2}[a_1(\boldsymbol{e}^n + \boldsymbol{e}^{n-1}, \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{v})\\
&+a_1(\bar{\boldsymbol{u}}^n, \boldsymbol{e}^n + \boldsymbol{e}^{n-1}, \boldsymbol{v})] - \frac{k^2}{48}(\boldsymbol{u}_{ttt}(\xi_{7n}), \boldsymbol{v}) - \frac{k^2}{48}(\boldsymbol{u}_{ttt}(\xi_{8n}), \boldsymbol{v})\\
&-\frac{\nu k^2}{16}(\nabla \boldsymbol{u}_{tt}(\xi_{9n}), \nabla \boldsymbol{v}) - \frac{\nu k^2}{16}(\nabla \boldsymbol{u}_{tt}(\xi_{10n}), \nabla \boldsymbol{v}) + \frac{k}{2}b(\boldsymbol{v}, p_t(\xi_{9n}))\\
&-\frac{k^2}{16}a_1(\boldsymbol{u}_{tt}(\xi_{10n}), \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{v}) - \frac{k^2}{16}a_1(\bar{\boldsymbol{u}}^n, \boldsymbol{u}_{tt}(\xi_{11n}), \boldsymbol{v})\\
&= b(\boldsymbol{v}, p(t_n) - p^n), \quad \forall \boldsymbol{v} \in U.
\end{aligned} \tag{26}$$

Then, with (25), (26), (8), Hölder inequality, and Cauchy inequality, we have that

$$\begin{aligned}
&\|p(t_n) - p^n\|_0 \leqslant \beta^{-1}\sup_{\boldsymbol{v}\in U}\frac{b(\boldsymbol{v}, p(t_n) - p^n)}{\|\nabla v\|_0}\\
&= \beta^{-1}k[2C(\boldsymbol{f}) + (\nu k)^{1/2}C(\boldsymbol{f}) + \frac{(\nu k)^{1/2}}{4}C(\boldsymbol{f}) + \frac{1}{2}\|p\|_{W^{1,\infty}(L^2)}\\
&+\frac{k}{24}\|\boldsymbol{u}\|_{W^{3,\infty}(L^2)} + \frac{\nu k}{8}\|\nabla \boldsymbol{u}\|_{W^{2,\infty}(L^2)} + \frac{\nu k}{64}\|\nabla \boldsymbol{u}\|_{W^{2,\infty}(L^2)}]\\
&+\beta^{-1}k\left(C(\boldsymbol{f}) + \frac{k^{1/2}}{16\nu^{1/2}}\|\nabla \boldsymbol{u}\|_{W^{2,\infty}(L^2)}\right)\left[\|\boldsymbol{u}^0\|_0 + \left(\nu^{-1}k\sum_{i=0}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|^2_{-1}\right)^{1/2}\right]\\
&\equiv \tilde{C}(\boldsymbol{f})k,
\end{aligned} \tag{27}$$

which completes the proof of Theorem 1.

3 Fully discrete SCNFVE formulation with the second-order time accuracy

3.1 Theory of FVE method

In order to get the numerical solutions of SCNFVE for Problem III, it is necessary to introduce the FVE approximation for the spatial variables of Problem II by means of the idea in Reference Li, Chen, and Wu (2002) [Li, Chen, and Wu (2002)].

Let $\Im_h = \{K\}$ be the quasi-uniform triangulation of Ω [Luo (2006) or Brezzi and Fortin (1991)] and $\Im_h^*$ the dual partition based on $\Im_h$. The elements in $\Im_h^*$, called the control volumes, are formed by means of the same approach as that in Cai and McCormick (1990) or Li, Chen, and Wu (2002). Let $\boldsymbol{z}_K$ be the barycenter of

$K \in \Im_h$. By connecting $\boldsymbol{z}_K$ with line segments to the midpoints of the edges of K, it is subdivided into three quadrilaterals K_z ($\boldsymbol{z} = (x_z, y_z) \in Z_h(K)$, where $Z_h(K)$ is a set of the vertices of K). Then the control volume V_z is formed by the sub-regions K_z of the sharing vertex $\boldsymbol{z} \in Z_h = \bigcup_{K \in \Im_h} Z_h(K)$ (see Figure 1). Thus, all control volumes covering the domain Ω constitute the dual partition $\Im_h^*$ based on $\Im_h$. Z_h° represents the set of interior vertices in Z_h.

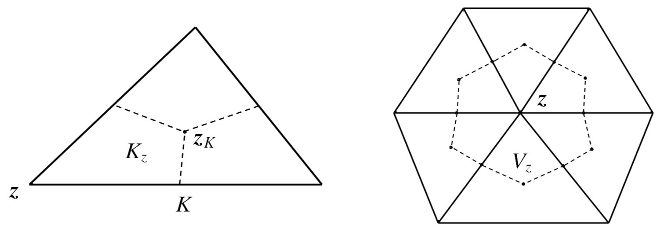

Figure 1: Left chart is a triangle K subdivided into three sub-regions K_z and right chart is a sample region with dotted lines indicating the corresponding control volume V_z.

The partition $\Im_h^*$ is known as regular or quasi-uniform, if there exist two positive constants C_1 and C_2, being independent of the spatial mesh size h and temporal mesh size k, such that

$$C_1 h^2 \leqslant \text{mes}(V_z) \leqslant C_2 h^2, \quad \forall V_z \in \Im_h^*, \tag{28}$$

where $\text{mes}(V_z)$ denotes the measure of V_z. Since the FE triangulation $\Im_h$ is quasi-uniform, so the dual partition $\Im_h^*$ is also quasi-uniform [Li, Chen, and Wu (2002)].

The trial function spaces U_h and M_h of velocity and pressure are respectively defined as follows:

$$U_h = \left\{ \boldsymbol{v}_h \in X \cap C(\overline{\Omega})^2 : \ \boldsymbol{v}_h|_K \in \mathscr{P}_1^2(K), \forall K \in \Im_h \right\},$$

$$M_h = \{ q_h \in M : \ q_h|_K \in \mathscr{P}_1(K), \ \forall K \in \Im_h \},$$

where $\mathscr{P}_1$ is 1-th polynomial space on K. It is obvious that $U_h \subset U = H_0^1(\Omega)^2$. For $\boldsymbol{u} \in U = H_0^1(\Omega)^2$, let $\Pi_h \boldsymbol{u}$ be the interpolating operator of $\boldsymbol{u}$ onto the trial function space U_h. If $\boldsymbol{u} \in H^2(\Omega)^2$, it follows from the interpolating theorem of Sobolev spaces [Li, Chen, and Wu (2002); Luo (2006)] that

$$\|\Pi_h \boldsymbol{u}\|_0 \leqslant C \|\boldsymbol{u}\|_0, \quad |\boldsymbol{u} - \Pi_h \boldsymbol{u}|_m \leqslant C h^{2-m} |\boldsymbol{u}|_2, \ \ m = 0, 1, \tag{29}$$

where C in this context indicates the positive constant which is possibly different at different occurrences, being independent of the spatial mesh size h and temporal mesh size k.

