

On Solving Linear and Nonlinear Sixth-Order Two Point Boundary Value Problems Via an Elegant Harmonic Numbers Operational Matrix of Derivatives

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Abstract: This paper is concerned with developing two new algorithms for direct solutions of linear and nonlinear sixth-order two point boundary value problems. These algorithms are based on the application of the two spectral methods namely, collocation and Petrov-Galerkin methods. The suggested algorithms are completely new and they depend on introducing a novel operational matrix of derivatives which is expressed in terms of the well-known harmonic numbers. The basic idea for the suggested algorithms rely on reducing the linear or nonlinear sixth-order boundary value problem governed by its boundary conditions to a system of linear or nonlinear algebraic equations which can be efficiently solved by a suitable solver. The algorithms are supported by investigating the convergence and the error analysis of the used expansion. Some illustrative examples are considered aiming to ascertain the wide applicability, and the high efficiency of the suggested algorithms. The obtained numerical results are convincing and the proposed approximate solutions are very close to the analytical ones.

Keywords: Harmonic numbers, Legendre polynomials, Petrov-Galerkin method, collocation method, sixth-order boundary value problems.

1 Introduction

The spectral methods aim to approximate functions (solutions of differential equations) by means of truncated series of orthogonal polynomials. There are three well-known versions of spectral methods, namely, tau, collocation and Galerkin methods, see for example [Canuto, Hussaini, Quarteroni, and Zang (1988); Doha and Abd-Elhameed (2002); Doha, Abd-Elhameed, and Youssri (2013)]. There are several computational methods can be employed for solving boundary value problems, see for example [Atluri (2005); Dong, Alotaibi, Mohiuddine, and Atluri

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(2014); Dai, Schnoor, and Atluri (2012); Elgohary, Dong, Junkins, and Atluri (2014)]. The main advantage of spectral methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In contrast, finite difference and finite-element methods yield only algebraic convergence rates. The choice of the suitable spectral method suggested for solving the given equation depends certainly on the type of the differential equation and the type of the boundary conditions governed by it. Collocation methods [Costabile and Napoli (2012, 2014, 2015); Guo and Yan (2009); Mai-Duy (2006); Kasi Viswanadham and Murali Krishna (2010)], have become increasingly popular for solving various kinds of differential equations. In particular, they are very useful in providing highly accurate solutions to nonlinear differential equations. Petrov-Galerkin method is widely used for solving ordinary and partial differential equations, see for example [Doha, Abd-Elhameed, and Youssri (2012); Abd-Elhameed, Doha, and Youssri (2013); Doha and Abd-Elhameed (2009); Geyikli and Karakoç (2012); Roshan (2012)]. The Petrov-Galerkin methods [Yu and Heinrich (1986)] have generally come to be known as "stabilized" formulations because they prevent the spatial oscillations and sometimes yield nodally exact solutions where the classical Galerkin method would fail badly. The difference between Galerkin and Petrov-Galerkin methods, is that the test and trial functions in Galerkin method are the same, while in Petrov-Galerkin method, they are not (see, [Abd-Elhameed (2009)]).

There is a great number of authors interested in solving high even-order boundary value problems. The reason for such interest is that this kind of BVPs arise in various applications in physics, engineering disciplines and applied mathematics. The interested reader for various applications for even-order boundary value problems can be referred for example to [Wazwaz (2000)]. In the sequence of papers [Doha and Abd-Elhameed (2002, 2009); Doha, Abd-Elhameed, and Bassuony (2013)], the authors handled such equations by means of the Galerkin method. They constructed suitable basis functions which satisfy the boundary conditions of the given differential equation. For this purpose, they used compact combinations of various orthogonal polynomials. The suggested algorithms in these articles are suitable for handling one and two dimensional linear high even-order boundary value problems. In particular, sixth-order BVPs are of interest and they arise in astrophysics. The narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modelled by sixth-order boundary value problems (see, [Chandrasekhar (1961)]). Sixth-order BVPs were handled by numerous numerical techniques, among of these techniques are, spline collocation method in [Lamni, Mraoui, Sbibih, Tijini, and Zidna (2008)], Adomian decomposition method with Green's function in [Al-Hayani (2011)], parametric quintic spline solution in

[Khan and Sultana (2012)], Sinc-Galerkin method in [El-Gamel, Cannon, and Zayed (2004)], homotopy perturbation method in [Noor and Mohyud-Din (2008)], and fourth order finite difference method in [Pandey (2013)].

The employment of operational matrices for solving different kinds of differential equations is considered as a common technique. This is due to the simplicity of this technique and also to the accuracy of the approximate solutions resulted from their uses. We refer here that a large number of authors follow this approach. For example, in [Doha, Abd-Elhameed, and Youssri (2013)], the authors employ the operational matrices of derivatives of Chebyshev polynomials of second kind to solve the singular Lane-Emden type equations. Other studies in [Öztürk and Gülsu (2014); Pandey, Kumar, Bhardwaj, and Dutta (2012)] employ operational matrices of derivatives for solving the same type of equations. Other kinds of differential equations were handled by the same technique (see, for example [Saadatmandi and Dehghan (2010); Maleknejad, Basirat, and Hashemizadeh (2012); Zhu and Fan (2012)]).

In this paper, we aim to give new algorithms for handling both of linear and nonlinear sixth-order boundary value problems based on introducing a new operational matrix of derivatives expressed in terms of harmonic numbers, then applying Petrov-Galerkin method on linear equations and collocation method on nonlinear equations.

For more clarification, the main objectives in the present paper can be summarized in the following threefold:

- Introducing a novel operational matrix of derivatives based on using shifted Legendre polynomials and harmonic numbers.
- Employing the Petrov-Galerkin method together with the introduced operational matrix of derivatives for handling linear sixth-order BVPs.
- Employing the collocation method together with the introduced operational matrix of derivatives for converting the nonlinear sixth-order BVP into a system of nonlinear algebraic equations, hence obtaining new approximate solutions for this type of equations.

The contents of the paper are arranged as follows. In Section 2, an overview on shifted Legendre polynomials and harmonic numbers is given. Section 3 is concerned with introducing in detail a novel operational matrix of derivatives with the aid of some properties of the shifted Legendre polynomials and harmonic numbers. The reduction for linear sixth-order BVPs to systems of linear algebraic equations based on the application of Petrov-Galerkin operational matrix method (PGOMM)

is described in detail in Section 4. Section 5 is devoted to handling nonlinear sixth-order BVPs based on the application of the collocation operational matrix method (COMM). Convergence and error analysis of the suggested approximate expansion are carefully investigated in Section 6. Numerical examples and discussions are given in Section 7 aiming to show the efficiency and the applicability of the suggested algorithms. Conclusions are given in Section 8.

