 CMES, vol.101, no.2, pp.97-111, 2014

Legendre Polynomials Method for Solving a Class of Variable Order Fractional Differential Equation

Lifeng Wang[1], **Yunpeng Ma**[1,2] **and Yongqiang Yang**[1]

Abstract: In this paper, a numerical method based on the Legendre polynomials is presented for a class of variable order fractional differential equation. We adopt the Coimbra variable order fractional operator, which can be viewed as a Caputo-type definition. Three different kinds of operational matrixes with Legendre polynomials are derived. A truncated the Legendre polynomials series together with the products of several dependent matrixes are utilized to reduce the variable order fractional differential equation to a system of algebraic equations. The solution of this system gives the approximation solution for the truncated limited n. An error analysis technique is also given. Some examples are included to demonstrate the validity and applicability of the approach.

Keywords: Legendre polynomials, variable order fractional, error analysis, numerical solution.

1 Introduction

A lot of scientific and engineering problems involving fractional phenomenon is already very large and still growing. One of the main advantages of the fractional phenomenon is that the fractional derivatives and fractional integrals provide an excellent approach for the different kinds of physical fields, such as dispersive transports in amorphous semiconductors, tracer transfer in underground water, seepage in soil or rocks, etc [Anh, Angulo and Ruiz-Medina(2005); Chen(2006); Meerschaert and Tadjeran(2004); Sun, Chen, Sheng and Chen(2010)]. Many of the numerical methods using various fractional derivative operators and integral operators for solving fractional differential equations have been proposed. Podlubny [Podlubny(1999)] used the Laplace transform method to solve the fractional partial differential equations with constant coefficients. Zaid Odibat and Shaher Momani [Odibat and Momani(2008)] applied generalized differential transform method to

[1] School of Aeronautic Science and Technology, Beihang University, Beijing, China.
[2] Corresponding author.

solve the numerical solution of linear partial differential equations of fractional order. Zhang [Zhang(2009)] discussed a practical implicit method to solve a class of initial boundary value space-time fractional convection-diffusion equations with variable coefficients.

In recent years, more and more researchers are finding that a variety of dynamical problems exhibit fractional order behavior. This indicates that variable order calculus is an effective mathematical framework to describe the complex dynamical problems [Anh and Leonenko(2001); Blaszczyk and Ciesielski(2011); Chen, Liu, Anh and Turner(2010)]. In order to deal with the diffusion processes in which the diffusion behaviors based on time evolution, space variation or system parameters, the variable order diffusion models were proposed [Chen, Sun and Zhang (2010); Schulz and Schulz(2006)]. The concept of variable order operator was first introduced by Samko [Samko and Ross(1993)] in 1993 and has received much attention in the fields of viscoelastic deformation, viscoelasticity, viscous fluid, etc. The variable order operator definitions recently proposed [Samko (1995)] in the literature include Caputo definition, Marchaud definition, Coimbra definition and Riemann Liouvile definition. Based on different definition, many numerical methods have been proposed correspondingly. Lin et al. [Lin, Liu and Anh (2009)] applied an explicit finite difference method to investigate stability and convergence of approximation for the variable order nonlinear fractional diffusion equation. Zhuang et al. [Zhuang, Liu, Anh and Turner(2009)] proposed explicit and implicit Euler method for the variable order fractional advection-diffusion equation. Chen et al. [Chen, Liu, Anh and Turner(2010)] used two numerical methods to solve the variable order anomalous sub-diffusion equation.

The main contribution of this paper is that we explored other method to solve the variable order fractional differential equation. As we all know, there are a lot of methods to obtain the numerical solution of the constant order fractional differential equation[Yi and Chen (2012); Wei, Chen, Li and Yi(2012); Wang, Meng, Ma, and Wu (2013)]. However, the main method to solve the variable order fractional differential equation is finite difference method. We consider the following variable order fractional differential equation by using Legendre polynomials.

$$D^{\alpha(t)}\left(u\left(t\right)g\left(t\right)\right)+D^{\beta(t)}\left(u\left(t\right)\right)+u'\left(t\right)=f\left(t\right) \tag{1}$$

with initial condition

$$u(0)=a \tag{2}$$

where $0<\alpha(t)<1$ and $D^{\alpha(t)}$ denotes the variable order fractional derivative defined by Coimabra [Coimbra (2003)]. $f(t), g(t)$ is the known function, $u(t)$ is the unknown function.