The test space $\tilde{U}_h$ of fluid velocity is defined as follows:

$$\begin{aligned}\tilde{U}_h = \{ & \boldsymbol{v}_h \in L^2(\Omega)^2 : \boldsymbol{v}_h|_{V_z} \in \mathscr{P}_0(V_z)^2 \ (V_z \cap \partial\Omega = \emptyset); \\ & \boldsymbol{v}_h|_{V_z} = \mathbf{0} \ (V_z \cap \partial\Omega \neq \emptyset), \forall V_z \in \Im_h^* \}\end{aligned} \tag{30}$$

spanned by the following basis functions

$$\phi_z(x,y) = \begin{cases} 1, & (x,y) \in V_z, \\ 0, & \text{elsewhere}, \end{cases} \quad \boldsymbol{z} \in Z_h^\circ. \tag{31}$$

For $\boldsymbol{w} \in U$, let $\Pi_h^* \boldsymbol{w}$ be the interpolating operator of $\boldsymbol{w}$ onto the test space $\tilde{U}_h$, i.e.,

$$\Pi_h^* \boldsymbol{w} = \sum_{z \in Z_h^\circ} \boldsymbol{w}(\boldsymbol{z}) \phi_{\boldsymbol{z}}. \tag{32}$$

It follows from the interpolating theorem of Sobolev spaces [Li, Chen, and Wu (2002); Luo (2006)] that the following error estimate

$$\|\boldsymbol{w} - \Pi_h^* \boldsymbol{w}\|_0 \leqslant Ch|\boldsymbol{w}|_1. \tag{33}$$

Moreover, there are the following properties for the interpolating operator Π_h^* [Shen, Li, and Chen (2009); Yang and Song (2009)].

Lemma 2 *If $\boldsymbol{v}_h \in U_h$, then*

$$\int_K (\boldsymbol{v}_h - \Pi_h^* \boldsymbol{v}_h) \mathrm{d}x\mathrm{d}y = 0, K \in \Im_h; \|\boldsymbol{v}_h - \Pi_h^* \boldsymbol{v}_h\|_{L^r(\Omega)} \leqslant Ch\|\boldsymbol{v}_h\|_{W^{1,r}(\Omega)}, 1 \leqslant r \leqslant \infty. \tag{34}$$

3.2 Fully discrete SCNFVE formulation based on two local Gauss integrals and parameter-free with the second-order time accuracy

Though the trial function space U_h satisfies $U_h \subset U$ like FE methods, the test space $\tilde{U}_h \not\subset U_h$. As in the case of nonconforming FE methods, this is due to the loss of continuity of the vector functions in $\tilde{U}_h$ on the boundary of two neighboring elements. So the bilinear forms $a(\boldsymbol{u}, \boldsymbol{v})$ and $b(\boldsymbol{v}, p)$ must be revised accordingly. It is obtained by Green's formula that

$$\begin{aligned}&\int_\Omega \Delta \boldsymbol{u} \cdot \boldsymbol{v} \mathrm{d}x\mathrm{d}y = \sum_{V_z \in \Im_h^*} \int_{V_z} \Delta \boldsymbol{u} \cdot \boldsymbol{v} \mathrm{d}x\mathrm{d}y \\ &= -\sum_{V_z \in \Im_h^*} \int_{V_z} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \mathrm{d}x\mathrm{d}y + \sum_{V_z \in \Im_h^*} \int_{\partial V_z} (\boldsymbol{v} \nabla \boldsymbol{u}) \cdot \boldsymbol{n} \mathrm{d}s;\end{aligned} \tag{35}$$

$$\int_\Omega \nabla p \cdot \boldsymbol{v} \mathrm{d}x\mathrm{d}y = -\sum_{V_z \in \Im_h^*} \int_{V_z} p \mathrm{div} \boldsymbol{v} \mathrm{d}x\mathrm{d}y + \sum_{V_z \in \Im_h^*} \int_{\partial V_z} p \boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}s, \tag{36}$$

where $\int_{\partial V_z}$ denotes the line integral, with the counter clockwise direction, on the boundary ∂V_z of the dual element; $\boldsymbol{n} = (n_1, n_2)$ is the unit outer normal vector to ∂V_z. So the bilinear forms $a(\boldsymbol{u}, \boldsymbol{v})$ and $b(\boldsymbol{v}, p)$ are respectively rewritten as

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu \sum_{V_z \in \Im_h^*} \left[\int_{V_z} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \mathrm{d}x\mathrm{d}y - \int_{\partial V_z} (\boldsymbol{v} \nabla \boldsymbol{u}) \cdot \boldsymbol{n} \mathrm{d}s \right]; \tag{37}$$

$$b(\boldsymbol{v},p) = \sum_{V_z \in \Im_h^*} \left[\int_{\partial V_z} p\boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}s - \int_{V_z} p \mathrm{div} \boldsymbol{v} \mathrm{d}x\mathrm{d}y \right]. \tag{38}$$

Since $\tilde{U}_h$ is the piecewise constant vector function space with the characteristic functions of the dual elements V_z as the basis functions, there hold

$$a(\boldsymbol{u},\boldsymbol{v}) = -\nu \sum_{V_z \in \Im_h^*} \int_{\partial V_z} (\boldsymbol{v} \nabla \boldsymbol{u}) \cdot \boldsymbol{n} \mathrm{d}s, \forall \boldsymbol{v} \in \tilde{U}_h, \boldsymbol{u} \in U_h; \tag{39}$$

$$b(\boldsymbol{v},p) = \sum_{V_z \in \Im_h^*} \int_{\partial V_z} p\boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}s, \forall \boldsymbol{v} \in \tilde{U}_h, \forall p \in M_h. \tag{40}$$

Then, the fully discrete SCNFVE formulation based on two local Gauss integrals and parameter-free with the second-order time accuracy for the non-stationary Navier-Stokes equations is read as follows.

Problem IV. Find $(\boldsymbol{u}_h^n, p_h^n) \in U_h \times M_h$ $(1 \leqslant n \leqslant N)$ such that

$$\begin{cases} (\bar{\partial}_t \boldsymbol{u}_h^n, \Pi_h^* \boldsymbol{v}_h) + a_h(\bar{\boldsymbol{u}}_h^n, \Pi_h^* \boldsymbol{v}_h) + a_{1h}(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \Pi_h^* \boldsymbol{v}_h) + b_h(\Pi_h^* \boldsymbol{v}_h, p_h^n) \\ = (\boldsymbol{f}^{n-\frac{1}{2}}, \Pi_h^* \boldsymbol{v}_h), \ \forall \boldsymbol{v}_h \in U_h, \\ b(\boldsymbol{u}_h^n, q_h) + D_h(p_h^n, q_h) = 0, \ \ \forall q_h \in M_h, \\ \boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}^0, \ \ (x,y) \in \Omega, \end{cases} \tag{41}$$

where $\bar{\boldsymbol{u}}_h^n = [\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}]/2$;

$$a_h(\boldsymbol{u}_h^n, \Pi_h^* \boldsymbol{v}_h) = -\nu \sum_{\boldsymbol{z}_j \in Z_h^\circ} \int_{\partial V_{\boldsymbol{z}_j}} (\boldsymbol{v}_h(\boldsymbol{z}_j) \nabla \boldsymbol{u}_h^n) \cdot \boldsymbol{n} \mathrm{d}s; \tag{42}$$

$$b_h(\Pi_h^* \boldsymbol{v}_h, q_h) = \sum_{\boldsymbol{z}_j \in Z_h^\circ} \boldsymbol{v}_h(\boldsymbol{z}_j) \int_{\partial V_{\boldsymbol{z}_j}} q_h \boldsymbol{n} \mathrm{d}s; \tag{43}$$