2 An overview on shifted Legendre polynomials and harmonic numbers

2.1 Shifted Legendre polynomials

The shifted Legendre polynomials $L_i^*(x)$ are defined on $[a, b]$ as:

$$L_i^*(x) = L_i\left(\frac{2x - a - b}{b - a}\right), \quad i = 0, 1, \dots,$$

where $L_i(x)$ are the Legendre polynomials. They may be generated by using the recurrence relation

$$(i + 1)L_{i+1}^*(x) = (2i + 1)\left(\frac{2x - b - a}{b - a}\right)L_i^*(x) - iL_{i-1}^*(x), \quad i = 1, 2, \dots, \quad (1)$$

with $L_0^*(x) = 1, L_1^*(x) = \frac{2x - a - b}{b - a}$. These polynomials are orthogonal on $[a, b]$, with respect to the weight function $w(x) = 1$, i.e.,

$$\int_a^b L_i^*(x)L_j^*(x) dx = \begin{cases} \frac{b - a}{2i + 1}, & i = j, \\ 0, & i \neq j. \end{cases} \quad (2)$$

The polynomials $L_i^*(x)$ are eigenfunctions of the following singular Sturm-Liouville equation:

$$-D[(x - a)(x - b)D\phi_i(x)] + i(i + 1)\phi_i(x) = 0,$$

where $D \equiv \frac{d}{dx}$.

The following theorem and lemma are essential in the sequel.

Theorem 1. [Abd-Elhameed, Doha, and Bassuony (2014)] *If the q times repeated integration of $L_i^*(x)$ is denoted by*

$$J_i^{(q)}(x) = \overbrace{\int \int \dots \int}^{q \text{ times}} L_i^*(x) \overbrace{dx dx \dots dx}^{q \text{ times}},$$

then

$$J_i^{(q)}(x) = \frac{(b-a)^q \sum_{m=0}^q \frac{(-1)^m \binom{q}{m} (i+q-2m+\frac{1}{2}) \Gamma(i-m+\frac{1}{2})}{\Gamma(i+q-m+\frac{3}{2})} L_{i+q-2m}^*(x) + \pi_{q-1}(x), \tag{3}$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Lemma 1. For all $b > a$ and $j \geq 0$, the following relation holds

$$(b-x)^3(x-a)^3 L_j^*(x) = \sum_{m=0}^6 \xi_{m,j} L_{j-6+2m}^*(x), \tag{4}$$

where

$$\begin{aligned} \xi_{0,j} &= \alpha_{j-4} \alpha_{j-2} \alpha_j, & \xi_{1,j} &= \alpha_{j-2} \alpha_j (\beta_{j-4} + \beta_{j-2} + \beta_j), \\ \xi_{2,j} &= \alpha_j (\alpha_{j-2} \theta_{j-4} + \alpha_j \theta_{j-2} + \alpha_{j+2} \theta_j + \beta_{j-2}^2 + \beta_j \beta_{j-2} + \beta_j^2), \\ \xi_{3,j} &= 2\alpha_j \beta_j \theta_{j-2} + 2\alpha_{j+2} \beta_j \theta_j + \alpha_j \beta_{j-2} \theta_{j-2} + \alpha_{j+2} \beta_{j+2} \theta_j + \beta_j^3, \\ \xi_{4,j} &= \theta_j (\alpha_j \theta_{j-2} + \alpha_{j+2} \theta_j + \alpha_{j+4} \theta_{j+2} + \beta_j^2 + \beta_{j+2} \beta_j + \beta_{j+2}^2), \\ \xi_{5,j} &= (\beta_j + \beta_{j+2} + \beta_{j+4}) \theta_j \theta_{j+2}, & \xi_{6,j} &= \theta_j \theta_{j+2} \theta_{j+4}, \end{aligned} \tag{5}$$

and

$$\alpha_j = \frac{-(b-a)^2 j(j-1)}{4(2j-1)(2j+1)}, \beta_j = \frac{(b-a)^2 (j^2+j-1)}{2(2j-1)(2j+3)}, \theta_j = \frac{-(b-a)^2 (j+1)(j+2)}{4(2j+1)(2j+3)}. \tag{6}$$

Proof. Since we have (see, [Doha and Abd-Elhameed (2002)])

$$\begin{aligned} (1-x^2)L_j(x) &= \frac{-(j-1)j}{(2j-1)(2j+1)} L_{j-2}(x) + \frac{2(j^2+j-1)}{(2j-1)(2j+3)} L_j(x) \\ &\quad - \frac{(j+1)(j+2)}{(2j+1)(2j+3)} L_{j+2}(x), \end{aligned} \tag{7}$$

then the following relation can be immediately obtained (by replacing x by $\frac{2x-a-b}{b-a}$)

$$(b-x)(x-a)L_j^*(x) = \alpha_j L_{j-2}^*(x) + \beta_j L_j^*(x) + \theta_j L_{j+2}^*(x), \tag{8}$$

where α_j, β_j and θ_j are given in (6).

Now, relation (8)-after performing some manipulations-lead to relation (4). □

2.2 Harmonic numbers

It is well-known that the n th harmonic number is defined as (see, [Rainville (1960)]):

$$H_n = \sum_{i=1}^n \frac{1}{i}. \tag{9}$$

The recurrence relation satisfied by H_n is

$$H_n - H_{n-1} = \frac{1}{n}, \quad n = 1, 2, \dots,$$

and they have the integral representation

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx.$$

The following lemma is of fundamental importance in the sequel.

