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On the Numerical Solution of the Laplace Equation with Complete and Incomplete Cauchy Data Using Integral Equations

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Abstract: We consider the numerical solution of the Laplace equations in planar bounded domains with corners for two types of boundary conditions. The first one is the mixed boundary value problem (Dirichlet-Neumann), which is reduced, via a single-layer potential ansatz, to a system of well-posed boundary integral equations. The second one is the Cauchy problem having Dirichlet and Neumann data given on a part of the boundary of the solution domain. This problem is similarly transformed into a system of ill-posed boundary integral equations. For both systems, to numerically solve them, a mesh grading transformation is employed together with trigonometric quadrature methods. In the case of the Cauchy problem the Tikhonov regularization is used for the discretized system. Numerical examples are included both for the well-posed and ill-posed cases showing that accurate numerical solutions can be obtained with small computational effort.

Keywords: Laplace equation; Cauchy problem; Corner domain; Mixed problem; Mesh grading transform; Single-layer potential; Tikhonov regularization.

1 Introduction

Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary ∂D, which is divided into the two C^2-smooth curves (arcs) Γ_1 and Γ_2 having the points (the endpoints) P_1 and P_{-1} in common. It is assumed that these two points are corner points of the boundary ∂D, with interior angles θ_1 and θ_{-1}, and θ_1, $\theta_{-1} \in (0,\pi)$.

We consider the following linear inverse ill-posed problem: Construct the function

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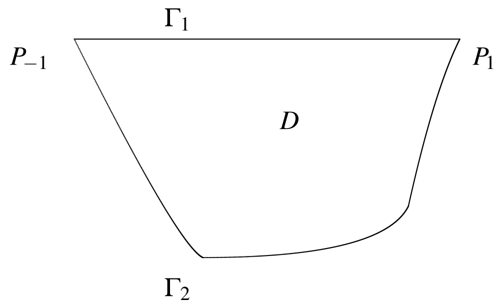


Figure 1: A solution domain D

$u : D \to \mathbb{R}$ satisfying the Laplace equation

$$\Delta u = 0 \text{ in } D \tag{1}$$

from the given Cauchy data on Γ_2:

$$u = f_2 \text{ on } \Gamma_2 \tag{2}$$

and

$$\frac{\partial u}{\partial \nu} = g_2 \text{ on } \Gamma_2. \tag{3}$$

Here f_2 and g_2 are given functions and ν is the outward unit normal to Γ_2. It is assumed that data are such that there exists a solution. Related to this problem is the case of incomplete Cauchy data, where for example the Dirichlet data f_1 is known on Γ_1 and the Neumann data g_2 is given on the boundary part Γ_2. This is known as a mixed problem and is well-posed; for physical applications and history of mixed problems going back to the model of Nobili's rings (where on the boundary of a material there are regions of zero and of very high conductivity, respectively), see Chapter 2 in Duffy (2008). Both the Cauchy problem and the mixed problem will be treated in this study in a unified approach.

We shall numerically solve these problems using boundary integral equations. This seems like a natural choice since only data on the boundary is given, and avoiding domain discretisation reduces the dimension of the problem. For the Cauchy problem it is interesting to note that although this is classical, see Hadamard (1923), and very well researched, see for example Payne (1975) and Cao, Klibanov and Pereverzev (2009) and references therein, most numerical results are for smooth domains without corners, and it also seems that the simple and straightforward approach of representing the solution in terms of a layer potential ansatz and then discretising has not been much investigated numerically. It seems like the work of Cakoni and Kress (2007) is the first where such an approach is mentioned. Theoretical properties of the method such as solvability of the obtained integral equations was shown there. However, no numerical results were presented for the Cauchy problem but for a related non-linear inverse problem (and further investigations for that problem were done in Cakoni, Kress and Schuft (2010 a,b)). Thus, the main aim and novelty of the present work is to numerically implement and investigate the potential approach both for mixed problems and Cauchy problem in domains with corners. In a previous work Chapko, Johansson and Savka (2012), we utilized this strategy for the slightly simpler situation for the Cauchy problem in an annular planar domain (separated boundaries), with Cauchy given on one of these (closed) boundary curves, obtaining reasonable accurate results with small computational effort. It is this approach that we shall further adjust and apply (also for the mixed problem) to the above situation for a simply connected domain with two corner points. Note that there are of course other direct boundary integral approaches that are possible as well, although not as straightforward; in Chapko and Johansson (2008) a Green's function technique was derived and in Chapko and Johansson (2012) potential theory was used to reduce the Cauchy problem to a boundary integral equation. Note that the approach we propose, as was pointed out in Chapko, Johansson and Savka (2012), can be used to obtain in a fast and computationally efficient way an approximation that can then be used as an initial guess in more involved methods for the Cauchy problem such as iterative gradient type minimization procedures. Note also that integral equation techniques can be adjusted to for example Cauchy problems in domains with cuts, see Chapko and Johansson (2008). Furthermore, recently some interesting integral approaches for mixed problems in corner domains that could potentially be applied for the Cauchy problem have been proposed, Bremer and Rokhlin (2010); Helsing and Ojala (2008).