$$a_{1h}(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \Pi_h^* \boldsymbol{v}_h) = ((\boldsymbol{u}_h^n \cdot \nabla) \boldsymbol{u}_h^n, \boldsymbol{\Pi}_h^* \boldsymbol{v}_h) + ((\mathrm{div} \boldsymbol{u}_h^n) \boldsymbol{u}_h^n, \boldsymbol{\Pi}_h^* \boldsymbol{v}_h)/2; \tag{44}$$

$$D_h(p_h^n, q_h) = \varepsilon \sum_{K \in \Im_h} \left\{ \int_{K,2} p_h^n q_h \mathrm{d}x\mathrm{d}y - \int_{K,1} p_h^n q_h \mathrm{d}x\mathrm{d}y \right\}, p_h, q_h \in M_h, \tag{45}$$

here ε is a positive real number and $\int_{K,i} g(x,y)\mathrm{d}x\mathrm{d}y$ $(i=1,2)$ indicate the appropriate Gauss integrals over K which are exact for polynomials of degree i $(i=1,2)$ and $g(x,y)=p_hq_h$ is a polynomial of degree not more than i $(i=1,2)$. Thus, for all test functions $q_h \in M_h$, the trial function $p_h \in M_h$ must be piecewise constant when $i=1$.

Further, we define the L^2-projection operator $\rho_h : L^2(\Omega) \to W_h$ such that $\forall p \in L^2(\Omega)$ satisfying

$$(p,q_h)=(\rho_h p,q_h),\ \ \forall q_h \in W_h, \tag{46}$$

where $W_h \subset L^2(\Omega)$ denotes the piecewise constant space associated with $\Im_h$. The projection operator ρ_h has the following properties [Shen, Li, and Chen (2009); He and He (2007); An, Sun, Luo, and Huang (2011)]:

$$\|\rho_h p\|_0 \leqslant C\|p\|_0, \quad \forall p \in L^2(\Omega), \tag{47}$$

$$\|p-\rho_h p\|_0 \leqslant Ch\|p\|_1, \quad \forall p \in H^1(\Omega). \tag{48}$$

Now, by using the definition of ρ_h, the bilinear form $D_h(\cdot,\cdot)$ can be rewrite as follows:

$$D_h(p_h,q_h)=\varepsilon(p_h-\rho_h p_h,q_h)=\varepsilon(p_h-\rho_h p_h,q_h-\rho_h q_h). \tag{49}$$

4 Existence and error estimates of fully discrete SCNFVE solutions

In order to discuss the existence, the uniqueness, the stability, and the error estimates of the solutions for fully discrete SCNFVE formulation with the second-order time accuracy or Problem IV, it is necessary to introduce some preliminary lemmas.

From Cai and McCormick (1990), Jones and Menziest (2000), Li, Chen, and Wu (2002), or Li, Luo, and Li (2007) we have the following three lemmas.

Lemma 3 *Let $\boldsymbol{z} \in Z_h$, $S_z^* = \mathrm{mes}(V_z)$, $S_K = \mathrm{mes}(K)$, $\boldsymbol{z}_i$, $\boldsymbol{z}_j$, and $\boldsymbol{z}_k$ be the vertices of $K \in \Im_h$, and define as follows:* $\forall \boldsymbol{u}_h \in U_h$,

$$\begin{aligned}\|\boldsymbol{u}_h\|_{0,h} &\equiv \|\Pi_h^*\boldsymbol{u}_h\|_0\\ &=\left\{\frac{1}{3}\sum_{K\in\Im_h}[\boldsymbol{u}_h^2(\boldsymbol{z}_i)+\boldsymbol{u}_h^2(\boldsymbol{z}_j)+\boldsymbol{u}_h^2(\boldsymbol{z}_k)]S_K\right\}^{1/2},\end{aligned} \tag{50}$$

$$|\boldsymbol{u}_h|_{1,h} \equiv \left\{\sum_{z\in K\in\Im_h}\left[\left(\frac{\partial \boldsymbol{u}_h(\boldsymbol{z})}{\partial x}\right)^2+\left(\frac{\partial \boldsymbol{u}_h(\boldsymbol{z})}{\partial y}\right)^2\right]S_K\right\}^{1/2}, \tag{51}$$

$$\|\boldsymbol{u}_h\|_{1,h} = \left[\|\boldsymbol{u}_h\|_{0,h}^2 + |\boldsymbol{u}_h|_{1,h}^2\right]^{1/2}. \tag{52}$$

Then the pairs of norms $|\cdot|_{1,h}$ and $|\cdot|_1$, $\|\cdot\|_{0,h}$ and $\|\cdot\|_0$, and $\|\cdot\|_{1,h}$ and $\|\cdot\|_1$ are equivalent on U_h, respectively.

Lemma 4 *There hold the following results:*

$$a_h(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{v}_h) = a(\boldsymbol{u}_h, \boldsymbol{v}_h),\ \forall \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h \in U_h, \tag{53}$$

$$b_h(\Pi_h^* \boldsymbol{v}_h, p_h) = -b(\boldsymbol{v}_h, p_h),\ \ \forall \boldsymbol{v}_h \in U_h, \forall p_h \in M_h. \tag{54}$$

Further, $a_h(\boldsymbol{u}_h, \Pi_h^* v_h)$ is symmetric, bounded, and positive definite, i.e.,

$$a_h(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{v}_h) = a_h(\boldsymbol{v}_h, \Pi_h^* \boldsymbol{u}_h),\ \ \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in U_h, \tag{55}$$

and there exist a positive constants $h_0 \geqslant h >$ such that

$$a_h(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{u}_h) \geqslant \nu |\boldsymbol{u}_h|_1^2, |a_h(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{v}_h)| \leqslant \nu \|\boldsymbol{u}_h\|_1 \|\boldsymbol{v}_h\|_1, \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in U_h. \tag{56}$$

Lemma 5 *There holds the following statement:*

$$(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{v}_h) = (\boldsymbol{v}_h, \Pi_h^* \boldsymbol{u}_h), \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in U_h. \tag{57}$$

For any $\boldsymbol{u} \in H^m(\Omega)^2$ $(m = 0,1)$ and $\boldsymbol{v}_h \in U_h$,

$$|(\boldsymbol{u}, \boldsymbol{v}_h) - (\boldsymbol{u}, \Pi_h^* v_h)| \leqslant C h^{m+n} \|\boldsymbol{u}\|_m \|\boldsymbol{v}_h\|_n,\ \ n = 0,1. \tag{58}$$

Set $|||\boldsymbol{u}_h|||_0 = (\boldsymbol{u}_h, \Pi_h^* \boldsymbol{u}_h)^{1/2}$, then $|||\cdot|||_0$ is equivalent to $\|\cdot\|_0$ on U_h, namely there exist two positive constants C_3 and C_4 such that

$$C_3 \|\boldsymbol{u}_h\|_0 \leqslant |||\boldsymbol{u}_h|||_0 \leqslant C_4 \|\boldsymbol{u}_h\|_0,\ \ \forall \boldsymbol{u}_h \in U_h. \tag{59}$$

The following discrete Gronwall Lemma [Luo (2006)] is useful for the proofs of the existence, the uniqueness, the stability, and the error estimates of the solutions for Problem IV.