Lemma 2. *The harmonic numbers satisfy the following three-term recurrence relation:*

$$(2i - 1)H_{i-1} - (i - 1)H_{i-2} = iH_i, \quad i \geq 2. \tag{10}$$

Proof. The recurrence relation (10) can be easily proved with the aid of relation (9). □

3 A shifted Legendre operational matrix of derivatives

In this section, we select the following set of basis functions

$$\phi_i(x) = (x - a)^3 (b - x)^3 L_i^*(x), \quad i = 0, 1, 2, \dots \tag{11}$$

It is to be noted that the polynomials $\{\phi_i(x) : i = 0, 1, 2, \dots\}$ are linearly independent and orthogonal with respect to the weight function $w(x) = \frac{1}{(x - a)^6 (b - x)^6}$, i.e.

$$\int_a^b \frac{\phi_i(x) \phi_j(x) dx}{(x - a)^6 (b - x)^6} = \begin{cases} 0, & i \neq j, \\ \frac{b - a}{2i + 1}, & i = j. \end{cases}$$

Let us denote $H_w^r(I)(r = 0, 1, 2, \dots)$, as the weighted Sobolev spaces, whose inner products and norms are denoted by $(\cdot, \cdot)_{r,w}$, and $\|\cdot\|_{r,w}$, respectively (see, [Canuto,

Hussaini, Quarteroni, and Zang (1988)]. To account for homogeneous boundary conditions, we define

$$H_{0,w}^3(I) = \{v \in H_w^3(I) : v^{(j)}(a) = v^{(j)}(b) = 0, \quad 0 \leq j \leq 2\},$$

where $v^{(j)}(x) = \frac{d^j v}{dx^j}$, and $I = (a, b)$. Now, define the following subspace of $H_{0,w}^3(I)$

$$V_N = \text{span}\{\phi_0(x), \phi_1(x), \dots, \phi_N(x)\}.$$

Any function $u(x) \in H_{0,w}^3(I)$ can be expanded in terms of the polynomials $\phi_i(x)$ as

$$u(x) = \sum_{i=0}^{\infty} c_i \phi_i(x), \tag{12}$$

where

$$c_i = \frac{2i+1}{b-a} \int_a^b \frac{u(x) \phi_i(x)}{(x-a)^6(b-x)^6} dx. \tag{13}$$

The function $u(x)$ in Eq. (12) can be approximated by the first $(N + 1)$ terms, that is

$$u(x) \simeq u_N(x) = \sum_{i=0}^N c_i \phi_i(x) = \mathbf{C}^T \mathbf{\Phi}(x), \tag{14}$$

where

$$\mathbf{C}^T = [c_0, c_1, \dots, c_N], \quad \mathbf{\Phi}(x) = [\phi_0(x), \phi_1(x), \dots, \phi_N(x)]^T. \tag{15}$$

Now, we state and prove the basic theorem, from which a new operational matrix of derivatives can be introduced.

Theorem 2. *If the polynomials $\phi_i(x)$ are selected as in (11), then the following relation holds for all $i \geq 1$,*

$$D \phi_i(x) = \frac{2}{b-a} \sum_{\substack{j=0 \\ (i+j) \text{ odd}}}^{i-1} (2j+1) (1 + 6H_i - 6H_j) \phi_j(x) + \eta_i(x), \tag{16}$$

where $\eta_i(x)$ is given by

$$\eta_i(x) = \begin{cases} 3(x-a)^2(b-x)^2(a+b-2x), & i \text{ even,} \\ 3(a-b)(x-a)^2(b-x)^2, & i \text{ odd.} \end{cases} \tag{17}$$

Proof. First consider the case $[a, b] \equiv [-1, 1]$. We will show that the following relation holds for all $i \geq 1$

$$D\psi_i(x) = \sum_{\substack{j=0 \\ (i+j) \text{ odd}}}^{i-1} (2j+1)(1+6H_i-6H_j)\psi_j(x) + \gamma_i(x), \tag{18}$$

where

$$\psi_i(x) = (1-x^2)^3 L_i(x),$$

and $\gamma_i(x)$ is given by

$$\gamma_i(x) = -6(1-x^2)^2 \begin{cases} x, & i \text{ even,} \\ 1, & i \text{ odd.} \end{cases}$$

We proceed by induction on i . For $i = 1$, it is clear that each of the two sides of (18) is equal to $(1-x^2)^2(1-7x^2)$. Now, assume the validity of relation (18) for $(i-2)$ and $(i-1)$, and we have to show that it is valid for i . It is clear that the polynomials $\psi_i(x)$ satisfy the same recurrence relation of Legendre polynomials, that is

$$\psi_i(x) = \frac{2i-1}{i} x \psi_{i-1}(x) - \frac{(i-1)}{i} \psi_{i-2}(x), \quad i \geq 2. \tag{19}$$

Differentiation of the last recurrence relation immediately gives

$$D\psi_i(x) = \frac{2i-1}{i} x D\psi_{i-1}(x) + \frac{2i-1}{i} \psi_{i-1}(x) - \frac{(i-1)}{i} D\psi_{i-2}(x). \tag{20}$$

The application of the induction hypothesis on $D\psi_{i-1}(x)$ and $D\psi_{i-2}(x)$ in (20), yields

$$\begin{aligned} D\psi_i(x) &= \frac{(2i-1)x}{i} \sum_{\substack{j=0 \\ (i+j) \text{ even}}}^{i-2} (2j+1)(1-6H_j+6H_{i-1})\psi_j(x) \\ &\quad - \frac{(i-1)}{i} \sum_{\substack{j=0 \\ (i+j) \text{ odd}}}^{i-3} (2j+1)(1-6H_j+6H_{i-2})\psi_j(x) + \frac{2i-1}{i} \psi_{i-1}(x) \\ &\quad + \left(\frac{2i-1}{i}\right) x \gamma_{i-1}(x) - \left(\frac{i-1}{i}\right) \gamma_{i-2}(x). \end{aligned} \tag{21}$$

If we substitute by the recurrence relation (19) written in the form

$$x\psi_j(x) = \frac{j+1}{2j+1} \psi_{j+1}(x) + \frac{j}{2j+1} \psi_{j-1}(x),$$

into relation (21), then after performing some rather lengthy manipulations, we get

$$\begin{aligned}
 D\psi_i(x) = & \sum_{\substack{j=1 \\ (i+j) \text{ odd}}}^{i-3} p_{ij} \psi_j(x) + \frac{1}{i} [6(i-1)(2i-1)(H_{i-1} - H_{i-2}) + i(2i-1)] \psi_{i-1}(x) \\
 & + \frac{\mu_i}{i} [(2i-1)(6H_{i-1} - 5) - (i-1)(6H_{i-2} + 1)] \psi_0(x) \\
 & + \left(\frac{2i-1}{i}\right) x \gamma_{i-1}(x) - \left(\frac{i-1}{i}\right) \gamma_{i-2}(x),
 \end{aligned} \tag{22}$$

where

$$\begin{aligned}
 p_{ij} = 2j + 1 - \frac{6}{i} [& j(2i-1)H_{j-1} - (2j+1)(i-1)H_j + (j+1)(2i-1)H_{j+1} \\
 & + (2j+1)(i-1)H_{i-2} - (2j+1)(2i-1)H_{i-1}],
 \end{aligned} \tag{23}$$

and

$$\mu_i = \begin{cases} 1, & i \text{ odd,} \\ 0, & i \text{ even.} \end{cases}$$