The paper is organized in the following way. In Section 2, we start by considering the numerical solution of the situation with incomplete Cauchy data, i.e. the well-posed mixed Dirichlet-Neumann boundary value problem in simply connected domains having two corner points (the boundary condition changes type at these points as well). We start with this case since it is a well-posed problem. Although the mixed problem is well researched, we think it is valuable to collect and give an overview and outline of some (known) results as well as outlining a numerical procedure for it in corner domains. Having these results in the same paper as the results for the Cauchy problem highlights the similarities of the difficulties that one faces when solving these problems numerically.

Thus, in Section 2, we show how to reduce the direct mixed problem to a system of boundary integral equations having various singularities: a logarithmic singularity in the kernel and weak singularities in the densities at the corner points. For the numerical solution of the obtained integral equations we use a nonlinear mesh grading transform for weakening of the singularities in densities, and then a quadrature method with trigonometric quadratures for the full-discretization of the integral equations (extended to a closed curve).

In Section 3, we investigate a corresponding integral equation approach for the Cauchy problem (1)–(3) as for the mixed problem. Following Section 2, via logarithmic potentials, we reduce the Cauchy problem to a similar system of integral equations with the additional difficulty that this system consists of ill-posed equations. Singularities at the corner points are handled via the same mesh grading transform as in the incomplete case. Tikhonov regularization is used to obtain a stable solution to the linear system that is obtained via discretisation of the boundary integrals. Numerical examples are presented in Section 4, showing that accurate results, both for the direct and inverse problems, can be obtained in corner domains with small computational effort.

2 A boundary integral equation approach for a mixed problem in a domain with corner points

We first consider the following mixed Dirichlet-Neumann boundary value problem:

$$\Delta u = 0 \ \text{ in } \ D, \tag{4}$$

$$u = f_1 \ \text{ on } \ \Gamma_1 \tag{5}$$

and

$$\frac{\partial u}{\partial \nu} = g_2 \ \text{ on } \ \Gamma_2, \tag{6}$$

where f_1 and g_2 are given functions. We assume that $\operatorname{cap}(\partial D) \neq 1$. This condition for the logarithmic capacity of the boundary is necessary to obtain uniqueness in the integral equations we use. It is not a severe restriction to put this condition on the boundary ∂D, since, if necessary, a preliminary rescaling of the solution domain D can be done such that the condition will hold (examples of boundaries satisfying the given condition are boundaries of solution domains D contained inside a circle of radius $r < 1$, respectively, convex domains containing a circle of radius $r > 1$). It is then known (see for example McLean (2000)) that for $f_1 \in H^{1/2}(\Gamma_1)$ and $g_2 \in H^{-1/2}(\Gamma_2)$ the mixed problem (4)–(6) has an unique solution $u \in H^1(D)$.

2.1 Logarithmic potential approach for the mixed problem (4)–(6)

For the bounded domain D it is known that the solution of the boundary value problem (4)–(6) can be represented as a single-layer potential

$$u(x) = \int_{\partial D} \mu(y)\Phi(x,y)\,ds(y). \tag{7}$$

Here, $\Phi(x,y) = (2\pi)^{-1}\ln|x-y|^{-1}$ is the fundamental solution of (4) and μ is an unknown density. Denote by $\mu_1 = \mu|_{\Gamma_1}$ and put $\mu_2 = \mu|_{\Gamma_2}$. Using the representation (7) and imposing the boundary conditions for the mixed problem (4)–(6), this mixed problem can be reduced to the following system of boundary integral equations

$$\begin{cases} \displaystyle\int_{\Gamma_1} \mu_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_2} \mu_2(y)\Phi(x,y)\,ds(y) = f_1(x),\ x \in \Gamma_1, \\ \displaystyle\frac{1}{2}\mu_2(x) + \int_{\Gamma_1} \mu_1(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) + \int_{\Gamma_2} \mu_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) = g_2(x),\ x \in \Gamma_2. \end{cases} \tag{8}$$

Note that from results in Costabel and Stephan (1985) it is known that for smooth boundary data f_1 and g_2, the densities μ_1 and μ_2 in (7) have singularities of the form

$$\mu(x) = O(|x-P_i|^{\lambda_i}), \quad x \to P_i, \quad \lambda_i = \min\left\{\frac{\pi}{2\theta_i}, \frac{\pi}{2(2\pi-\theta_i)}\right\} - 1, \quad i = -1, 1,$$

near the corner points P_i. Therefore, the well-posedness of the system (8) can be shown in a weighted L_2-space, see Grisvard (1985), or, in a Sobolev space of negative order, see Costabel and Stephan (1985).