Lemma 6 (discrete Gronwall Lemma) *If $\{a_n\}$ and $\{b_n\}$ are two positive sequences, $\{c_n\}$ is a monotone positive sequence, and they satisfy*

$$a_n + b_n \leqslant c_n + \bar{\lambda} \sum_{i=0}^{n-1} a_i,\ \ \bar{\lambda} > 0,\ \ a_0 + b_0 \leqslant c_0, \tag{60}$$

then

$$a_n + b_n \leqslant c_n \exp(n\bar{\lambda}),\ \ n \geqslant 0. \tag{61}$$

It is obtained by Lemma 4 and using the same approach as the proof of Theorem 4.1 in An, Sun, Luo, and Huang (2011) that the following inequalities

$$\begin{aligned}&\sup_{(\boldsymbol{v}_h,q_h)\in U_h\times M_h}\{2(\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)+k[a(\boldsymbol{u}_h^n,\boldsymbol{v}_h)-2b(\boldsymbol{v}_h,p_h^n)\\&+2b(\boldsymbol{u}_h^n,q_h)+2D_h(p_h^n,q_h)]\}/[\|\boldsymbol{v}_h\|_1+\|q_h\|_0]\\&\geqslant\tilde{\beta}(\|\boldsymbol{u}_h^n\|_0+k\|\nabla\boldsymbol{u}_h^n\|_0+k\|p_h^n\|_0),\ \forall(\boldsymbol{u}_h^n,p_h^n)\in U_h\times M_h,\end{aligned}\tag{62}$$

where $\tilde{\beta}$ is a constant independent of h and k and $1\leqslant n\leqslant N$.

There are the following results of the existence, the uniqueness, and the stability of the solution for Problem IV.

Theorem 7 *Under the hypotheses of Theorem 1, there exists a unique series of solutions* $(\boldsymbol{u}_h^n,p_h^n)$ $(n=1,2,\cdots,N)$ *to the fully discrete SCNFVE formulation with the second-order time accuracy, i.e., Problem IV satisfying*

$$\|\boldsymbol{u}_h^n\|_0^2+k\nu\sum_{i=1}^n|\bar{\boldsymbol{u}}_h^i|_1^2\leqslant C\left(\|\boldsymbol{u}^0\|_0^2+k\sum_{i=1}^n\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right),\tag{63}$$

$$k\|p_h^n\|_0\leqslant C\left(\|\boldsymbol{u}^0\|_0+k^{1/2}\sum_{i=1}^n\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}+\|\boldsymbol{u}^0\|_0^2+k\sum_{i=1}^n\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2\right),\tag{64}$$

where C is the constant independent of h and k, which shows that the series of solutions of Problem IV is stable.

Proof Problem IV has a unique series of solutions $(\boldsymbol{u}_h^n,p_h^n)$ $(n=1,2,\cdots,N)$ by means of mixed FE methods (see Heywood and Rannacher (1982), Heywood and Rannacher (1990), Luo (2006), or Brezzi and Fortin (1991)) due to inequality (62). It is obtained by taking $\boldsymbol{v}_h=\bar{u}_h^n$ in the first equation of Problem IV and $q_h=p_h^n$ in the second equation of Problem IV and by using Lemmas 3–5, Hölder inequality, and Cauchy inequality that

$$\begin{aligned}&\frac{1}{2}(|||\boldsymbol{u}_h^n|||_0^2-|||\boldsymbol{u}_h^{n-1}|||_0^2)+k\nu|\bar{\boldsymbol{u}}_h^n|_1^2+k\varepsilon\|p_h^n-\rho_hp_h^n\|_0^2\\&=k(\boldsymbol{f}^{n-\frac{1}{2}},\Pi_h^*\bar{\boldsymbol{u}}_h^n)\leqslant\frac{k}{2\nu}\|\boldsymbol{f}^{n-\frac{1}{2}}\|_{-1}^2+\frac{k\nu}{2}|\bar{\boldsymbol{u}}_h^n|_1^2.\end{aligned}\tag{65}$$

It follows from (65) that

$$\begin{aligned}&|||\boldsymbol{u}_h^n|||_0^2-|||\boldsymbol{u}_h^{n-1}|||_0^2+k\nu|\bar{\boldsymbol{u}}_h^n|_1^2\\&\leqslant|||\boldsymbol{u}_h^n|||_0^2-|||\boldsymbol{u}_h^{n-1}|||_0^2+2k\nu|\bar{\boldsymbol{u}}_h^n|_1^2\leqslant k\nu^{-1}\|\boldsymbol{f}^{n-\frac{1}{2}}\|_{-1}^2.\end{aligned}\tag{66}$$

It is gotten by summing (66) from 1 to n that

$$
\begin{aligned}
&|||\boldsymbol{u}_h^n|||_0^2 + k\nu\sum_{i=1}^{n}|\bar{\boldsymbol{u}}_h^i|_1^2 \leqslant |||\boldsymbol{u}_h^0|||_0^2 + k\nu^{-1}\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2 \\
&\leqslant |||\boldsymbol{u}_h^0|||_0^2 + k\nu^{-1}\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2,
\end{aligned} \tag{67}
$$

which yields (63) from (29). It is obtained by using (62), the first equation of Problem IV, Hölder inequality, inverse estimate, and (63) that

$$
\begin{aligned}
&\tilde{\beta}k\|p_h^n\|_0 \\
&\leqslant \sup_{(\boldsymbol{v}_h,q_h)\in U_h\times M_h}\frac{2(\boldsymbol{u}_h^n,\Pi_h^*\boldsymbol{v}_h)+k[a(\boldsymbol{u}_h^n,\boldsymbol{v}_h)-2b(\boldsymbol{v}_h,p_h^n)+2b(\boldsymbol{u}_h^n,q_h)+2D_h(p_h^n,q_h)]}{\|\boldsymbol{v}_h\|_1+\|q_h\|_0} \\
&\leqslant \sup_{\boldsymbol{v}_h\in U_h}\frac{2ka_{1h}(\bar{\boldsymbol{u}}_h^n,\bar{\boldsymbol{u}}_h^n,\Pi_h^*\boldsymbol{v}_h)-ka_h(\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h)+2k(\boldsymbol{f}^{n-\frac{1}{2}},\Pi_h^*\boldsymbol{v}_h)+2(\boldsymbol{u}_h^{n-1},\Pi_h^*\boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_1} \\
&\leqslant 2kN_0\|\bar{\boldsymbol{u}}_h^n\|_0^2 + C[\|\boldsymbol{u}_h^{n-1}\|_0 + k\|\boldsymbol{f}^{n-\frac{1}{2}}\|_{-1}] \\
&\leqslant C\left(\|\boldsymbol{u}_h^0\|_0^2+\|\boldsymbol{u}_h^0\|_0+k\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}^2+k^{1/2}\sum_{i=1}^{n}\|\boldsymbol{f}^{i-\frac{1}{2}}\|_{-1}\right),
\end{aligned} \tag{68}
$$

which yields (64) from (29) and completes the proof of Theorem 8.

The following Lemma 8 is obtained by means of the SCNFE methods [Li, Shen, and Chen (2010); Luo (2006); Brezzi and Fortin (1991)] for the non-stationary Navier-Stokes equations.