2 Definitions and properties of variable order operator

Here we just recall the most typical definitions which are easy to use in physics. The Caputo type variable order derivative definition presented by Coimbra is stated as following [Sun, Chen, Sheng and Chen(2010)]:

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0^+}^{t}\frac{u'(\tau)}{(t-\tau)^{\alpha(t)}}d\tau + \frac{(u(0^+)-u(0^-))}{\Gamma(1-\alpha(t))t^{\alpha(t)}}, \quad 0<\alpha(t)<1 \quad (3)$$

We suppose the property of function $u(t)$ at $t=0$ is good enough , then we can write the following Caputo type definition

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0}^{t}\frac{u'(\tau)}{(t-\tau)^{\alpha(t)}}d\tau, \quad 0<\alpha(t)<1 \quad (4)$$

The definition of variable order integration proposed by Samko as follows

$$I^{\alpha(t)}u(t) = \frac{1}{\Gamma(\alpha(t))}\int_{0}^{t}(t-\tau)^{\alpha(t)-1}u(\tau)d\tau, \quad Re(\alpha(t))>0 \quad (5)$$

Then we give following properties for the above definitions which will be used in this paper.

Property 1: $D^{\alpha(t)}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(t)+1)}t^{\beta-\alpha(t)}$.

Property 2: $I^{\alpha(t)}\left(D^{\alpha(t)}u(t)\right) = u(t)-u(0)$.

3 Legendre polynomials and their some properties

The Legendre basis polynomials of degree n in $[0,1]$ (see [Saadatmandi (2010)]) are defined by

$$P_{i+1}(t) = \frac{(2i+1)(2t-1)}{(i+1)}P_i(t) - \frac{i}{i-1}P_{i-1}(t), \quad i=1,2,\ldots \quad (6)$$

where $P_0(t)=1, P_1(t)=2t-1$. The Legendre polynomials of degree i can be also written as

$$P_i(t) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{t^k}{(k!)^2} \quad (7)$$

Let

$$\mathbf{\Phi}(t) = \left[P_0(t), P_1(t), \cdots, P_n(t)\right]^T \quad (8)$$

The Legendre polynomials given by Eq.(7) can be expressed in the matrix form

$$\mathbf{\Phi}(t) = \mathbf{A}\mathbf{T}_n(t) \tag{9}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & (-1)^2 2! & 0 & \cdots & 0 \\ (-1)^2 & (-1)^3 \frac{3!}{1!} & (-1)^4 \frac{4!}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^n & (-1)^{n+1} \frac{(n+1)!}{(n-1)!} & (-1)^{n+1} \frac{(n+2)!}{(n-2)!2!} & \cdots & (-1)^{2n} \frac{(2n)!}{n!} \end{bmatrix}, \tag{10}$$

$$\mathbf{T}_n(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \tag{11}$$

A function $u(t) \in L^2(0,1)$ can be expressed in terms of the Legendre basis. In practice, only the first $(n+1)$ term of Legendre polynomials are considered. Hence

$$u(t) \cong \sum_{i=0}^{n} c_i P_i(t) = c^T \mathbf{\Phi}(t) \tag{12}$$

where $\mathbf{c} = [c_0, c_1, \cdots, c_n]^T$, $c_i\ (i = 0, 1, 2, \cdots, n)$ are called Legendre coefficients, and $\mathbf{c} = \mathbf{Q}^{-1}(u, \mathbf{\Phi}(t))$. The dimension of $\mathbf{Q}$ is $(n+1) \times (n+1)$, it is called as the inner product matrix which is given by

$$\begin{aligned} \mathbf{Q} &= \textstyle\int_0^1 \mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\,dx = \int_0^1 (\mathbf{A}\mathbf{T}_n(t))(\mathbf{A}\mathbf{T}_n(t))^T\,dt \\ &= \mathbf{A}\left(\textstyle\int_0^1 \mathbf{T}_n(t)\mathbf{T}_n^T(t)\,dt\right)\mathbf{A}^T = \mathbf{A}\mathbf{H}\mathbf{A}^T \end{aligned} \tag{13}$$

where $\mathbf{H} = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{bmatrix}$.

For the function $u(x,t) \in L^2([0,1] \times [0,1])$, we can also obtain its approximation by using Legendre polynomials

$$u(x,t) \cong \sum_{i=0}^{n} \sum_{j=0}^{n} u_{ij} P_i(x) P_j(t) = \mathbf{\Phi}^T(x)\,\mathbf{U}\mathbf{\Phi}(t) \tag{14}$$

where $\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ u_{10} & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} \end{bmatrix}$.