2.2 Parametrization and a mesh grading transformation for (8)

For the numerical solution of the integral equations (8), we are going to implement a quadrature method but first, as mentioned in the introduction, we make a special nonlinear mesh grading transformation (for details see Chapko (2004); Elschner and Graham (1997); Elschner and Jeon (1997); Kress (1990); Kress and Tran (2000)). For this transformation to be possible, we have to parametrize the integral equations (8). First, we consider a parametrization $\tilde{z} : [0,2\pi] \to \partial D$ and $\tilde{z}([0,\pi]) \equiv \Gamma_1$ and $\tilde{z}([\pi,2\pi]) \equiv \Gamma_2$. Introducing the cubic polynomial

$$v(s) = \left(\frac{1}{q} - \frac{\pi}{2}\right)\left(\frac{\pi - 2s}{\pi}\right)^3 - \frac{1}{q}\left(\frac{\pi - 2s}{\pi}\right) + \frac{\pi}{2}, \tag{9}$$

where $q \geq 2$, and setting

$$w(s) = \pi \frac{[v(s)]^q}{[v(s)]^q + [v(\pi - s)]^q}, \quad 0 \leq s \leq \pi, \tag{10}$$

we define the mesh grading transformation

$$\gamma(s) = \begin{cases} \gamma_1(s) = w(s), & 0 \leq s \leq \pi, \\ \\ \gamma_2(s) = \pi + w(s - \pi), & \pi \leq s \leq 2\pi. \end{cases}$$

Then, clearly

$$\gamma \in C^{q-1}[0,2\pi], \quad \gamma^{(\ell)}(0) = \gamma^{(\ell)}(\pi) = \gamma^{(\ell)}(2\pi) = 0, \quad \ell = 1,\dots,q-1.$$

Now, we consider a new parametrization of the boundary ∂D, given by

$$z(s) = \begin{cases} z_1(s) = \tilde{z}(\gamma_1(s)), & 0 \leq s \leq \pi, \\ \\ z_2(s) = \tilde{z}(\gamma_2(s)), & \pi \leq s \leq 2\pi, \end{cases}$$

and can then rewrite the integral equations (8) in the parametric form

$$\begin{cases} \dfrac{1}{2\pi}\left[\displaystyle\int_0^{\pi} \psi_1(\sigma) L_{11}(s,\sigma)\, d\sigma + \int_{\pi}^{2\pi} \psi_2(\sigma) L_{12}(s,\sigma)\, d\sigma\right] = f_1(s), \\ \dfrac{1}{2}\psi_2(s) + \dfrac{|z_2'(s)|}{2\pi}\left[\displaystyle\int_0^{\pi} \psi_1(\sigma) K_{21}(s,\sigma)\, d\sigma + \int_{\pi}^{2\pi} \psi_2(\sigma) K_{22}(s,\sigma)\, d\sigma\right] = g_2(s), \end{cases} \tag{11}$$

where in the first equation $s \in [0,\pi]$ and in the second $s \in [\pi,2\pi]$. Here, we introduced the functions $f_1(s) = f_1(z_1(s))$, $g_2(s) = g_2(z_2(s))|z_2'(s)|$ and $\psi_\ell(s) = \mu_\ell(z_\ell(s))|z_\ell'(s)|$, $\ell = 1,2$, and the kernels have the form

$$L_{i\ell}(s,\sigma) = -\ln|z_i(s) - z_\ell(\sigma)| \quad \text{for} \quad s \neq \sigma \quad i,\ell = 1,2$$

and

$$K_{i\ell}(s,\sigma) = \frac{< z_\ell(\sigma) - z_i(s), \nu(z_i(s)) >}{|z_i(s) - z_\ell(\sigma)|^2} \quad \text{for} \quad s \neq \sigma \quad i,\ell = 1,2,$$

with the diagonal term

$$K_{ii}(s,s) = \frac{< z_i''(s), \nu(z_i(s)) >}{2|z_i'(s)|^2}.$$

The logarithmic singularity in L_{ii} shall be dealt with later on in this paper.

In the next step, we shall extend each of the parametrizations z_1 and z_2 to become 2π-periodic. For this, in addition, we define

$$z_1(s) = \begin{cases} z_1(s), & 0 \le s \le \pi, \\ z_1(2\pi - s), & \pi \le s \le 2\pi, \end{cases} \quad \text{and} \quad z_2(s) = \begin{cases} z_2(2\pi - s), & 0 \le s \le \pi, \\ z_2(s), & \pi \le s \le 2\pi, \end{cases}$$

together with $z_\ell(s+2\pi) = z_\ell(s), \quad \ell = 1,2.$

Thus, each z_ℓ is even and 2π-periodic, and clearly these properties then extend to all functions in the system (11). Now, we can write the system (11) as

$$\begin{cases} \dfrac{1}{2\pi}\left[\displaystyle\int_0^{2\pi} \psi_1(\sigma) L_{11}(s,\sigma)\, d\sigma + \int_0^{2\pi} \psi_2(\sigma) L_{12}(s,\sigma)\, d\sigma\right] = 2f_1(s), \\ \psi_2(s) + \dfrac{|z_2'(s)|}{2\pi}\left[\displaystyle\int_0^{2\pi} \psi_1(\sigma) K_{21}(s,\sigma)\, d\sigma + \int_0^{2\pi} \psi_2(\sigma) K_{22}(s,\sigma)\, d\sigma\right] = 2g_2(s), \end{cases} \tag{12}$$

where $s \in \mathbb{R}$.

Returning to the logarithmic kernel L_{ii}, we perform the transformation

$$L_{ii}(s,\sigma) = -\frac{1}{2}\ln\left[\frac{4}{e^2}(\cos s - \cos\sigma)^2\right] + b_i(s,\sigma), \quad i = 1,2,$$

where

$$b_i(s,\sigma) = \ln \frac{2|\cos s - \cos\sigma|}{e|z_i(s) - z_i(\sigma)|},$$

with the diagonal term

$$b_i(s,s) = \ln \frac{2|\sin s|}{e|z_i'(s)|}.$$

The functions b_i are, as can be seen, not defined at the four corners and the centre of the square $[0,2\pi] \times [0,2\pi]$, and we shall take this into account later on.