Lemma 8. *Let $(S_h\boldsymbol{u}^n, Q_hp^n)$ be the Navier-Stokes projection of the solutions $(\boldsymbol{u}^n, p^n)$ for Problem III on $U_h\times M_h$, i.e., for the solutions $(\boldsymbol{u}^n, p^n)\in U\times M$ for Problem III, there exist $(S_h\boldsymbol{u}^n, Q_hp^n)$ ($n=1,2,...,N$) such that, for $n=1,2,...,N$,*

$$
\begin{aligned}
&k\mathscr{A}((S_h\bar{\boldsymbol{u}}^n,Q_hp^n);(\boldsymbol{v}_h,q_h))+(S_h\boldsymbol{u}^n-S_h\boldsymbol{u}^{n-1},\boldsymbol{v}_h)+kD_h(Q_hp^n,q_h)= \\
&k\mathscr{A}((\bar{\boldsymbol{u}}^n,p^n);(\boldsymbol{v}_h,q_h))+(\boldsymbol{u}^n-\boldsymbol{u}^{n-1},\boldsymbol{v}_h), \forall(\boldsymbol{v}_h,q_h)\in U_h\times M_h,
\end{aligned} \tag{69}
$$

$$
S_h\boldsymbol{u}^0=\Pi_h\boldsymbol{u}^0(x,y),\quad \boldsymbol{u}^0=\boldsymbol{u}^0(x,y),\quad (x,y)\in\Omega, \tag{70}
$$

where $\mathscr{A}((S_h\bar{\boldsymbol{u}}^n,Q_hp^n);(\boldsymbol{v}_h,q_h))=a(S_h\bar{\boldsymbol{u}}^n,\boldsymbol{v}_h)-b(\boldsymbol{v}_h,Q_hp^n)+b(S_h\boldsymbol{u}^n,q_h)+a_1(S_h\bar{\boldsymbol{u}}^n, S_h\bar{\boldsymbol{u}}^n,\boldsymbol{v}_h)$. Then, there hold

$$
\|S_h\boldsymbol{u}^n\|_1+\|Q_hp^n\|_0\leqslant C(\|\boldsymbol{u}^n\|_1+\|p^n\|_0),\quad 1\leqslant n\leqslant N. \tag{71}
$$

If the solution $(\boldsymbol{u}^n, p^n)\in H^2(\Omega)^2\times H^1(\Omega)$ ($n=1,2,\cdots,N$) for Problem III, then there hold the following error estimates

$$
\begin{aligned}
&\|\boldsymbol{u}^n-S_h\boldsymbol{u}^n\|_0^2+k\nu\sum_{i=1}^{n}\|\boldsymbol{u}^i-S_h\boldsymbol{u}^i\|_1^2\leqslant Ch^4, \\
&\|p^n-Q_hp^n\|_0\leqslant Ch, n=1,2,...,N.
\end{aligned} \tag{72}
$$

Remark 1 In fact, (69) and (70) are the system of error equations between standard SCNFE formulation and Problem III, thus (71) and (72) are directly obtained from SCNFE method (see, *e.g.*, Li, Shen, and Chen (2010), Luo (2006), or Brezzi and Fortin (1991)) like the approaches in Heywood and Rannacher (1990).

Theorem 9 *Let $(\boldsymbol{u},p)$ be the solution for Problem II and $(\boldsymbol{u}_h^n,p_h^n)$ the solution of fully discrete SCNFVE formulation with the second-order time accuracy (i.e., Problem IV). Then, under the hypotheses of Theorems 1 and 7, if $p_h^0=p^0=0$ (or $p_h^0=Q_hp^0$), $h=O(k)$, $N\nu^{-1}\|\nabla\bar{\boldsymbol{u}}_h^n\|_0\leqslant 1/4$, $\boldsymbol{u}^0\in H^2(\Omega)^2$, and $\boldsymbol{f}\in W^{1,\infty}(0,T;H^1(\Omega)^2)^2$, there hold the following error estimates*

$$\begin{aligned}&\|\boldsymbol{u}(t_n)-\boldsymbol{u}_h^n\|_0+k[\|p(t_n)-p_h^n\|_0+\|\boldsymbol{u}(t_n)-\boldsymbol{u}_h^n\|_1]\\&\leqslant C\left(h^2+k^2\right),n=1,2,...,N.\end{aligned}\tag{73}$$

Proof Subtracting Problem IV from Problem III taking $\boldsymbol{v}=\boldsymbol{v}_h$ and $q=q_h$, and using Lemmas 4 and 2 yield the following system of error equations:

$$\begin{cases}(\boldsymbol{u}^n-\boldsymbol{u}_h^n,\boldsymbol{v}_h)+(\boldsymbol{u}_h^n-\Pi_h^*\boldsymbol{u}_h^n,\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h)+ka(\bar{\boldsymbol{u}}^n-\bar{\boldsymbol{u}}_h^n,\boldsymbol{v}_h)\\\quad+ka_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\boldsymbol{v}_h)-ka_{1h}(\bar{\boldsymbol{u}}_h^n,\bar{\boldsymbol{u}}_h^n,\Pi_h^*\boldsymbol{v}_h)-kb(\boldsymbol{v}_h,p^n-p_h^n)\\\quad=k(\boldsymbol{f}^{n-\frac{1}{2}}-\Pi_h^*\boldsymbol{f}^{n-\frac{1}{2}},\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h)+(\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h)\\\quad+(\boldsymbol{u}_h^{n-1}-\Pi_h^*\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h-\Pi_h^*\boldsymbol{v}_h),\ \forall\boldsymbol{v}_h\in U_h,n=1,2,...,N;\\b(\boldsymbol{u}^n-\boldsymbol{u}_h^n,q_h)-\varepsilon(p_h^n-\rho_hp_h^n,q_h-\rho_hq_h)=0,\forall q_h\in M_h,n=1,2,...,N,\\\boldsymbol{u}^0-\boldsymbol{u}_h^0=\boldsymbol{u}^0(x,y)-\Pi_h\boldsymbol{u}^0(x,y),\ \ (x,y)\in\Omega.\end{cases}\tag{74}$$

Let $\zeta^n=Q_hp^n-p_h^n$, $\boldsymbol{E}^n=S_h\boldsymbol{u}^n-\boldsymbol{u}_h^n$, and $\bar{\boldsymbol{E}}^n=S_h\bar{\boldsymbol{u}}^n-\bar{\boldsymbol{u}}_h^n$. On the one hand, it is obtained by using (69), the system of error equations (74), and Lemmas 4 and 5 that

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{E}^n\|_0^2+k\nu|\bar{\boldsymbol{E}}^n|_1^2\\=&(S_h\boldsymbol{u}^n-\boldsymbol{u}^n,\bar{\boldsymbol{E}}^n)+ka(S_h\bar{\boldsymbol{u}}^n-\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{E}}^n)\\&+(\boldsymbol{u}^n-\boldsymbol{u}_h^n,\bar{\boldsymbol{E}}^n)+ka(\bar{\boldsymbol{u}}^n-\bar{\boldsymbol{u}}_h^n,\bar{\boldsymbol{E}}^n)-\frac{1}{2}(\boldsymbol{E}^{n-1},\boldsymbol{E}^n)\\=&(S_h\boldsymbol{u}^{n-1}-\boldsymbol{u}^{n-1},\bar{\boldsymbol{E}}^n)+kb(\bar{\boldsymbol{E}}^n,Q_hp^n-p^n)+ka_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{E}}^n)\\&-ka_1(S_h\bar{\boldsymbol{u}}^n,S_h\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{E}}^n)-ka_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{E}}^n)+ka_{1h}(\bar{\boldsymbol{u}}_h^n,\bar{\boldsymbol{u}}_h^n,\Pi_h^*\bar{\boldsymbol{E}}^n)\\&+kb(\bar{\boldsymbol{E}}^n,p^n-p_h^n)-(\boldsymbol{u}_h^n-\Pi_h^*\boldsymbol{u}_h^n,\bar{\boldsymbol{E}}^n-\Pi_h^*\bar{\boldsymbol{E}}^n)\\&+(\boldsymbol{u}_h^{n-1}-\Pi_h^*\boldsymbol{u}_h^{n-1},\bar{\boldsymbol{E}}^n-\Pi_h^*\bar{\boldsymbol{E}}^n)+(\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\bar{\boldsymbol{E}}^n)\\&-\frac{1}{2}(S_h\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\boldsymbol{E}^n)+k(\boldsymbol{f}^{n-\frac{1}{2}}-\Pi_h^*\boldsymbol{f}^{n-\frac{1}{2}},\bar{\boldsymbol{E}}^n-\Pi_h^*\bar{\boldsymbol{E}}^n)\end{aligned}$$