Now, the elements p_{ij} in (23) after making use of Lemma 2, can be simplified to take the formula

$$p_{ij} = (2j + 1)(1 + 6H_i - 6H_j). \tag{24}$$

Repeated use of Lemma 2 in (22), and after performing some rather manipulation, lead to

$$\begin{aligned}
 D\psi_i(x) = & \sum_{\substack{j=0 \\ (i+j) \text{ odd}}}^{i-1} (2j + 1)(1 + 6H_i - 6H_j) \psi_j(x) - \frac{6(2i-1)}{i} \mu_i \psi_0(x) \\
 & + \left(\frac{2i-1}{i}\right) x \gamma_{i-1}(x) - \left(\frac{i-1}{i}\right) \gamma_{i-2}(x),
 \end{aligned} \tag{25}$$

and by noting that

$$\left(\frac{2i-1}{i}\right) x \gamma_{i-1}(x) - \left(\frac{i-1}{i}\right) \gamma_{i-2}(x) - \frac{6(2i-1)}{i} \mu_i \psi_0(x) = \gamma_i(x),$$

this proves relation (18).

Now, if x in (18) is replaced by $\frac{2x-a-b}{b-a}$, then after performing some manipulations, we get

$$D\phi_i(x) = \frac{2}{b-a} \sum_{\substack{j=0 \\ (i+j) \text{ odd}}}^{i-1} (2j + 1)(1 + 6H_i - 6H_j) \phi_j(x) + \eta_i(x),$$

where $\eta_i(x)$ is given by

$$\eta_i(x) = \begin{cases} 3(x-a)^2(b-x)^2(a+b-2x), & i \text{ even,} \\ 3(a-b)(x-a)^2(b-x)^2, & i \text{ odd.} \end{cases}$$

This completes the proof of Theorem 2. □

Now, and with the aid of Theorem 2, one can deduce that the first derivative of the vector $\Phi(x)$ defined in (15) can be expressed in the following matrix form:

$$\frac{d\Phi(x)}{dx} = G\Phi(x) + \eta(x), \tag{26}$$

where $\eta(x) = (\eta_0(x), \eta_1(x), \dots, \eta_N(x))^T$, and $G = (g_{ij}^{(1)})_{0 \leq i, j \leq N}$, is an $(N+1) \times (N+1)$ matrix whose nonzero elements can be given explicitly from relation (16) as:

$$g_{i,j}^{(1)} = \begin{cases} \frac{2}{b-a}(2j+1)(1+6H_i-6H_j), & i > j, (i+j) \text{ odd,} \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

For example, for $N = 6$, we have

$$G = \frac{2}{b-a} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 15 & 0 & 0 & 0 & 0 \\ 0 & \frac{45}{2} & 0 & \frac{35}{2} & 0 & 0 & 0 \\ \frac{147}{10} & 0 & \frac{57}{2} & 0 & \frac{99}{5} & 0 & 0 \\ 0 & \frac{291}{10} & 0 & \frac{329}{10} & 0 & 22 & 0 \end{pmatrix}_{7 \times 7}.$$

Corollary 1. The q th-derivative of the vector $\Phi(x)$ is given by

$$\frac{d^q\Phi(x)}{dx^q} = G^q\Phi(x) + \sum_{m=0}^{q-1} G^{q-m-1} \frac{d^m}{dx^m} \eta(x). \tag{28}$$

4 Treatment of linear sixth-order two point BVPs

In this section, we are interested in introducing and analyzing spectral solutions for linear sixth-order BVPs by employing the operational matrix of derivatives that introduced in Section 3. For this purpose, the Petrov-Galerkin method is applied on the linear sixth-order BVPs.

Now, consider the linear sixth-order BVPs

$$u^{(6)}(x) + \sum_{q=0}^5 f_q(x) u^{(q)}(x) = g(x), \quad x \in (a, b), \tag{29}$$

governed by the homogeneous boundary conditions

$$u^{(q)}(a) = u^{(q)}(b) = 0, \quad q = 0, 1, 2. \tag{30}$$

If $u(x)$ is approximated as in (14), then with the aid of Corollary 1, the derivative $u^{(q)}(x)$, $1 \leq q \leq 6$, can be approximated as:

$$u^{(q)}(x) \simeq \mathbf{C}^T \left(G^q \Phi(x) + \sum_{m=0}^{q-1} G^{q-m-1} \frac{d^m}{dx^m} \boldsymbol{\eta}(x) \right). \tag{31}$$

If we substitute by relations (14) and (31) into Eq. (29), then the residual $R(x)$, of this equation is given by:

$$R(x) = \sum_{q=1}^6 f_q(x) \mathbf{C}^T \left(G^q \Phi(x) + \sum_{m=0}^{q-1} G^{q-m-1} \frac{d^m}{dx^m} \boldsymbol{\eta}(x) \right) + f_0(x) \mathbf{C}^T \Phi(x) - g(x). \tag{32}$$

The application of Petrov-Galerkin method (see, [Canuto, Hussaini, Quarteroni, and Zang (1988)]) yields the following $(N + 1)$ linear equations in the unknown expansion coefficients, c_i , namely

$$\int_a^b R(x) L_i^*(x) dx = 0, \quad 0 \leq i \leq N. \tag{33}$$

It is clear from Eq. (33) that, a set consists of $(N + 1)$ linear equations is generated. These equations can be solved for the unknown components of the vector \mathbf{C} by any suitable solver, and hence the approximate spectral solution $u_N(x)$ given in (14) can be obtained.

In the following three subsections, we give a comprehensive study on the linear system (33) for the case in which $f_q(x) = v_q$, $1 \leq q \leq 6$, v_q are real constants. Moreover, we comment on the other cases of the coefficients $f_q(x)$.

4.1 Sixth-order two point BVPs with constant coefficients

This subsection is concerned with analyzing the system in (33) which resulted from the application of Petrov-Galerkin method on the linear sixth-order two point BVPs:

$$u^{(6)}(x) + \sum_{q=0}^5 v_q u^{(q)}(x) = g(x), \quad x \in (a, b), \tag{34}$$

governed by the homogeneous boundary conditions

$$u^{(q)}(a) = u^{(q)}(b) = 0, \quad q = 0, 1, 2, \tag{35}$$

where v_q are real constants, $0 \leq q \leq 5$.