We then use that all functions in (12) are even implying that the following identity holds

$$\int_0^{2\pi} \mu(\sigma) \ln\left[\frac{4}{e^2}(\cos s - \cos\sigma)^2\right] d\sigma = 2\int_0^{2\pi} \mu(\sigma) \ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) d\sigma, \; s \in [0,2\pi].$$

The Dirichlet data on the boundary part Γ_2 can, according to (7), be found from the representation

$$u(z_2(s)) = \frac{1}{2\pi}\left[\int_0^{\pi} \psi_1(\sigma)L_{21}(s,\sigma)\,d\sigma + \int_{\pi}^{2\pi} \psi_2(\sigma)L_{22}(s,\sigma)\,d\sigma\right]$$

and the Neumann data on the boundary part Γ_1 is given by

$$\frac{\partial u}{\partial \nu}(z_1(s)) = \frac{\psi_1(s)}{2|z_1'(s)|} + \frac{1}{2\pi}\left[\int_0^{\pi} \psi_1(\sigma)K_{11}(s,\sigma)\,d\sigma + \int_{\pi}^{2\pi} \psi_2(\sigma)K_{12}(s,\sigma)\,d\sigma\right].$$

Thus, one can complete the data from the mixed problem to have full Cauchy data on the boundary ∂D.

2.3 Solution properties of the integral equations (12)

Note that the mesh grading transform gives the possibility to analyse the solvability of the obtained integral equations in L_2-spaces. Therefore, introduce the integral operators

$$(S_{11}\varphi)(s) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right)d\sigma,$$

$$(A_{11}\varphi)(s) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\sigma)b_1(s,\sigma)\,d\sigma,$$

$$(B_{12}\varphi)(s) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\sigma)L_{12}(s,\sigma)\,d\sigma$$

and

$$(B_{2\ell}\varphi)(s) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\sigma)K_{2\ell}(s,\sigma)\,d\sigma,\ \ell = 1,2.$$

Denote by $H_e^p[0,2\pi]$, $p \geq 0$, the standard Sobolev spaces of even 2π-periodic functions. It is clear that the operator $S_{11} : H_e^p[0,2\pi] \to H_e^{p+1}[0,2\pi]$ is bounded and has a bounded inverse. The operators $B_{i\ell} : H_e^p[0,2\pi] \to H_e^p[0,2\pi]$, $i,\ell = 1,2, i\ell \neq 1$, have smooth kernels and are therefore compact. Thus, we can rewrite the system (12) in the following operator form

$$(\mathscr{A} + \mathscr{B})\,\vec{\psi} = \vec{g},$$

where $\vec{\psi} = (\psi_1,\psi_2)^\top$, $\vec{g} = (2f_1, 2g_2)^\top$ and

$$\mathscr{A} = \begin{pmatrix} S_{11} & B_{12} \\ 0 & I \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} A_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

Clearly, $\mathscr{A} : H_e^0[0,2\pi] \times H_e^0[0,2\pi] \to H_e^1[0,2\pi] \times H_e^0[0,2\pi]$ is bounded and has a bounded inverse.

In Elschner and Graham (1997) it is shown by using a technique involving the Mellin transform that the integral operator $I + \mathscr{A}^{-1}\mathscr{B} : H_e^0[0,2\pi] \times H_e^0[0,2\pi] \to H_e^0[0,2\pi] \times H_e^0[0,2\pi]$ is bounded and has a bounded inverse. Thus, we have the following existence result.

Theorem 2.1 *Assume that $q \geq 3$. Then for every $f_1 \in H_e^1[0,2\pi]$ and $g_2 \in H_e^0[0,2\pi]$ there exist unique solutions $\psi_{1,n}, \psi_{2,n} \in H_e^0[0,2\pi]$ to the equations* (12).

2.4 Full discretization of the system (12) by a quadrature method

For the numerical solution of the integral equations (12) we use the quadrature method Atkinson (1997); Kress (1999). Introduce the equidistant grid

$$s_k = \frac{k\pi}{n} \quad \text{for} \quad k = 0, \ldots, 2n-1, \tag{13}$$

and for this grid consider the following two trigonometric quadrature rules

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma) \ln\left(\frac{4}{e}\sin^2\frac{s_i - \sigma}{2}\right) d\sigma \approx \sum_{k=0}^{2n-1} R_{|k-i|} f(s_k), \tag{14}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(s)\, ds \approx \frac{1}{2n}\sum_{k=0}^{2n} f(s_k), \tag{15}$$

with the weights

$$R_j = -\frac{1}{n}\Big(1 + 2\sum_{m=1}^{n-1}\frac{1}{m}\cos\frac{mj\pi}{n} + \frac{(-1)^j}{n^2}\Big), \quad j = 0, \ldots, 2n-1.$$