$$
\begin{aligned}
&= ka_1(\bar{\boldsymbol{E}}_h^n, \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n) \\
&\quad - (\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1} - \Pi_h^*(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}), \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n) \\
&\quad + \frac{1}{2}(\boldsymbol{E}^{n-1}, \boldsymbol{E}^{n-1}) + kb(\bar{\boldsymbol{E}}^n, \zeta^n) + ka_{1h}(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n).
\end{aligned} \tag{75}
$$

On the other hand, it is first obtained by using (58), Hölder inequality, and Cauchy inequality that

$$
|k(\boldsymbol{f}^{n-\frac{1}{2}} - \Pi_h^* \boldsymbol{f}^{n-\frac{1}{2}}, \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n)| \leqslant Ckh^2 \|\boldsymbol{f}^{n-\frac{1}{2}}\|_1 |\bar{\boldsymbol{E}}^n|_1 \leqslant Ckh^4 + \frac{\nu k}{8} |\bar{\boldsymbol{E}}^n|_1^2. \tag{76}
$$

And then, if $k = O(h)$, by using inverse error estimate and Taylor's formula, we obtain that

$$
\begin{aligned}
&|(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1} - \Pi_h^*(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}), \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n)| \leqslant Ch^2 \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_1 |\bar{\boldsymbol{E}}^n|_1 \\
&\leqslant Ch^3 (\|\nabla \boldsymbol{E}^n\|_0^2 + \|\nabla(S_h \boldsymbol{u}^n - S_h \boldsymbol{u}^{n-1})\|_0^2 + \|\nabla \boldsymbol{E}^{n-1}\|_0^2) + \frac{k\nu}{8} |\bar{\boldsymbol{E}}^n|_1^2 \\
&\leqslant Ch \|\boldsymbol{E}^n\|_0^2 + Ck^2 h^3 + Ch \|\boldsymbol{E}^{n-1}\|_0^2 + \frac{k\nu}{8} |\bar{\boldsymbol{E}}^n|_1^2.
\end{aligned} \tag{77}
$$

Next, noting that $b(S_h \boldsymbol{u}^n - \boldsymbol{u}^n, q_h) = -k\varepsilon(Q_h p^n - \rho_h(Q_h p^n), q_h - \rho_h q_h)$, by the properties of operator ρ_h and the second equation of (74), we have that

$$
\begin{aligned}
&b(\bar{\boldsymbol{E}}^n, \zeta^n) = b(S_h \bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}^n, \zeta^n) + b(\bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n, \zeta^n) \\
&= -\frac{\varepsilon}{2}(\zeta^n - \rho_h \zeta^n, \zeta^n - \rho_h \zeta^n) - \frac{\varepsilon}{2}(\zeta^{n-1} - \rho_h \zeta^{n-1}, \zeta^n - \rho_h \zeta^n) \\
&\leqslant \frac{\varepsilon}{4} \|\zeta^{n-1} - \rho_h \zeta^{n-1}\|_0^2 - \frac{\varepsilon}{4} \|\zeta^n - \rho_h \zeta^n\|_0^2.
\end{aligned} \tag{78}
$$

Finally, if $N_0 \nu^{-1} \|\bar{\boldsymbol{u}}_h^n\| \leqslant 1/4$ $(1, 2, \ldots, N)$, then we get by Lemma 5, (2), and (5) that

$$
k|a_{1h}(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n) - a_1(\bar{\boldsymbol{E}}^n, \bar{\boldsymbol{u}}_h^n, \bar{E}^n)| \leqslant Ckh^4 + \frac{k\nu}{4} |\bar{\boldsymbol{E}}^n\|_0^2. \tag{79}
$$

Thus, it is obtained by combining (76)–(79) with (75) that

$$
\begin{aligned}
&\|\boldsymbol{E}^n\|_0^2 + k\nu |\bar{\boldsymbol{E}}^n|_1^2 + \frac{\varepsilon}{2} \|\zeta^n - \rho_h \zeta^n\|_0^2 - \frac{\varepsilon}{2} \|\zeta^{n-1} - \rho_h(\zeta^{n-1}\|_0^2 \\
&\leqslant Ckh^4 + Ck^2 h^3 + \|E^{n-1}\|_0^2 + Ch \|\boldsymbol{E}^{n-1}\|_0^2 + Ch \|\boldsymbol{E}^n\|_0^2.
\end{aligned} \tag{80}
$$

If h is sufficiently small such that $Ch \leqslant 1/2$ in (80) and $p_h^0 = p^0 = 0$ (or $p_h^0 = Q_h p^0$), summing (80) from 1 to n yields that

$$
\begin{aligned}
&\|S_h \boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0^2 + k\nu \sum_{i=1}^{n} |S_h \bar{\boldsymbol{u}}^i - \bar{\boldsymbol{u}}_h^i|_1^2 \\
&\leqslant Ch^4 + \|S_h \boldsymbol{u}^0 - \boldsymbol{u}_h^0\|_0^2 + Ch \sum_{i=0}^{n-1} \|S_h \boldsymbol{u}^i - \boldsymbol{u}_h^i)\|_0^2.
\end{aligned} \tag{81}
$$

By Gronwall Lemma 6, Lemma 8, and (29), it follows from (81) that

$$\begin{aligned}
&\|\boldsymbol{E}^n\|_0^2 + k\nu\sum_{i=1}^{n}|\bar{\boldsymbol{E}}^i|_1^2 \\
&\leqslant C\left[kh^4\sum_{i=1}^{n}\|\boldsymbol{f}^i\|_1^2 + nk^2h^3\|u_t\|_{L^\infty(H^1)}^2 + h^4\|\boldsymbol{u}^0\|_2^2\|_0^2\right]\exp(Cnh) \\
&\leqslant Ch^4.
\end{aligned} \tag{82}$$

Noting that $\sum_{i=1}^{n} a_i^2 \geqslant (\sum_{i=1}^{n} a_i)^2/n$ and $|a+b|_1 \geqslant |a|_1 - |b|_1$, by using (72) and triangle inequality, we obtain that

$$\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0 + k\nu\sum_{i=1}^{n}|\boldsymbol{u}^i - \boldsymbol{u}_h^i|_1 \leqslant Ch^2. \tag{83}$$