In such case, the Petrov-Galerkin formulation in Eq. (33) can be written as

$$(D^6 u_N(x), L_i^*(x)) + \sum_{q=1}^5 v_q (D^q u_N(x), L_i^*(x)) + v_0 (u_N(x), L_i^*(x)) = (g(x), L_i^*(x)). \tag{36}$$

Now, if we denote

$$\begin{aligned} g_i &= (g(x), L_i^*(x)), & \mathbf{g} &= (g_0, g_1, \dots, g_N), \\ B &= (b_{ij})_{0 \leq i, j \leq N}, & B^{(q)} &= (b_{ij}^{(q)})_{0 \leq i, j \leq N}, \quad 1 \leq q \leq 6, \end{aligned}$$

then (36) can be written alternatively in the following matrix system

$$\left(B^{(6)} + \sum_{q=1}^5 v_q B^{(q)} + v_0 B \right) \mathbf{c} = \mathbf{g}, \tag{37}$$

where the nonzero elements of the matrices B and $B^{(q)}$, $1 \leq q \leq 6$ are given explicitly in the following theorem.

Theorem 3. *If the basis $\phi_i(x)$ are selected as in (11), and if we denote $b_{ij} = (\phi_j(x), L_i^*(x))$, $b_{ij}^{(q)} = (D^q \phi_j(x), L_i^*(x))$, $1 \leq q \leq 6$, then the nonzero elements of the matrices $B, B^{(q)}$, are given explicitly as follows:*

$$\begin{aligned} b_{ij} &= \frac{b-a}{2i+1} \xi_{\frac{i-j+6}{2}, j}, & (i+j) \text{ even}, |j-i| \leq 6, \\ b_{ij}^{(1)} &= \frac{2}{2i+1} \sum_{r=0}^{j-1} \sum_{m=0}^6 (2r+1)(1+6H_j-6H_r) \xi_{m,r} \delta_{r-6+2m,i} + d_{ij}^{(1)}, & (i+j) \text{ odd}, |j-i| \leq 5, \\ b_{ij}^{(2)} &= \frac{b-a}{2i+1} \sum_{r=0}^{j-1} \sum_{m=0}^6 g_{r,j}^{(2)} \xi_{m,r} \delta_{r-6+2m,i} + d_{ij}^{(2)}, & (i+j) \text{ even}, |j-i| \leq 4, \\ b_{ij}^{(3)} &= \frac{b-a}{2i+1} \sum_{r=0}^{j-1} \sum_{m=0}^6 g_{r,j}^{(3)} \xi_{m,r} \delta_{r-6+2m,i} + d_{ij}^{(3)}, & (i+j) \text{ odd}, |j-i| \leq 3, \\ b_{ij}^{(4)} &= \frac{b-a}{2i+1} \sum_{r=0}^{j-1} \sum_{m=0}^6 g_{r,j}^{(4)} \xi_{m,r} \delta_{r-6+2m,i} + d_{ij}^{(4)}, & (i+j) \text{ even}, j = i + 2s + 2, s \leq 0, \end{aligned}$$

$$b_{ij}^{(5)} = \frac{b-a}{2i+1} \sum_{r=0}^{j-1} \sum_{m=0}^6 g_{r,j}^{(5)} \xi_{m,r} \delta_{r-6+2m,i} + d_{ij}^{(5)}, \quad (i+j) \text{ odd}, j = i + 2s + 1, s \leq 0,$$

$$b_{ii}^{(6)} = \frac{(a-b)(i+1)6}{2i+1},$$

$$b_{ij}^{(6)} = \frac{b-a}{2i+1} \sum_{r=0}^{j-1} \sum_{m=0}^6 g_{r,j}^{(6)} \xi_{m,r} \delta_{r-6+2m,i} + d_{ij}^{(6)}, \quad (i+j) \text{ even}, j = i + 2s, s \leq -1,$$

where $g_{r,j}^{(q)} = (G^q)_{r,j}$, $1 \leq q \leq 6$, is the (r, j) entry of the matrix G^q , where G is the operational matrix whose nonzero elements are given explicitly in (27), q denotes the notation of the matrix power, $\xi_{m,r}$ are given by (5), $\delta_{m,n}$ is the well-known Kronecker delta function. Moreover, $d_{ij}^{(q)}$ are given by

$$d_{ij}^{(q)} = \int_a^b \left(\sum_{m=1}^{q-1} \sum_{j=0}^{i-m} g_{ij}^{(m)} D^{q-m-1} \eta_j(x) + D^{q-1} \eta_i(x) \right) L_i^*(x) dx,$$

and $\eta_i(x)$ are given by (17).

Proof. At first, the nonzero elements of the matrix $B = (b_{ij}) = (\phi_j(x), L_i^*(x))$ can be obtained with the aid of Lemma 1, along with the orthogonality relation (2). To obtain the nonzero elements of the matrix $B^{(1)} = (b_{ij}^{(1)}) = (D\phi_j(x), L_i^*(x))$, we make use of Theorem 2. The nonzero elements of the other matrices $B^{(q)}$, $2 \leq q \leq 6$, can be obtained, after some lengthy manipulations, if Eq. (28) is written alternatively in the form

$$D^q \phi_i(x) = \sum_{j=0}^{i-q} g_{ij}^{(q)} \phi_j(x) + \pi_{i,q}(x), \tag{38}$$

where $\pi_{i,q}(x)$ is a polynomial of degree at most five for all $1 \leq q \leq 6$, and it can be given explicitly as

$$\pi_{i,q}(x) = \sum_{m=1}^{q-1} \sum_{j=0}^{i-m} g_{ij}^{(m)} D^{q-m-1} \eta_j(x) + D^{q-1} \eta_i(x). \tag{39}$$

Remark 1. The elements $d_{ij}^{(q)}$ that appears in Theorem 3 can be obtained explicitly, by expressing the polynomials $\pi_{i,q}(x)$, $1 \leq q \leq 6$, in terms of the shifted Legendre

polynomials. For example, $d_{i,j}^{(1)}$ are given explicitly as follows:

$$d_{i,j}^{(1)} = (b-a)^6 \begin{cases} \frac{-1}{10}, & i = 0, j \text{ odd}, \\ \frac{-1}{70}, & i = 1, j \text{ even}, \\ \frac{1}{35}, & i = 2, j \text{ odd}, \\ \frac{1}{105}, & i = 3, j \text{ even}, \\ \frac{-1}{210}, & i = 4, j \text{ odd}, \\ \frac{-1}{462}, & i = 5, j \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

□

4.2 Structure of the coefficient matrices in the linear system (37)

In this subsection, we are interested in investigating the structure of the matrices B and $B^q, 1 \leq q \leq 6$ which appear in the linear system (37). Moreover, the structure of the combined matrix $D = B^{(6)} + \sum_{q=1}^5 v_q B^{(q)} + v_0 B$ is also discussed.