The use of the quadrature formulas (14) and (15) for the integrals in (12) and collocation at the quadrature points, lead to a linear system. As remarked earlier all functions in (12) are even with respect to the midpoint of the interval $[0,2\pi]$, and $\psi_\ell(0) = 0$ for $\ell = 1,2$. Therefore, we can write the linear system (12) in the form

$$\begin{cases} \displaystyle\sum_{i=1}^{n-1}\left[\psi_{1,i}\left(R_{|i-k|} + R_{i+k} + \frac{1}{n}b_1(s_k,s_i)\right) + \psi_{2,i}\frac{1}{n}L_{12}(s_k,s_i)\right] = 2f_1(s_k), \\ \\ \displaystyle\psi_{2,k} + \frac{|z_2'(s_k)|}{n}\sum_{i=1}^{n-1}\left[\psi_{1,i}K_{21}(s_k,s_i) + \psi_{2,i}K_{22}(s_k,s_i)\right] = 2g_2(s_k), \end{cases} \tag{16}$$

where $\psi_{\ell,k} \approx \psi_\ell(s_k)$, $k = 1,\dots,n-1$, $\ell = 1,2$. Here, we note that we do not need to calculate the function b_1 at the singular points.

The numerical solution of (4)–(6) can be calculated as

$$\tilde{u}(x) = -\frac{1}{2n}\sum_{k=1}^{n-1}\sum_{\ell=1}^{2}\psi_{\ell,k}\ln|x - z_\ell(s_k)|, \quad x \in D \tag{17}$$

and values of the Dirichlet data on Γ_2 can be calculated from

$$\tilde{u}(z_2(s_k)) = \sum_{i=1}^{n-1}\left(\psi_{1,i}\frac{1}{2n}L_{21}(s_k,s_i) + \psi_{2,i}[\frac{1}{2}(R_{|i-k|} + R_{i+k}) + \frac{1}{2n}b_2(s_k,s_i)]\right),$$

and the Neumann data on Γ_1 is given by

$$\frac{\partial\tilde{u}}{\partial\nu}(z_1(s_k)) = \frac{\psi_{1,k}}{|z_1'(s_k)|} + \frac{1}{2n}\sum_{i=1}^{n-1}\sum_{\ell=1}^{2}\psi_{\ell,i}K_{1\ell}(s_k,s_i)$$

for $k = 1,\dots,n-1$. Thus, it is possible to numerically complete the data of the mixed problem (4)–(6) to obtain values of the Cauchy data on all of the boundary ∂D.

Convergence analysis and error estimates for the above quadrature method can be carried out much in the same way as in Kress and Tran (2000) to obtain the following.

Theorem 2.2 *Let $\Gamma_1,\Gamma_2 \in C^\infty$, and let the two corner points of ∂D have interior angles $(1-\beta_1)\pi$ and $(1-\beta_2)\pi$ with $0 < |\beta_1|,|\beta_2| < 1$, and assume that $f_1 \in H^{m+5/2}(\Gamma_1)$, $g_2 \in H^{m+5/2}(\Gamma_2)$, $m \in \mathbb{N}$, where q is sufficiently large. Then*

$$||\psi_\ell - \tilde{\psi}_\ell||_{H_e^0[0,2\pi]} \le C_\ell n^{-m}, \quad \ell = 1,2.$$

Here, $\tilde{\psi}_\ell$ are the trigonometric interpolation functions obtained from (16).

3 A boundary integral equation approach for the Cauchy problem (1)–(3)

Now, we consider the original Cauchy problem (1)–(3). Similar to the previous section, we represent the solution in the following integral form

$$u(x) = \int_{\Gamma_1}\phi_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_2}\phi_2(y)\Phi(x,y)\,ds(y), \quad x \in D, \tag{18}$$

with unknown densities ϕ_1 on Γ_1 and ϕ_2 on Γ_2. Imposing the boundary conditions on Γ_2 constituting the given Cauchy data give the following system of boundary integral equations

$$\begin{cases} \displaystyle\int_{\Gamma_1} \phi_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_2} \phi_2(y)\Phi(x,y)\,ds(y) = f_2(x),\ x\in\Gamma_2, \\ \displaystyle\frac{1}{2}\phi_2(x) + \int_{\Gamma_1} \phi_1(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) + \int_{\Gamma_2} \phi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) = g_2(x),\ x\in\Gamma_2. \end{cases} \tag{19}$$

Clearly, the system (19) can by rewritten as

$$A\vec{\phi} = \vec{f}$$

with $\vec{\phi} = (\phi_1,\phi_2)^\top$ and $\vec{f} = (f_2,g_2)^\top$.

One can show, see Theorem 4.1 in Cakoni and Kress (2007), that the operator $A : L^2(\Gamma_1)\times L^2(\Gamma_2) \to L^2(\Gamma_2)\times L^2(\Gamma_2)$ is injective and has dense range. Therefore, the standard Tikhonov regularization approach can by applied to the system (19).

We point out that once the densities ϕ_1 and ϕ_2 have been found from (18), using the integral representation for the solution u one can construct, for example, the function value f_1 on Γ_1. Then our densities also satisfies the system (8). Thus, as was mentioned after (8), even for smooth data, solutions to (8) will have singularities near the corner points. Therefore, it is expected that solutions to (18) have singularities near the same points, motivating the use of a mesh grading transform also when numerically solving the Cauchy problem.