4 The operational matrix of Legendre polynomials

4.1 The first order operational matrix of the form $u'(t)$

Let $u(t) = \mathbf{c}_1^T \mathbf{\Phi}(t)$, the first order derivative of $\mathbf{\Phi}(t)$ is given by

$$\mathbf{\Phi}'(t) = \mathbf{D}\mathbf{\Phi}(t) \tag{15}$$

where the order of matrix $\mathbf{D}$ is $(n+1) \times (n+1)$. Using Eq.(9), we have

$$\mathbf{\Phi}'(t) = \mathbf{A} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ nt^{n-1} \end{bmatrix} \tag{16}$$

Suppose the form of matrix $\mathbf{V}_{(n+1)\times n}$ and vector $\mathbf{T}_n^*(t)$ is as following

$$\mathbf{V}_{(n+1)\times n} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}, \quad \mathbf{T}_n^*(x) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{bmatrix} \tag{17}$$

Applying $\mathbf{\Phi}(t)$ to express the $\mathbf{T}_n^*(t)$, we get

$$\mathbf{T}_n^*(t) = \mathbf{B}^* \mathbf{\Phi}(t) \tag{18}$$

where $\mathbf{B}^* = \begin{bmatrix} \mathbf{A}_{[1]}^{-1} & \mathbf{A}_{[2]}^{-1} & \cdots & \mathbf{A}_{[n]}^{-1} \end{bmatrix}^T$, and $\mathbf{A}_{[k]}^{-1}$ is the kth row of matrix $\mathbf{A}^{-1}$. Then we obtain

$$\mathbf{\Phi}'(t) = \mathbf{A}\mathbf{V}_{(n+1)\times n}\mathbf{B}^*\mathbf{\Phi}(t) \tag{19}$$

The operational matrix of first order derivative of Legendre polynomials is

$$\mathbf{D} = \mathbf{A}\mathbf{V}_{(n+1)\times n}\mathbf{B}^* \tag{20}$$

So we have

$$u'(t) = \mathbf{c}_1^T \mathbf{A}\mathbf{V}_{(n+1)\times n}\mathbf{B}^*\mathbf{\Phi}(t) \tag{21}$$

4.2 The variable order fractional operational matrix of the form $D^{\beta(t)}$

According to the Eq.(4), and property 1 and 2, we can get

$$
\begin{aligned}
D^{\beta(t)}u(t) &= D^{\beta(t)}\mathbf{c}_1^T\mathbf{\Phi}(t) = \mathbf{c}_1^T D^{\beta(t)}\mathbf{\Phi}(t) \\
&= \mathbf{c}_1^T D^{\beta(t)}\mathbf{A}\mathbf{T}_n(t) \\
&= \mathbf{c}_1^T\mathbf{A}D^{\beta(t)}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} = \mathbf{c}_1^T\mathbf{A}\begin{bmatrix} 0 \\ \frac{\Gamma(2)}{\Gamma(2-\beta(t))}x^{-\beta(t)} \\ \frac{\Gamma(3)}{\Gamma(3-\beta(t))}x^{2-\beta(t)} \\ \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\beta(t))}x^{n-\beta(t)} \end{bmatrix} \\
&= \mathbf{c}_1^T\mathbf{A}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta(t))}t^{-\beta(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta(t))}t^{-\beta(t)} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \\
&= \mathbf{c}_1^T\mathbf{A}\mathbf{N}\mathbf{A}^{-1}\mathbf{\Phi}(t)
\end{aligned} \tag{22}
$$

Let $\mathbf{N} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta(t))}t^{-\beta(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta(t))}t^{-\beta(t)} \end{bmatrix}$, then we acquire the differential operational matrix of $D^{\beta(t)}$:

$$
D^{\beta(t)}u(t) = \mathbf{c}_1^T\mathbf{A}\mathbf{N}\mathbf{A}^{-1}\mathbf{\Phi}(t) \tag{23}
$$

4.3 The variable order fractional operational matrix of the form $D^{\alpha(t)}(u(t)g(t))$

Similarly, let $g(t)=\mathbf{c}_2^T\mathbf{\Phi}(t)$, then we have the following

$$\begin{aligned}
&D^{\alpha(t)}(u(t)g(t))\\
&=D^{\alpha(t)}\left(\mathbf{c}_1^T\mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\mathbf{c}_2\right)\\
&=D^{\alpha(t)}\left(\mathbf{c}_1^T\mathbf{A}\mathbf{T}_n^*(t)\left(\mathbf{A}\mathbf{T}_n^*(t)\right)^T\mathbf{c}_2\right)\\
&=D^{\alpha(t)}\left(\mathbf{c}_1^T\mathbf{A}\mathbf{T}_n^*(t)\mathbf{T}_n^{*T}(t)\mathbf{A}^T\mathbf{c}_2\right)\\
&=\mathbf{c}_1^T\mathbf{A}D^{\alpha(t)}\left(\mathbf{T}_n^*(t)\mathbf{T}_n^{*T}(t)\right)\mathbf{A}^T\mathbf{c}_2\\
&=\mathbf{c}_1^T\mathbf{A}D^{\alpha(t)}\left(\begin{bmatrix}1\\ t\\ \vdots\\ t^n\end{bmatrix}\begin{bmatrix}1 & t & \dots & t^n\end{bmatrix}\right)\mathbf{A}^T\mathbf{c}_2\\
&=\mathbf{c}_1^T\mathbf{A}D^{\alpha(t)}\begin{bmatrix}1 & t & \dots & t^n\\ t & t^2 & \dots & t^{n+1}\\ \vdots & \vdots & \ddots & \vdots\\ t^n & t^{2n} & \dots & t^{2n}\end{bmatrix}\mathbf{A}^T\mathbf{c}_2\\
&=\mathbf{c}_1^T\mathbf{A}\begin{bmatrix}0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)}\\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{2-\alpha(t)} & \dots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} & \dots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))}t^{2n-\alpha(t)}\end{bmatrix}\mathbf{A}^T\mathbf{c}_2\\
&=\mathbf{c}_1^T\mathbf{A}\mathbf{M}\mathbf{A}^T\mathbf{c}_2
\end{aligned} \tag{24}$$