- The matrix $B^{(6)}$ is a nonsingular lower triangular matrix, and hence for the case corresponds to $\eta_q = 0, 1 \leq q \leq 6$, the linear system in (37) is reduced to a lower triangular system, which can be easily solved via the forward substitution procedure.
- The four matrices $B, B^{(1)}, B^{(2)}, B^{(3)}$ are, respectively, seven-, six-, five- and four-diagonals. Therefore, with respect to the case corresponds to $v_4 = v_5 = 0$, the matrix D is a combination of a lower triangular matrix of order $(N + 1) \times (N + 1)$ and a seven-diagonal matrix at most. In such case, we can form explicitly LU -factorization effectively, and the number of operations necessary to factorize D in the form $D = LU$ is of order $\frac{2}{3} (N + 1)^3$, and the total cost for solving the two triangular systems is of order $2(N + 1)^2$ for large values of N , (see, [Schatzman (2002)]).
- In the case of $v_q \neq 0, \forall 1 \leq q \leq 6$, then we from explicitly LU -factorization.

Remark 2. If the coefficients $f_q(x)$ which appear in (29) are polynomials in $x, 1 \leq q \leq 6$, then $f_q(x)$ can be expanded in terms of the shifted Legendre polynomials L_j^* , and hence the nonzero elements of the matrices involved in system (33) can be given explicitly. This approach is followed in [Doha and Abd-Elhameed (Accepted)].

Remark 3. *If the coefficients $f_q(x)$ are neither constants nor polynomials for $1 \leq q \leq 6$, then one can employ (33) for the sake of obtaining the desired spectral numerical solutions.*

4.3 Condition number of the resulting system (37)

It is well-known that the application of the direct collocation method on sixth-order BVPs leads to a condition number behaves like $O(N^{12})$ (N : maximal degree of polynomials). In this paper, we obtain an improved condition number with $O(N^5)$. The advantages with respect to propagation of rounding errors is demonstrated.

Now, with respect to the equation $u^{(6)} = g(x)$, the resulting linear system from the application of Petrov-Galerkin method is $B^{(6)} \mathbf{c} = \mathbf{g}$ where the matrix $B^{(6)}$ is a lower triangular matrix whose diagonal elements are $b_{ii}^{(6)}$, where

$$b_{ii}^{(6)} = \frac{(a-b) \prod_{r=1}^6 (i+r)}{2i+1}.$$

Thus we note that the condition number of the matrix $B^{(6)}$ behaves like $O(i^5)$ for large values of i . Table 1 ascertains this result in case of $[a, b] \equiv [-1, 1]$.

Table 1: Condition number for the matrix $B^{(6)}$.

N	Cond($B^{(6)}$)	Cond($B^{(6)}$)/ N^5
8	0.00566101	$1.72760 \cdot 10^{-7}$
12	0.00134669	$5.41206 \cdot 10^{-9}$
16	0.000442282	$4.21793 \cdot 10^{-10}$
20	0.000178083	$5.56509 \cdot 10^{-11}$

Remark 4. *If we add $\sum_{q=1}^5 v_q B^{(q)} + v_0 B$ to the matrix $B^{(6)}$, we find that the combined matrix given by: $D = B^{(6)} + \sum_{q=1}^5 v_q B^{(q)} + v_0 B$, also behaves like $O(N^5)$ for large values of N . This means that the matrix $B^{(6)}$, which resulted from the sixth-derivative of the sixth-order BVP under investigation, plays the essential influence in the propagation of the roundoff errors.*

4.4 Treatment of sixth-order two point BVPs governed by nonhomogeneous boundary conditions

Consider the linear sixth-order BVPs

$$u^{(6)}(x) + \sum_{q=0}^5 f_q(x) u^{(q)}(x) = g(x), \quad x \in (a, b), \quad (40)$$

governed by the nonhomogeneous boundary conditions

$$u^{(q)}(a) = \alpha_q, \quad u^{(q)}(b) = \beta_q, \quad q = 0, 1, 2. \quad (41)$$

then it is easy-with aid of a suitable transformation (see,[Doha, Abd-Elhameed, and Bassuony (2013)])-to convert problem (40), governed by the nonhomogeneous boundary conditions (41) into a modified one which is similar to (29)-(30).

At the end of this section, we give a summary to the steps required to solve linear sixth-order two point BVPs:

- Introducing the operational matrix of derivatives in terms of harmonic numbers.
- Expressing explicitly the derivatives of the selected basis functions in terms of the entries of the introduced operational matrices of derivatives.
- Applying Petrov-Galerkin method on Eq. (29) in order to convert this equation subject to its boundary conditions (30) into a system of linear algebraic equations.
- Solving the resulting linear system by any suitable numerical solver.
- Obtaining the desired spectral numerical solution.

5 Solution of nonlinear sixth-order two point BVPs

Consider the following nonlinear sixth-order boundary value problem

$$u^{(6)}(x) = F(x, u(x), u^{(1)}(x), u^{(2)}(x), u^{(3)}(x), u^{(4)}(x), u^{(5)}(x)), \quad (42)$$

governed by the homogenous boundary conditions

$$u^{(i)}(a) = u^{(i)}(b) = 0, \quad i = 0, 1, 2. \quad (43)$$

If $u(x)$ is approximated as in (14) and if the derivatives $u^{(\ell)}(x)$, $1 \leq \ell \leq 6$, are approximated as in (31), then the following nonlinear equations in the unknown vector \mathbf{C} can be obtained

$$\begin{aligned}
 & \mathbf{C}^T (G^6 \Phi(x) + \boldsymbol{\eta}_6(x)) = \\
 & F \left(x, \mathbf{C}^T \Phi(x), \mathbf{C}^T (G \Phi(x) + \boldsymbol{\eta}(x)), \mathbf{C}^T (G^2 \Phi(x) + \boldsymbol{\eta}_2(x)), \right. \\
 & \left. \mathbf{C}^T (G^3 \Phi(x) + \boldsymbol{\eta}_3(x)), \mathbf{C}^T (G^4 \Phi(x) + \boldsymbol{\eta}_4(x)), \mathbf{C}^T (G^5 \Phi(x) + \boldsymbol{\eta}_5(x)) \right), \tag{44}
 \end{aligned}$$

where $\boldsymbol{\eta}_q(x)$ is given by

$$\boldsymbol{\eta}_q(x) = \sum_{m=0}^{q-1} G^{q-m-1} \frac{d^m}{dx^m} \boldsymbol{\eta}(x), \quad 2 \leq q \leq 6,$$

and the components of the vector $\boldsymbol{\eta}(x)$ are given by (17).