We are then in a similar situation as for the mixed problem of the previous system, compare (19) with the system (8). Thus, the same techniques and quadratures can be applied. This shows that the Cauchy problem (1)–(3) and the mixed problem (4)–(6), are from the point of numerical solution closely connected via our approach. For the sake of completeness, we outline the details below for the solution of the system (19)

3.1 Parametrization of the boundary integrals (19)

Employing the mesh grading transformation in (19), given in detail in Section 2.2, leads to the following parametric system

$$\begin{cases} \displaystyle\frac{1}{2\pi}\left[\int_0^{2\pi} \varphi_1(\sigma)L_{21}(s,\sigma)\,d\sigma + \int_0^{2\pi} \varphi_2(\sigma)L_{22}(s,\sigma)\,d\sigma\right] = 2f_2(s), \\ \displaystyle\varphi_2(s) + \frac{|z_2'(s)|}{2\pi}\left[\int_0^{2\pi} \varphi_1(\sigma)K_{21}(s,\sigma)\,d\sigma + \int_0^{2\pi} \varphi_2(\sigma)K_{22}(s,\sigma)\,d\sigma\right] = 2g_2(s), \end{cases} \tag{20}$$

for $s\in\mathbb{R}$, where $f_2(s) = f_2(z_2(s))$, $g_2(s) = g_2(z_2(s))|z_2'(s)|$ and $\varphi_\ell(s) = \phi_\ell(z_\ell(s))$ $|z_\ell'(s)|$, $\ell = 1,2$. All kernels are defined as in Section 2.

3.2 Discretisation of (20)

The use of the quadrature method described in Section 2.4, applied to the integrals (20), leads to the following linear system

$$\begin{cases} \sum_{i=1}^{n-1} \left[\varphi_{1,i} \frac{1}{n} L_{21}(s_k, s_i) + \varphi_{2,i} \left(R_{|i-k|} + R_{i+k} + \frac{1}{n} b_2(s_k, s_i) \right) \right] = 2 f_2(s_k), \\ \varphi_{2,k} + \frac{|z_2'(s_k)|}{n} \sum_{i=1}^{n-1} \left[\varphi_{1,i} K_{21}(s_k, s_i) + \varphi_{2,i} K_{22}(s_k, s_i) \right] = 2 g_2(s_k), \end{cases} \tag{21}$$

where $\varphi_{\ell,k} \approx \varphi_\ell(s_k)$, $k = 1, \ldots, n-1$, $\ell = 1, 2$. Similar to (17) the numerical solution of (1)–(3) can be calculated as

$$\tilde{u}(x) = -\frac{1}{2n} \sum_{k=1}^{n-1} \sum_{\ell=1}^{2} \varphi_{\ell,k} \ln |x - z_\ell(s_k)|, \quad x \in D. \tag{22}$$

The matrix corresponding to the system (21) has a large condition number due to the ill-posedness of the Cauchy problem (1)–(3), and therefore to obtain a stable solution regularization of this system is necessary. One possible way is to apply to (21) Tikhonov regularization with a regularization parameter $\alpha > 0$. Although there are optimal choices for α (the discrepancy principle), it is often simpler and faster to use a heuristic choice such as the L-curve rule, see Hansen (2000).

4 Numerical examples

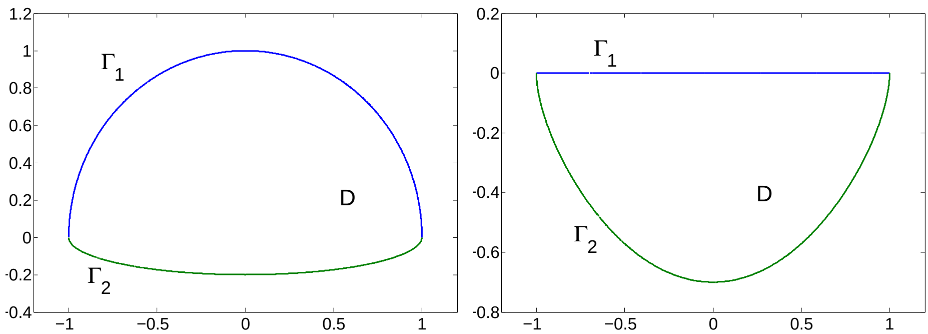
We then demonstrate the applicability of our approach for a direct mixed problem as well as two Cauchy problems. The following L^2-errors will be calculated

$$e_2^{(\ell)} = \sqrt{\frac{1}{n} \sum_{i=1}^{n-1} (u_{ex}(z_\ell(s_i)) - \tilde{u}(z_\ell(s_i)))^2 |z_\ell'(s_i)|}$$

and

$$de_2^{(\ell)} = \sqrt{\frac{1}{n} \sum_{i=1}^{n-1} (\frac{\partial u_{ex}}{\partial \nu}(z_\ell(s_i)) - \frac{\partial \tilde{u}}{\partial \nu}(z_\ell(s_i)))^2 |z_\ell'(s_i)|}$$

for $\ell = 1, 2$. Here, u_{ex} is the sought solution that is known analytically in the first two examples and only numerically in the third example. For both of the solution domains that we shall use, and which are introduced below, one can check that $\mathrm{cap}(\partial D) \neq 1$.



a). Example 1 b). Example 2

Figure 2: Solution domains for the numerical examples

Example 1. Let the bounded domain D be bounded by the union of the boundary curves (see Fig.2a)

$$\Gamma_1 = \{\tilde{z}_1(t) = (\cos t, \sin t), t \in [0,\pi]\}$$

and

$$\Gamma_2 = \{\tilde{z}_2(t) = (\cos t, 0.2\sin t), t \in [\pi, 2\pi]\}.$$

We consider the harmonic function $u_{ex}(x) = x_1^2 - x_2^2$, $x \in D$, and the necessary boundary data functions for the mixed problem (4)–(6) is generated as the necessary restrictions of u_{ex} and its normal derivative to the corresponding boundary parts.