Let

$$\mathbf{M}=\begin{bmatrix}0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)}\\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{2-\alpha(t)} & \dots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} & \dots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))}t^{2n-\alpha(t)}\end{bmatrix} \tag{25}$$

Hence, we obtain the differential operational matrix of $D^{\alpha(t)}(u(t)g(t))$:

$$D^{\alpha(t)}(u(t)g(t))=\mathbf{c}_1^T\mathbf{A}\mathbf{M}\mathbf{A}^T\mathbf{c}_2 \tag{26}$$

5 Numerical solution of Eq.(1)-(2)

Substituting Eq.(21), Eq.(23) and Eq.(26) into Eq.(1), we obtain

$$\mathbf{c}_1^T\mathbf{AMA}^T\mathbf{c}_2+\mathbf{c}_1^T\mathbf{ANA}^{-1}\mathbf{\Phi}(t)+\mathbf{c}_1^T\mathbf{AV}_{(n+1)\times n}\mathbf{B}^*\mathbf{\Phi}(t)=f(t) \tag{27}$$

In this paper, we use Legendre polynomials collocation method to determine the coefficients $\mathbf{c}_1$. By solving this system we can obtain the approximate solution of Eq.(1) as following

$$u(t)=\mathbf{c}_1^T\mathbf{\Phi}(t) \tag{28}$$

6 Error analysis

Suppose function $u:[x_0,1]\to R$ is $m+1$ times continuously differentiable $u\in C^{m+1}[0,1]$, and $\mathbf{Y}=Span\{P_0,P_1,P_2\cdots,P_n\}$. If $c^T\mathbf{\Phi}(x)$ is the best approximation of u out of $\mathbf{Y}$, then the mean error bound is presented as follows:

$$\left\|u-\mathbf{c}^T\mathbf{\Phi}(x)\right\|_2\le\frac{\sqrt{2}MS^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}} \tag{29}$$

where $M=\max\limits_{x\in[0,1]}\left|u^{(m+1)}(x)\right|,\ S=\max\{1-x_0,x_0\}$.

Proof. Considering the Taylor polynomials, we have

$$u_1(x)=u(x_0)+u'(x_0)(x-x_0)+u''(x_0)\frac{(x-x_0)^2}{2!}\cdots+u^{(m)}(x_0)\frac{(x-x_0)^m}{m!} \tag{30}$$

Using the mean value theorem, we have

$$|u(x)-u_1(x)|=\left|u^{(m+1)}(\varepsilon)\right|\frac{(x-x_0)^{m+1}}{(m+1)!}\quad\exists\varepsilon\in(0,1) \tag{31}$$

where $\varepsilon\in(0,1)$. Since $c^T\Phi(x)$ is the best approximation of$u$, then we get

$$\begin{aligned}\left\|u-c^T\Phi(x)\right\|_2^2&\le\|u-u_1\|_2^2=\int_0^1(u(x)-u_1(x))^2dx\\&=\int_0^1\left(\left|u^{(m+1)}(\varepsilon)\right|\frac{(x-x_0)^{m+1}}{(m+1)!}\right)^2dx\\&\le\frac{M^2}{[(m+1)!]^2}\int_0^1(x-x_0)^{2m+2}dx\\&\le\frac{2M^2S^{2m+3}}{[(m+1)!]^2(2m+3)}\end{aligned}$$

Therefore $\left\|u-c^T\Phi(x)\right\|_2\le\frac{\sqrt{2}MS^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}}$.

7 Numerical examples

To show the efficiency and the accuracy of the proposed method, we consider the following three examples.