To find the numerical solution $u_N(x)$, we enforce Eq. (44) to be satisfied exactly at $(N + 1)$ collocation points. There are many options to select these collocations points. Every choice leads to a numerical solution $u_N(x)$. Among of the possible choices for these collocation points are:

1. The $(N + 1)$ roots of the polynomial $L_{N+1}^*(x)$.
2. The $(N + 1)$ roots of the Chebyshev polynomial of the first kind $T_{N+1}^*(x)$.
3. The $(N + 1)$ roots of the Chebyshev polynomial of the second kind $U_{N+1}^*(x)$.

It is clear that for every choice of the collocation points, a set of $(N + 1)$ nonlinear equations is generated in the expansion coefficients, c_i . With the aid of the well-known Newton’s iterative method, this nonlinear system can be solved, and hence the corresponding approximate solution $u_N(x)$ can be obtained.

At the end of this section, the steps required to solve nonlinear sixth-order two point BVPs can be summarized in the following items:

- Introducing the operational matrix of derivatives in terms of harmonic numbers.
- Deducing the operational matrices of derivatives for all $q \geq 1$ as in (28).
- Applying the typical collocation method on the nonlinear sixth-order BVP for the sake of converting it into a nonlinear system of equations.
- Solving the resulting nonlinear system of equations by a suitable solver such as the well-known Newton’s iterative method.
- Obtaining the desired spectral numerical solution.

6 Convergence and error analysis of the suggested expansion

In this section, we discuss the convergence and error analysis of the suggested approximate solution. To be more precise, we will prove a theorem in which the expansion in (12) of a function $u(x) = (x - a)^3 (b - x)^3 f(x) \in H_{0,w}^3(I)$, where $f(x)$ is of bounded third derivative, converges uniformly to $u(x)$. Moreover, an upper bound for the error (in L_w^2 norm) is given.

Theorem 4. *A function $u(x) = (x - a)^3 (b - x)^3 f(x) \in H_{0,w}^3(I)$, $w(x) = \frac{1}{(x-a)^6 (b-x)^6}$ with $|f^{(3)}(x)| \leq M$, can be expanded as an infinite sum of the basis given in (12). This series converges uniformly to $u(x)$, and the coefficients in (12) satisfy the inequality*

$$|c_i| < \frac{M(b-a)^3}{2i^2}, \quad \forall i \geq 3. \tag{45}$$

Proof. From Eq. (13), and with the aid of (11), one has

$$c_i = \frac{2i+1}{b-a} \int_a^b f(x) L_i^*(x) dx. \tag{46}$$

Relation (46) gives after integration by parts three times and making use of Theorem 1 (for $q = 3$)

$$c_i = \frac{2i+1}{a-b} \int_a^b f^{(3)}(x) I^{(3)}(x) dx, \quad i \geq 3, \tag{47}$$

where

$$I^{(3)}(x) = \frac{(b-a)^3}{8} \left[\frac{-L_{i-3}^*(x)}{(2i-3)(2i-1)(2i+1)} + \frac{3L_{i-1}^*(x)}{(2i-3)(2i+1)(2i+3)} - \frac{3L_{i+1}^*(x)}{(2i-1)(2i+1)(2i+5)} + \frac{L_{i+3}^*(x)}{(2i+1)(2i+3)(2i+5)} \right],$$

hence the coefficients c_i can be written in the form

$$c_i = \frac{1}{8}(b-a)^2 \int_a^b \left[\frac{L_{i-3}^*(x)}{(2i-3)(2i-1)} - \frac{3L_{i-1}^*(x)}{(2i-3)(2i+3)} + \frac{3L_{i+1}^*(x)}{(2i-1)(2i+5)} - \frac{L_{i+3}^*(x)}{(2i+3)(2i+5)} \right] f^{(3)}(x) dx.$$

Now, making use of the inequality (see, [Rainville (1960)])

$$|L_i^*(x)| < 1, \quad a < x < b,$$

and with the aid of the assumption $|f^{(3)}(x)| \leq M$, it is not difficult to show that

$$|c_i| < \frac{M(b-a)^3}{(2i-3)(2i+5)}.$$

Since, $i \geq 3$, thus we get

$$|c_i| < \frac{M(b-a)^3}{2i^2},$$

and this completes the proof of Theorem 4. □

In the following, and based on Theorem 4, we give an estimation to the error (in L_w^2 norm). The following lemma is needed.

Lemma 3. (see, Stewart (2012)) *Let $f(x)$ be a continuous, positive, decreasing function for $x \geq n$. If $f(k) = a_k$, provided that $\sum a_n$ is convergent, and $R_n = \sum_{k=n+1}^{\infty} a_k$, then the following inequality holds:*

$$R_n \leq \int_n^{\infty} f(x) dx.$$

Theorem 5. *If u satisfy the hypothesis of Theorem 4, and if we consider the expansion $u_N(x) = \sum_{i=0}^N c_i \phi_i(x)$, then the following error estimate (in L_w^2 -norm, $w = \frac{1}{(x-a)^6(b-x)^6}$) is obtained*

$$\|u - u_N\|_w < \frac{M(b-a)^{\frac{7}{2}}}{5N^2}. \tag{48}$$

Proof. From Eq. (13), and making use of the orthogonality property of $\{\phi_i(x)\}$, we get

$$\|u - u_N\|_w^2 = \sum_{i=N+1}^{\infty} \frac{(b-a)}{(2i+1)} c_i^2.$$

In virtue of Theorem 4, one can write

$$\|u - u_N\|_w^2 < \frac{M^2(b-a)^7}{4} \sum_{i=N+1}^{\infty} \frac{1}{i^4(2i+1)} < \frac{M^2(b-a)^7}{8} \sum_{i=N+1}^{\infty} \frac{1}{i^5},$$

and the application of Lemma 3 leads to

$$\begin{aligned} \|u - u_N\|_w^2 &< \frac{M^2(b-a)^7}{8} \int_N^\infty x^{-5} dx \\ &= \frac{M^2(b-a)^7}{32 N^4} \\ &< \frac{M^2(b-a)^7}{25 N^4}, \end{aligned}$$

and hence

$$\|u - u_N\|_w < \frac{M(b-a)^{\frac{7}{2}}}{5N^2},$$

which completes the proof of the theorem. □

7 Numerical results and discussions

In this section, the two presented algorithms in Sections 4 and 5 are applied to solve linear and nonlinear sixth-order boundary value problems. We support our algorithms by presenting three numerical examples accompanied by some comparisons with some other methods in literature.