We then investigate our numerical procedure for the mixed boundary value problem (4)–(6) with the above constructed data. In Table 1 are errors for the calculated boundary data on Γ_1 and Γ_2 for various numbers n used to generate the grid points in (13), together with the absolute error $|u_{ex}(x^*) - \tilde{u}(x^*)|$ at the point $x^* = (0,0)$ in the domain D (with $\tilde{u}$ being given by (17)). All results were obtained for the degree $q = 7$ of the polynomial in (9) used for the mesh grading transformation.

Table 1: Errors for the numerical solution in Example 1 (mixed problem)

n	$e_2^{(2)}$	$de_2^{(1)}$	$\lvert u_{ex}(x^*) - \tilde{u}(x^*)\rvert$
8	7.101928E-03	3.202661E-01	3.837358E-03
16	2.934899E-04	5.990996E-02	4.540162E-05
32	1.335491E-07	1.775584E-04	2.861962E-10
64	1.727974E-10	3.578890E-06	7.370935E-13
128	8.345351E-15	3.028298E-07	2.259076E-15

The algebraic convergence of high order is clearly demonstrated. Varying the solution domain gives much the same results and no further issues can be reported, as expected, since the mixed problem is well-posed.

Example 2. We use the same analytical solution u_{ex} and solution domain D as in Example 1, and consider the Cauchy problem (1)–(3). The results of the numerical reconstruction $\tilde{u}$ given by (22) of the function u_{ex} with the proposed method, on the boundary part Γ_1 of the domain D, for the case of exact and noisy data, are presented in Figs.3–5. Here, the degree $q = 3$ in (9) and to generate the grid points $n = 64$ in (13). Note that for noisy data, random errors are added pointwise to the corresponding boundary function, with the percentage given in terms of the L_2 norm. As expected, the reconstruction of the normal derivative is less accurate. Note though, as mentioned in the introduction, our proposed approach is straightforward and requires little computational effort. Thus, we can not expect very accurate results but taking into account the simplicity of the proposed approach and comparing with results in the literature, the employed approach is doing well. One can, to try to improve the reconstructions, use some post processing or filtering, or use the obtained reconstruction as an initial guess in gradient type minimization procedures.

Example 3. Let now the boundary curves Γ_1 and Γ_2 be given as (see Fig.2b)

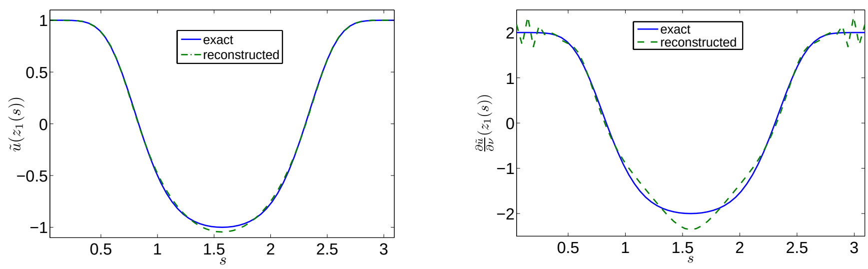
$$\Gamma_1 = \{\tilde{z}_1(t) = (-\frac{2t}{\pi} + 1, 0), t \in [0, \pi]\}$$

and

$$\Gamma_2 = \{\tilde{z}_2(t) = (\cos t, 0.4 \sin t - 0.3 \sin^2 t), t \in [\pi, 2\pi]\}.$$

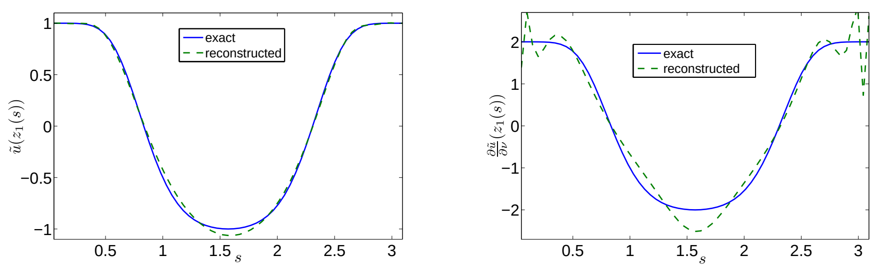
The Cauchy data on Γ_2 are generated by solving the mixed boundary value problem (4)–(6) with

$$f_1(x) = \cos(x_1 + x_2),\ x \in \Gamma_1, \quad g_2(x) = \sin(x_1 + x_2),\ x \in \Gamma_2,$$