Example 7.1. Consider the following variable order fractional differential equation:

$$\begin{aligned} &D^{\frac{t}{4}}\left(u(t)\,t^{\frac{7}{8}}\right)+D^{\cos t}u(t)+u'(t)=f(t) \\ &u(0)=0 \qquad t\in[0,1] \end{aligned} \tag{32}$$

where $f(t)=2t+\frac{23t^{\frac{23}{8}-\frac{t}{4}}\Gamma\left(\frac{23}{8}\right)}{8\Gamma\left(\frac{31}{8}-\frac{t}{4}\right)}+\frac{2t^{2-\cos t}}{(2-3\cos t+\cos^2 t)\Gamma(1-\cos t)}$. The exact solution of this problem is $u(t)=t^2$. Take $n=3$, and $t_i=\frac{k_i}{3}-\frac{1}{6}$, $(k_i=1,2,3)$, then we acquire the vector $c_1=\begin{bmatrix} 0 & 0.000528709 & 0.999136 & 1.00022 \end{bmatrix}^T$ by adopting the above method. **Fig.1** shows the absolute errors of exact solution and numerical solution for $n=3$.

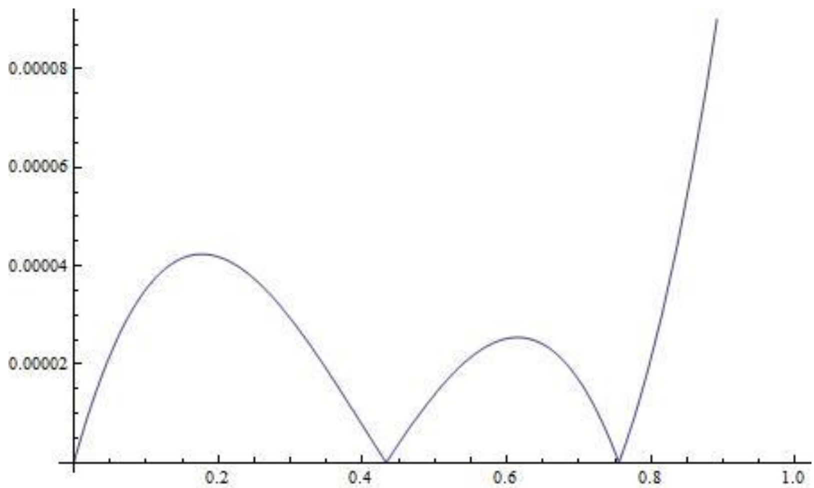

Figure 1: Absolute errors for exact and numerical solution of $n=3$.

Take $n=4$, and $t_i=\frac{k_i}{4}-\frac{1}{8}$, $(k_i=1,2,3,4)$, then we acquire the vector $c_1=\begin{bmatrix} 0 & 0.000528709 & 0.999136 & 1.00022 \end{bmatrix}^T$. **Fig.2** shows the absolute errors of exact solution and numerical solution for $n=4$.

Example 7.2. Consider this equation:

$$\begin{aligned} &D^{\frac{t}{4}}\left(u(t)(t+1)\right)+D^{\frac{t}{3}}u(t)+u'(t)=f(t) \\ &u(0)=1 \qquad t\in[0,1] \end{aligned} \tag{33}$$

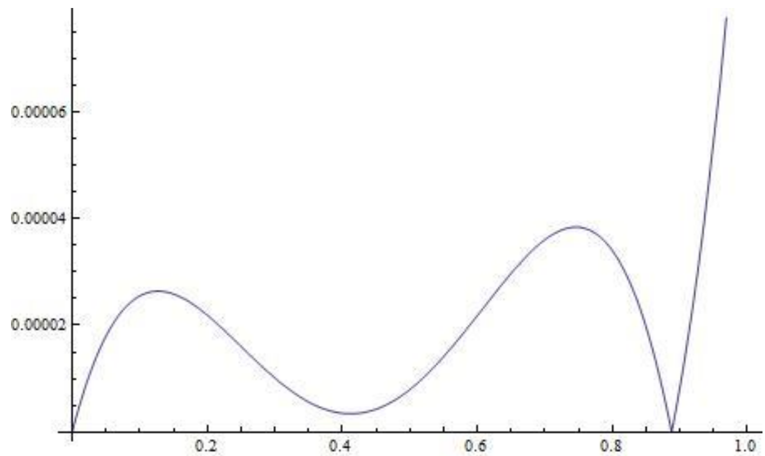

Figure 2: Absolute errors for exact and numerical solution of$n = 4$.

such that $f(t) = 2t + \frac{18t^{2-\frac{t}{3}}}{(18-9t+t^2)\Gamma\left(1-\frac{t}{3}\right)} - \frac{4t^{1-\frac{t}{4}}\left(96+76t+89t^2\right)}{(-12+t)(-8+t)(-4+t)\Gamma\left(1-\frac{t}{4}\right)}$.
We applied the Legendre polynomials approach to solve Eq.(33) for various values of n. Take $n = 3$ and $t_i = \frac{k_i}{3} - \frac{1}{6}$, $(k_i = 1,2,3)$, then we obtain $c_1 = \begin{bmatrix} 1 & 3 & 4 & 2 \end{bmatrix}^T$. The absolute errors for $n = 3$is shown in **Fig.3**.