If $h = O(k)$, it is gotten from (62), error equation (74), inverse error estimate, Lemma 8, and (83) that

$$\begin{aligned}
&\tilde{\beta}k\|\zeta^n\|_0 \\
&\leqslant \sup_{(\boldsymbol{v}_h,q_h)\in U_h\times M_h}\frac{2(\boldsymbol{E}^n,\Pi_h^*\boldsymbol{v}_h) + k[a(\boldsymbol{E}_h^n,\boldsymbol{v}_h) - 2b(\boldsymbol{v}_h,\zeta^n) + 2b(\boldsymbol{E}^n,q_h) + 2D_h(p_h^n,q_h)]}{\|\boldsymbol{v}_h\|_1 + \|q_h\|_0} \\
&\leqslant C\|\boldsymbol{E}^n\|_0 + 2k\sup_{\boldsymbol{v}_h\in U_h}\frac{|b(\boldsymbol{v}_h,p^n - p_h^n)|}{\|\boldsymbol{v}_h\|_1} + 2k\sup_{\boldsymbol{v}_h\in U_h}\frac{|b(\boldsymbol{v}_h,Q_hp^n - p^n)|}{\|\boldsymbol{v}_h\|_1} \\
&\leqslant 2k\sup_{\boldsymbol{v}_h\in U_h}\Big|k^{-1}\big[(\boldsymbol{u}^n - \boldsymbol{u}_h^n,\boldsymbol{v}_h) + (\boldsymbol{u}_h^n - \Pi_h^*\boldsymbol{u}_h^n,\boldsymbol{v}_h - \Pi_h^*\boldsymbol{v}_h) - (\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1},\boldsymbol{v}_h) \\
&\quad -(\boldsymbol{u}_h^{n-1} - \Pi_h^*\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h - \Pi_h^*\boldsymbol{v}_h)\big] + a(\bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n,\boldsymbol{v}_h) + a_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n,\boldsymbol{v}_h) \\
&\quad +a_{1h}(\bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n,\bar{\boldsymbol{u}}_h^n,\Pi_h^*\boldsymbol{v}_h) - (\boldsymbol{f}^n - \Pi_h^*\boldsymbol{f}^n,\boldsymbol{v}_h - \Pi_h^*\boldsymbol{v}_h)\Big|/\|\boldsymbol{v}_h\|_1 \\
&\quad +C(k\|Q_hp^n - p^n\|_0 + \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0) \\
&\leqslant Ck\big[k^{-1}\left(\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0 + \|\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}\|_0\right) + |\boldsymbol{u}^n - \boldsymbol{u}_h^n|_1 + |\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}|_1 \\
&\quad +h^2\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_1 + Ch^2\|\boldsymbol{f}^n\|_1\big] + C(k\|Q_hp^n - p^n\|_0 + \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0) \\
&\leqslant Ch^2.
\end{aligned} \tag{84}$$

Applying triangle inequality and Lemma 8 to (84) yields that

$$k\|p^n - p_h^n\|_0 \leqslant Ch^2. \tag{85}$$

Combining (83) and (85) with Theorem 1 yields (73).

5 Some numerical experiments

In this section, some numerical experiments with two squared cavities at the bottom and top of the channel are presented illustrating that the fully discrete SCNFVE formulation with the second-order time accuracy is more effective than the SFVE formulation with the first-order time accuracy for the non-stationary Navier-Stokes equations. Moreover, it is shown that the fully discrete SCNFVE method is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations.

Let computational domain $\bar{\Omega}$ consist of the width of the channel to 6 and its length to 20 and two squared cavities at the bottom and top of the channel all are the same width as length to 2 (see Fig. 2). Take $Re = 1000$, $f = g = 0$. Except inflow of left boundary with a velocity of $(u,v) = (0.1(y-2)(8-y),0)$ $(2 \leqslant y \leqslant 8)$ and outflow of right boundary with velocity of (u,v) satisfying $v = 0$ and $\partial u/\partial x = 0$, all initial and boundary value conditions are taken as $\mathbf{0}$.

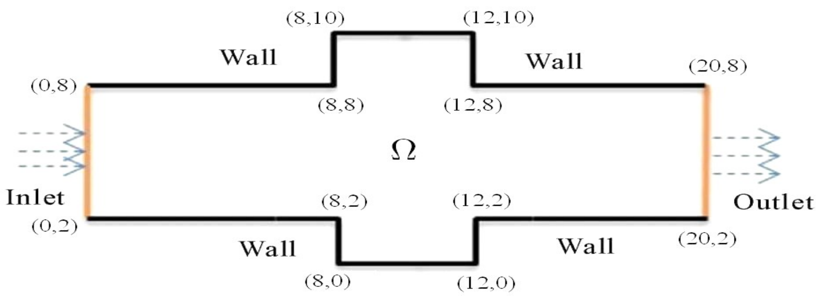

Figure 2: Physics model.

We divide the $\bar{\Omega}$ into small squares with side length $\Delta x = \Delta y = 10^{-2}$, and then link diagonal of the square to divide each square into two triangles in the same direction, which composes triangularizations $\Im_h$ $(h = \sqrt{2} \times 10^{-2})$. The dual decomposition $\Im_h^*$ is taken as barycenter dual decomposition, i.e., the barycenter of the right triangle $K \in \Im_h$ is taken as the node of the dual decomposition. In order to satisfy the condition $k = O(h)$ in Theorem 9, we also take a time step increment as $k = 10^{-2}$. The parameter-free $\varepsilon = 0.01$.

We find the SCNFVE solutions $(\boldsymbol{u}_h^n, p_h^n)$ by means of SCNFVE formulation with the second-order time accuracy when $n = 5 \times 10^5$ (i.e., $t = 5000$) and $n = 6 \times 10^5$ (i.e., $t = 6000$), which are depicted graphically at the the top charts in Figures 3 and 4 and at Figure 5, respectively.

By using the same $U_h \times M_h$ as Problem IV, the fully discrete SFVE formulation

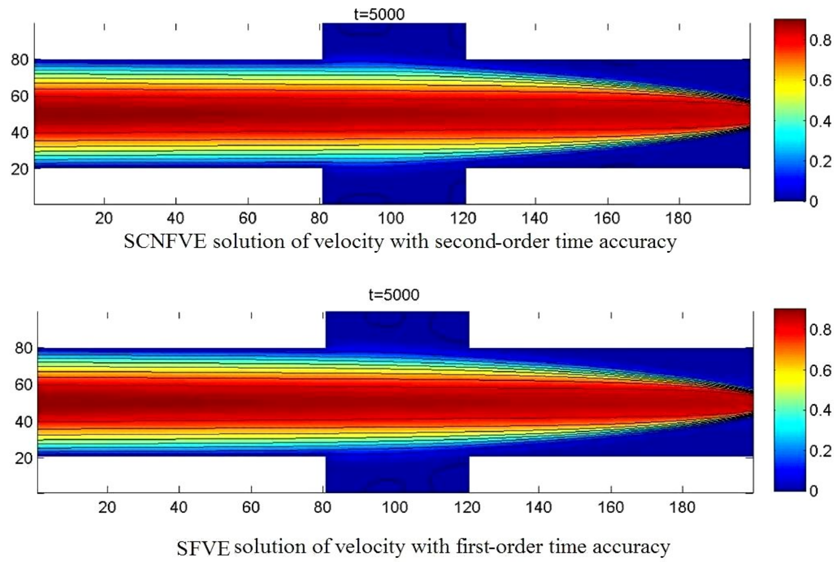

Figure 3: When $Re = 1000$, the top and bottom charts are the SCNFVE solution with the second-order time accuracy and the SFVE solution with the first-order time accuracy of the velocity at the time level $t = 5000$, respectively.

with the first-order time accuracy is denoted as follows.