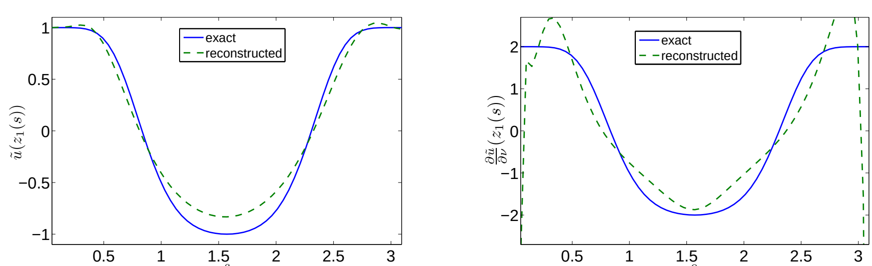
a). The function ($e_2^{(1)} = 0.016$) b). The normal derivative ($de_2^{(1)} = 0.12$)

Figure 3: Reconstruction on the boundary part Γ_1 in Example 2 (exact data, $\alpha = 10^{-10}$)



a). The function ($e_2^{(1)} = 0.034$) b). The normal derivative ($de_2^{(1)} = 0.22$)

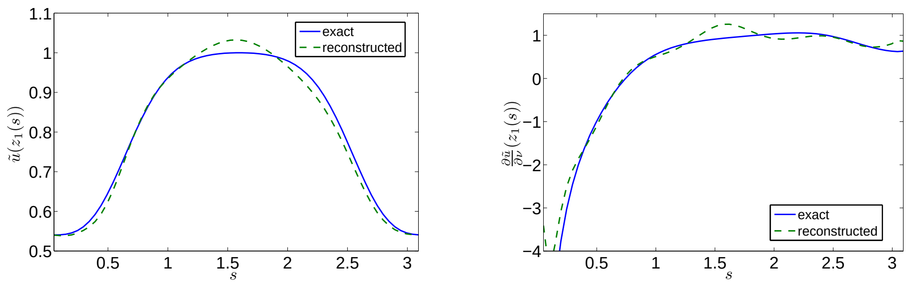
Figure 4: Reconstruction on the boundary Γ_1 in Example 2 (3% noise in the function g_2, $\alpha = 10^{-6}$)



a). The function ($e_2^{(1)} = 0.12$) b). The normal derivative ($de_2^{(1)} = 0.55$)

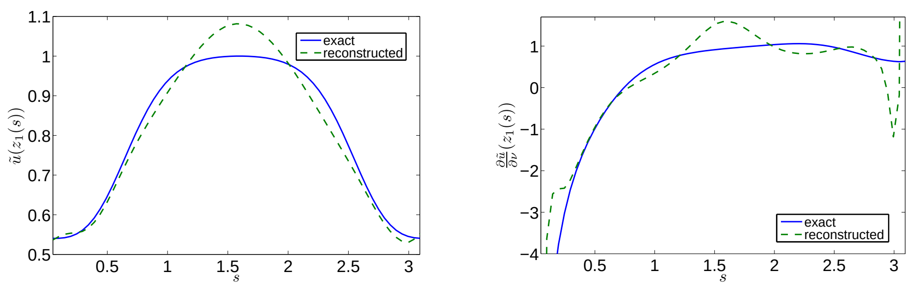
Figure 5: Reconstruction on the boundary Γ_1 in Example 2 (3% noise in the function f_2, $\alpha = 10^{-4}$)

and the obtained numerical solution is denoted u_{ex} (altought not analytically known). The results of the reconstructions for the Cauchy problem (1)–(3) with the proposed approach are presented in Fig.6 and Fig.7. Here, we used the parameters $q = 3$ in (9) and for generating the grid points $n = 64$ in (13). The numerical approximation $\tilde{u}$ is given by (22). Again, as pointed out in the previous example, taking into account the simplicity of the proposed approach, the results are good and comparable with results obtained with more involved methods.



a). The function ($e_2^{(1)} = 0.016$) b). The normal derivative ($de_2^{(1)} = 0.13$)

Figure 6: Reconstruction on the boundary Γ_1 in Example 3 (exact data, $\alpha = 10^{-4}$)



a). The function ($e_2^{(1)} = 0.030$) b). The normal derivative ($de_2^{(1)} = 0.38$)

Figure 7: Reconstruction on the boundary Γ_1 in Example 3 (3% noise in the function g_2, $\alpha = 10^{-3}$)

5 Conclusion

We considered a unified approach for mixed boundary value problems and Cauchy problems in planar simply connected domains having two corner points. The solution is sought as a single-layer potential over the boundary of the solution domain and imposing the given boundary conditions gives, for both problems, a system of boundary integral equations. Via a mesh grading transformation technique, singularities in these boundary integrals could be weakened and the integrals can be extended to all of the boundary, and therefore standard quadrature rules can be used for discretisation. For the Cauchy problem, it was shown that the integral equations had the necessary properties such that Tikhonov regularization could be applied to obtain a stable solution. Numerical examples were presented both for the direct mixed problem as well for Cauchy problems. These confirmed that the proposed approach can give accurate results with small computational effort both for direct and inverse problems. To further improve the reconstructions, some post processing or filtering can be used, or the obtained reconstruction can be used as an initial guess in gradient type minimization procedures.

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