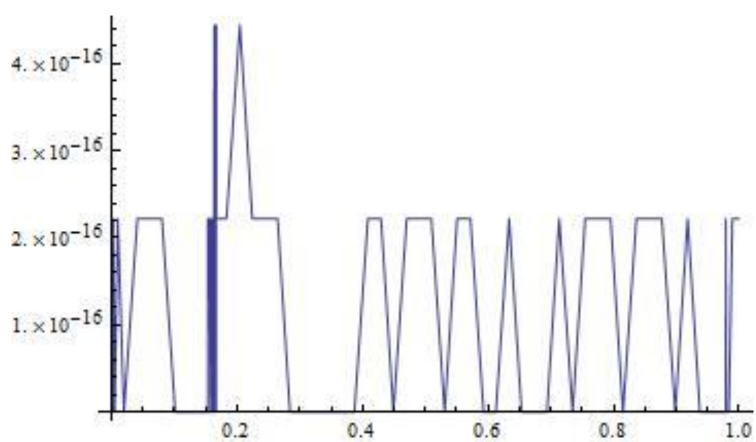

Figure 3: Absolute errors for exact and numerical solution of $n = 3$.

Take $n = 4$, and $t_i = \frac{k_i}{4} - \frac{1}{8}$, $(k_i = 1,2,3,4)$, then we acquire the vector $c_1 = \begin{bmatrix} 1 & 4.00019 & 6.99979 & 6.00026 & 2 \end{bmatrix}^T$. The absolute errors for $n = 4$is shown in **Fig.4**.

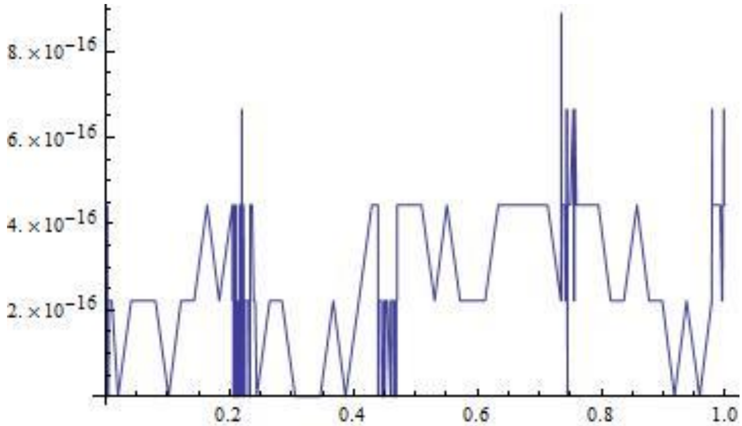

Figure 4: Absolute errors for exact and numerical solution of $n = 4$.

Example 7.3. Consider the below variable order fractional differential equation:

$$\begin{aligned} &D^{\frac{t}{4}}\left(u^2(t)\right) + D^{\frac{t}{3}}u(t) + u'(t) = f(t) \\ &u(0) = 0 \qquad t \in [0,1] \end{aligned} \tag{34}$$

where $f(t) = 2t + \frac{18t^{2-\frac{t}{3}}}{(18-9t+t^2)\Gamma\left(1-\frac{t}{3}\right)} + \frac{6144t^{4-\frac{t}{4}}}{(-16+t)(-12+t)(-8+t)(-4+t)\Gamma\left(1-\frac{t}{4}\right)}$.

The exact solution is $u(t) = t^2$.

Take $n = 2$ and $t_i = \frac{k_i}{2} - \frac{1}{4}$, $(k_i = 1,2)$, so we have $c_1 = \begin{bmatrix} 0 & -1.25\times 10^{-16} & 1 \end{bmatrix}^T$. **Fig.5** shows the absolute errors of exact solution and numerical solution.

Take $n = 3$ and $t_i = \frac{k_i}{3} - \frac{1}{6}$, $(k_i = 1,2,3)$, we can get $c_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$. The absolute errors of exact solution and numerical solution is shown in **Fig.6**.

Take $n = 4$ and $t_i = \frac{k_i}{4} - \frac{1}{8}$, $(k_i = 1,2,3,4)$, we can obtain $c_1 = \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \end{bmatrix}^T$. The absolute errors of exact solution and numerical solution is shown in **Fig.7**.

The calculating results show that combining with the Legendre polynomials operational matrix, the method in this paper can be effectively used in numerical calculus for variable order fractional differential equations. This problem was solved by other methods, we list their absolute errors in **Table 1-3**.