Problem V. Find $(\boldsymbol{u}_h^n, p_h^n) \in U_h \times M_h$ such that

$$\begin{cases} (\boldsymbol{u}_h^n, \boldsymbol{v}_h) + ka(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + ka_1(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{v}_h) - kb(p_h^n, \boldsymbol{v}_h) \\ = (\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h), \ \forall \boldsymbol{v}_h \in U_h, \ \ n = 1, 2, ..., N, \\ b(\boldsymbol{u}_h^n, q_h) + D_h(p_h^n, q_h) = 0, \ \ \forall q_h \in M_h, \ \ n = 1, 2, ..., N, \\ \boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}^0(x, y), \ \ (x, y) \in \Omega. \end{cases}$$

If $k = O(h)$, it is obtained by means of the same approach as Theorem 9 that the error estimates for Problem V as follows

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|_0 + k^{1/2}\|p(t_n) - p_h^n\|_0 \leqslant C(k + h), \ \ n = 1, 2, ..., N,$$

which is one-order lower than these of Problem IV. Thus, in order to get the same accuracy as SCNFVE formulation Problem IV with the second-order time accuracy, the spatial mesh size h and temporal mesh size k for Problem V have to be taken as $O(h) = k = 10^{-4}$. When we find the solution at $t = 5000$ by means of SFVE formulation with first-order time accuracy, depicted graphically at the the

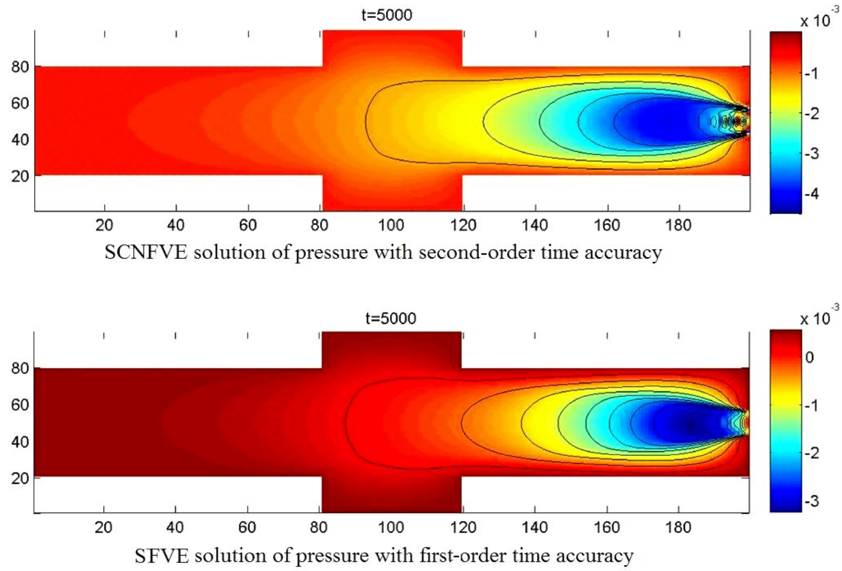

Figure 4: When $Re = 1000$, the top and bottom charts are the SCNFVE solution with the second-order time accuracy and the SFVE solution with the first-order time accuracy of the pressure at the time level $t = 5000$, respectively.

bottom charts in Figures 3 and 4, it is necessary to compute 5×10^7 steps, which are as 100 times as those of SCNFVE formulation with the second-order time accuracy. Especially, the SFVE formulation with first-order time accuracy includes $3\times136\times10^{10}$ degrees of freedom (unknown quantities) on each time-level, while the SCNFVE formulation with the second-order time accuracy does only contain $3\times136\times10^6$ degrees of freedom on each time-level, namely, the degrees of freedom of the SFVE formulation with first-order time accuracy on each time-level are as 10000 times as those of the SCNFVE formulation with the second-order time accuracy.

When we find the numerical solutions $(\boldsymbol{u}_h^n, p_h^n)$ by means of SFVE formulation with first-order time accuracy, it is nonconvergent at $t > 5500$ due to truncation error accumulation in computational process. However, the SCNFVE formulation with the second-order time accuracy at $t = 6000$ is still convergent (see Figure 5), it is shown that the fully discrete SCNFVE formulation with the second-order time accuracy is more effective than the SFVE formulation with the first-order time accuracy for the non-stationary Navier-Stokes equations. Moreover, it is shown that

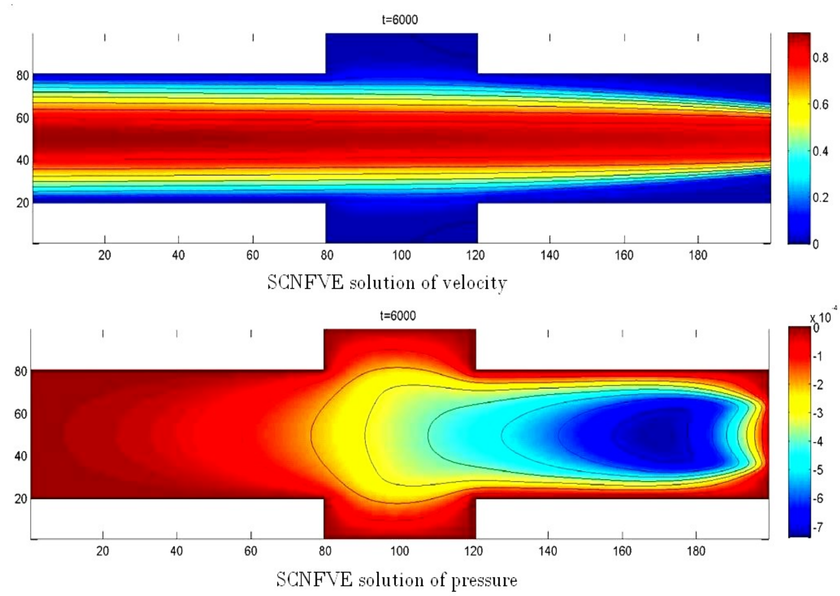

Figure 5: When $Re = 1000$, the top and bottom charts are SCNFVE solutions of the velocity and the pressure with second-order time accuracy at the time level $t = 6000$, respectively.

the fully discrete SCNFVE method with the second-order time accuracy is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations.

6 Conclusions and discussions

In this article, we have first established the time semi-discrete CN formulation with the second-order time accuracy for the non-stationary Navier-Stokes equations. Then, we have directly established the fully discrete SCNFVE formulation with the second-order time accuracy from the time semi-discrete CN formulation. Next, we have provided the error estimates between the fully discrete SCNFVE solutions with the second-order time accuracy and the exact solution. Finally, we have presented some numerical experiments illustrating that the fully discrete SCNFVE Formulation with the second-order time accuracy is more effective than the SFVE formulation with the first-order time accuracy for the non-stationary Navier-Stokes equations, thus validating that the fully discrete SCNFVE method with the second-order time accuracy is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations.

Especially, the fully discrete SCNFVE formulation with the second-order time accuracy is established from the time semi-discrete CN formulation with the second-order time accuracy directly and avoids the semi-discrete SCNFVE formulation with respect to space variables. It is unnecessary to discuss the semi-discrete SCNFVE formulation with respect to spatial variables. Consequently, it is a new type of study attempt for the non-stationary Navier-Stokes equations and the improvement and innovation for the existing methods.

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