Now, consider the following examples.

Example 1. Consider the following sixth-order linear boundary value problem (see, [Khandelwal and Sultana (2013)]):

$$\left. \begin{aligned} y^{(6)}(x) + xy(x) &= -(x^3 + 11x + 24)e^x, \quad x \in [0, 1], \\ y(0) = y(1) &= 0, \\ y'(0) = 1, \quad y'(1) &= -e, \\ y''(0) = 0, \quad y''(1) &= -4e. \end{aligned} \right\} \tag{49}$$

The exact solution of the above problem is

$$y(x) = x(1-x)e^x. \tag{50}$$

In Table 2, the maximum pointwise errors $E = |u - u_N|$ which resulted from the application of PGOMM, for various values of N are displayed. This table shows that by taking three terms only from the retained modes of the approximate expansion, an error which does not exceed $(2.64 \cdot 10^{-8})$ is achieved. In Table 3, we present a comparison between the best errors obtained from the application of PGOMM with the best errors obtained by applying the three methods namely, second-, fourth- and

sixth-order methods which presented in [Khandelwal and Sultana (2013)]. This table shows that the best accuracy obtained by PGOMM is better than the best accuracy obtained by the three methods developed in [Khandelwal and Sultana (2013)].

Table 2: Maximum pointwise error of $|u - u_N|$ for $N = 2, 4, 6, 8$

N	E
2	$2.64 \cdot 10^{-8}$
4	$2.06 \cdot 10^{-11}$
6	$9.19 \cdot 10^{-15}$
8	$2.13 \cdot 10^{-16}$

Table 3: Comparison between the best errors for Example 1 by different methods

Best error	[Khandelwal and Sultana (2013)]			
	2nd-order methods	4th-order methods	6th-order methods	PGOMM
E	9.37×10^{-8}	2.81×10^{-11}	4.29×10^{-13}	2.13×10^{-16}

Example 2. Consider the following linear sixth-order boundary value problem (see, Siddiqi and Akram [Siddiqi and Akram (2008)]):

$$y^{(6)}(x) + (5x + 1)y(x) = 5x(2x^3 - 5x + 37)\cos(x) + 18(15 - 2x^2)\sin(x), \quad x \in [-1, 1], \tag{51}$$

subject to the boundary conditions:

$$\begin{aligned} y(-1) &= 4\cos(1), & y(1) &= -2\cos(1), \\ y^{(1)}(-1) &= \cos(1) + 4\sin(1), & y^{(1)}(1) &= \cos(1) + 2\sin(1), \\ y^{(2)}(-1) &= -16\cos(1) + 2\sin(1), & y^{(2)}(1) &= 14\cos(1) - 2\sin(1). \end{aligned}$$

The exact solution of this problem is

$$y(x) = (2x^3 - 5x + 1)\cos(x).$$

In Table 4, the maximum pointwise errors of $|u - u_N|$ which resulted from the application of PGOMM are displayed for various values of N , while, Table 5, illustrates a comparison between the best errors obtained by PGOMM, with the septic spline method developed in [Siddiqi and Akram (2008)]. This table ascertains that the application of PGOMM gives more accurate results than those obtained by the algorithm developed in [Siddiqi and Akram (2008)].

Table 4: Maximum pointwise error of $|u - u_N|$ for $N = 4, 6, 8, 10, 12, 14$

N	E
4	4.48×10^{-7}
6	7.92×10^{-9}
8	1.90×10^{-12}
10	2.15×10^{-15}
12	1.77×10^{-15}
14	2.22×10^{-16}

Table 5: Comparison between the best errors for Example 2 by different methods

Best error	PGOMM	[Siddiqi and Akram (2008)]
E	2.22×10^{-16}	8.68×10^{-7}

Example 3. Consider the following sixth-order nonlinear boundary value problem on (see, El-Kady and Khalil [El-Kady and Khalil (2011)]):

$$\left. \begin{aligned} &y^{(6)}(x) + e^{-x}y^2(x) = e^{-x} + e^{-3x}, \quad 0 \leq x \leq 1, \\ &y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1, \\ &y(1) = \frac{1}{e}, \quad y'(1) = -\frac{1}{e}, \quad y''(1) = \frac{1}{e}, \end{aligned} \right\} \tag{52}$$

The exact solution of this problem is

$$y(x) = e^{-x}. \tag{53}$$

In Table 6, we list the maximum pointwise error using COMM for various values of N . We choose three kinds of collocation points. Let E_1, E_2 and E_3 denote the

maximum pointwise errors when the collocation points are respectively, Legendre, first and second kinds of Chebyshev polynomials. The numerical results show that the best results are achieved if the selected collocation points are the zeros of the second kind of Chebyshev polynomials. Table 7 displays a comparison between the best errors obtained by COMM and the method developed in [El-Kady and Khalil (2011)].

Table 6: Maximum pointwise error of $|u - u_N|$ for $N = 2, 4, 6, 8$

N	E_1	E_2	E_3
2	$3.91 \cdot 10^{-9}$	$5.67 \cdot 10^{-9}$	$2.74 \cdot 10^{-9}$
4	$5.90 \cdot 10^{-12}$	$1.10 \cdot 10^{-11}$	$3.01 \cdot 10^{-12}$
6	$2.06 \cdot 10^{-15}$	$4.92 \cdot 10^{-15}$	$7.77 \cdot 10^{-16}$
8	$1.60 \cdot 10^{-16}$	$1.60 \cdot 10^{-16}$	$1.60 \cdot 10^{-16}$

Table 7: Comparison between the best errors for Example 3 by different methods

Best error	COMM	Method in [El-Kady and Khalil (2011)]
E	1.60×10^{-16}	9.8×10^{-11}

8 Conclusions

In this paper, a novel operational matrix of derivatives is derived. Two algorithms based on employing a new operational matrix of derivatives together with the application of the two spectral methods namely, Petrov-Galerkin and collocation methods are analyzed and presented for handling linear and nonlinear sixth-order boundary value problems. Convergence and error analysis of the suggested expansion are carefully investigated. The suggested algorithms are applicable, efficient and easy in implementation. The main advantage of the two presented algorithms is their availability for application on both linear and non linear boundary value problems. Moreover, numerical experiments show that high accurate approximate solutions are achieved by using a few number of terms of the suggested expansion. The obtained approximate solutions are very close to the analytical ones.

Acknowledgment. The author would like to thank the anonymous referees for carefully reading the paper and also for their constructive comments which have improved the paper.

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