From the **Table1-3,** we can find that compared with existed method which the accuracy is about 10^{-4}, our method can acquire higher degree of accuracy when solving the same equation. They also demonstrate the simplicity, and powerfulness of the proposed method. What's more, the method in this paper is easy to implementation.

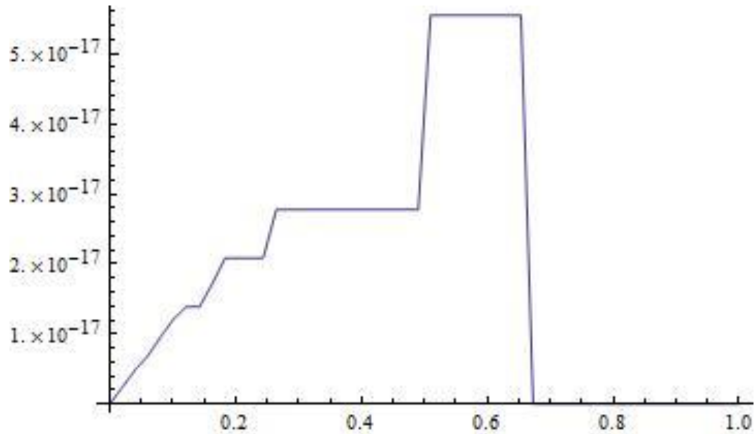

Figure 5: Absolute errors for exact and numerical solution of $n = 2$.

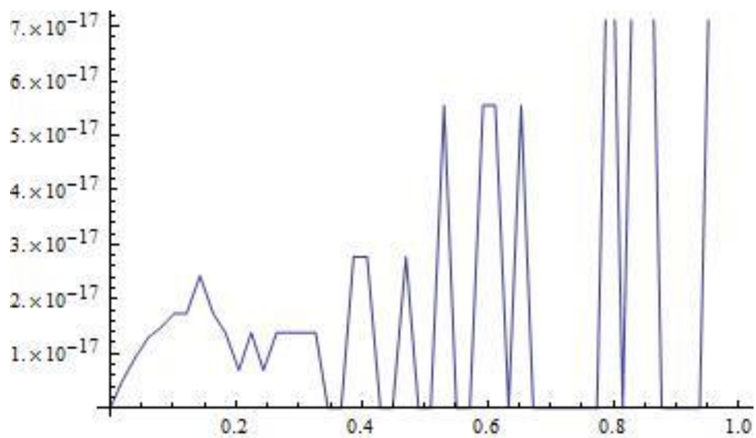

Figure 6: Absolute errors for exact and numerical solution of $n = 3$.

Table 1: The absolute errors analysis in Ref.[Lin, Liu, Anh and Turner (2009)].

x	Numerical solution	Exact solution	Absolute errors
0.80	0.11490000	0.11824342	0.00334342
2.40	0.80430000	0.80339167	0.00090833
4.80	1.83840001	1.83575041	0.00264950
6.40	1.63413334	1.63079520	0.00333810
7.20	1.03410000	1.03101940	0.00308060

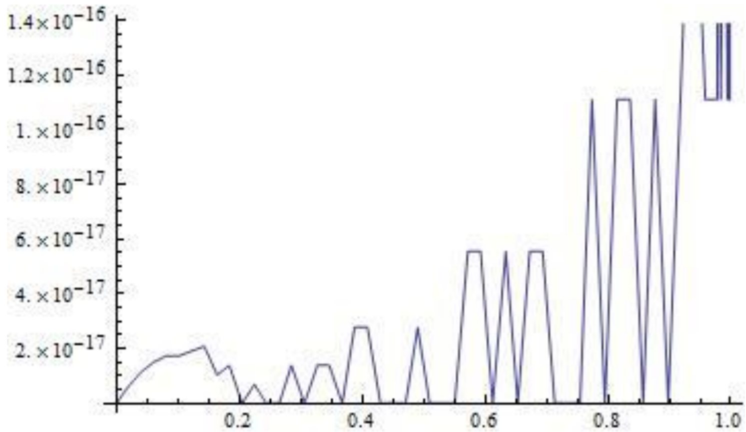

Figure 7: Absolute errors for exact and numerical solution of $n = 4$.

Table 2: The absolute errors analysis in Ref.[Zhuang, Liu, Anh and Turner (2009)].

x	Numerical solution	Exact solution	Absolute errors
0.2000	1.28005972	1.28000000	0.00005972
0.4000	3.84011251	3.84000000	0.00011251
0.5000	5.00012981	5.00000000	0.00012981
0.6000	5.76013595	5.76000000	0.00013595
0.7000	5.88012705	5.88000000	0.00012705

Table 3: The absolute errors analysis in Ref.[Shen (2011)].

x	Numerical solution	Exact solution	Absolute errors
0.1670	0.92605370	0.92592593	0.00012777
0.3333	2.96319736	2.92692696	0.00023440
0.5000	5.00028449	5.00000000	0.00028449
0.7500	5.62520916	5.62500000	0.00020916
0.8330	4.62977787	4.62962963	0.00014824

8 Conclusion

In this paper, a numerical method is presented for numerical solutions of variable order fractional differential equation. Taking full advantage of the definition of Caputo type fractional derivative and the properties of Legendre polynomials, we propose three kinds of differential operational matrix and transform the initial problem into a linear algebraic system equations. By solving the linear system, numerical solutions are obtained. The numerical results show that the approximation is in very good coincidence with the exact solution.

Acknowledgement: This work was supported by the National Natural Science Foundation of China under Grant No. 51307004.

References

Anh, V. V.; Angulo, J. M.; Ruiz-Medina, M. D. (2005): Diffusion on multifractals. *Nonlinear Analysis*, vol. 63, pp. e2053-e2056.

Anh, V. V.; Leonenko, N. N. (2001): Spectral analysis of fractional kinetic equations with random data. *J. Statis. Phys.* vol. 104, pp. 1349-1387.

Blaszczyk, T.; Ciesielski, M. et al. (2011): Numerical solution of fractional oscillator equation. *Appl. Math. Comput.*, vol. 218, pp. 2480-2488.

Chen, W. (2006): A speculative study of 2/3-order fractional Laplacian modeling of turbulence: Some thoughts and conjectures. *Chaos*, vol. 16, pp. 023126.

Chen, C.; Liu, F.; Anh, V.; Turner, I. (2010): Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation. *SIAM J. Sci. Comput.*, vol. 34, no. 4, pp. 1740- 1760.

Chen, W.; Sun, H. G.; Zhang, X. D. et al. (2010): Anomalous diffusion modeling by fractal and fractional derivatives. *Comput. Math. Appl.*, vol. 59, pp. 1754-1758.

Coimbra, C. F. M. (2003): Mechanics with variable-order differential operators. *Ann. Phys.* vol.12, no. 11-12, pp. 692-703.

Lin, R.; Liu, F.; Anh, V.; Turner, I. (2009): Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. *Appl. Math. Comput.*, vol. 212, pp. 435-445.

Meerschaert, M. M.; Tadjeran, C. (2004): Finite difference approximations for fractional advection dispersion flow equations. *Journal of Computational and Applied Mathematics*, vol. 172, pp. 65-77.

Odibat, Z.; Momani, S. (2008): A generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Let-*

ters, vol. 21, pp. 194-199.

Podlubny, I. (1999): *Fractional Differential Equations*, Academic press.

Sun, H. G.; Chen, W.; Sheng, H.; Chen, Y. Q. (2010): On mean square displacement behaviors of anomalous diffusions with variable and random order. *Physics Letter A*, vol. 374, pp. 906-910.

Schulz, B. M.; Schulz, M. (2006): Numerical investigations of anomalous diffusion effects in glasses. *Journal of Non-Crystalline Solids*, vol. 352, pp. 4884-4887.

Samko, S. G.; Ross, B. (1993): Integration and differentiation to a variable fractional order. *Integral Trans. Special Func.*, vol. 1, no. 4, pp. 277-300.

Samko, S. G. (1995): Fractional integration and differentiation of variable order. *Anal. Math.*,vol. 21, pp. 213-236.

Saadatmandi, A. (2010): A new operational matrix for solving fractional-order differential equations. *Computers and Mathematics with Applications*, vol. 59, pp. 1326-1336.

Shen, S. J. (2011): Numerical simulation for the variable order time fractional diffusion equation. *Journal of Putian University*, vol. 18, no. 5, pp. 5-9.

Wang, L. F.; Meng, Z. J.; Ma, Y. P,; Wu, Z. Y. (2013): Numerical solution of fractional partial differential equations using Haar wavelets. *Computer Modeling in Engineering & Sciences*, vol. 91, no. 4, pp. 269-287.

Wei, J. X.; Chen, Y. M.; Li, B. F.; Yi, M. X. (2012): Numerical Solution of Space-Time Fractional Convection-Diffusion Equations with Variable Coefficients Using Haar Wavelets. *Computer Modeling in Engineering & Sciences*, vol. 89, no. 6, pp. 481-495.

Yi, M. X.; Chen, Y. M. (2012): Haar wavelet operational matrix method for solving fractional partial differential equations. *Computer Modeling in Engineering & Sciences*, vol. 88, no. 3, pp. 229-244.

Zhang, Y. (2009): A finite difference method for fractional partial differential equation. *Applied Mathematics and Computation*, vol.215, pp. 524-529.

Zhuang, P.; Liu, F.; Anh, V.; Turner, I. (2009): Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. *SIAM J. Numer. Anal.*, vol. 47, pp. 1760-1781.