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Collocation Methods to Solve Certain Hilbert Integral Equation with Middle Rectangle Rule

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Abstract: The generalized composite middle rectangle rule for the computation of Hilbert integral is discussed. The pointwise superconvergence phenomenon is presented, i.e., when the singular point coincides with some a priori known point, the convergence rate of the rectangle rule is higher than what is global possible. We proved that the superconvergence rate of the composite middle rectangle rule occurs at certain local coordinate of each subinterval and the corresponding superconvergence error estimate is obtained. By choosing the superconvergence point as the collocation points, a collocation scheme for solving the relevant Hilbert integral equation is presented and an error estimate is established. At last, some numerical examples are provided to validate the theoretical analysis.

Keywords: Hilbert integral, Composite middle rectangle rule , Boundary integral equation, Superconvergence, Error expansion, Collocation methods.

1 Introduction

Consider the Hilbert integral

$$I(f,s) = \fint_{c}^{c+2\pi} \cot\frac{x-s}{2} f(x)dx, s \in (0,2\pi) \tag{1}$$

where $\fint_{c}^{c+2\pi}$ denotes a Hilbert integral and s is the singular point.

There are several different definitions which can be proved equally, such as the definition of subtraction of the singularity, regularity definition, direct definition

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and so on. In this paper we adopt the following one

$$\int_{c}^{c+2\pi} \cot\frac{x-s}{2} f(x)dx = \lim_{\varepsilon\to 0}\left\{\int_{0}^{s-\varepsilon} \cot\frac{x-s}{2} f(x)dx + \int_{s+\varepsilon}^{2\pi} \cot\frac{x-s}{2} f(x)dx\right\}, \quad (2)$$

Recently, more and more mathematicians are interested in numerical approximation (1), such as the Gaussian method [Criscuolo and Mastroianni (1989); Diethelm (1995a); Hasegawa (2004);Monegato (1984)] the Newton-Cote methods [Koler (1997); Amari (1994); Diethelm (1994); Liu Zhang and Wu (2010); Li and Yu (2011a); Li and Yu (2011b); Li Zhang and Yu (2013); Li Yang and Yu (2014); Li Rui and Yu (2014)], spline methods [Orsi (1990);Dagnino and Santi (1990)] and some other method [Natarajan and Mohankumar (1995); Kim and Choi (2000); Kim and Yun (2002); Behforooz (1992); Xu and Yao (1998); Junghanns and Silbermann (1998); Chen and You (1999); Chen and Hong (1999); Chen Yang Lee and Chang (2014)]. The main reason for this interest is probably due to the fact that integral equations have shown to be an adequate tool for the modeling of many physical situations [Yu (2002)], such as acoustics, fluid mechanics, elasticity, fracture mechanics and electromagnetic scattering problems.

The superconvergence phenomenon for hypersingular integral was studied with the density function is replaced by the approximation function while the singular kernel is computed analysis in each subinterval. This methods may be considered as the semi-discrete methods and the order of singularity kernel can be reduced somehow. This idea was firstly presented by [Linz (1985)] in the paper to calculated the hypersingular integral on interval. The superconvergence of composite Newton-Cote rules for Hadamard finite-part integrals and Cauchy principal value integrals were studied in [Wu and Sun (2008)] and [Liu Zhang and Wu (2010)], where the superconvergence rate and the superconvergence point were presented, respectively. In the reference [Feng Zhang and Li (2012)],the midpoint rule for evaluating finite-part integral with the hypersingular kernel $\sin^{-2}\frac{x-s}{2}$ is studied and the pointwise superconvergence phenomenon is also obtained.

This paper focuses on the superconvergence of middle rectangle rule to compute the Hilbert integral on a circle. It is the aim of this paper to investigate the superconvergence phenomenon of rectangle rule for it and, in particular, to derive error estimates. Based on the error functional of the middle rectangle rule, we prove both theoretically and numerically that the composite middle rectangle rule reach the superconvergence rate $O(h^2)$ when the local coordinate of the singular point s is $\pm\frac{2}{3}$. Then a collocation scheme for solving a certain kind of Hilbert integral equation is presented and an optimal error estimate is established.

The rest of this paper is organized as follows. In Sect.2, after introducing some basic formulas of the rectangle rule, the main results is presented. In Sect.3, the proof of the superconvergence phenomenon is completed. In Sect. 4, we present a collocation scheme for solving a certain kind of Hilbert singular integral equation. Based on the superconvergence result, an error estimate of the Hilbert integral equation is presented. Finally, several numerical examples are provided to validate our analysis.

2 Main result

Let $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = c + 2\pi$ be a uniform partition of the interval $[c, c + 2\pi]$ with mesh size $h = 2\pi/n$. Define by $f_C(x)$ the piecewise constant interpolant for $f(x)$

$$f_C(x) = f(\hat{x}_i), \ \ \hat{x}_i = x_{i-1} + h/2 \ \ i = 1, 2, \cdots, n \tag{3}$$

and a linear transformation

$$x = \hat{x}_i(\tau) := (\tau + 1)(x_{i+1} - x_i)/2 + x_i, \ \ \tau \in [-1, 1], \tag{4}$$

from the reference element $[-1, 1]$ to the subinterval $[x_i, x_{i+1}]$. Replacing $f(x)$ in Eq. 1 with $f_C(x)$ gives the composite rectangle rule:

$$I_n(f;s) := \fint_c^{c+2\pi} \cot\frac{x-s}{2} f_C(x)dx = \sum_{i=0}^{n-1} \omega_i(s) f(x_i) = I(f,s) - E_n(f;s), \tag{5}$$

where $\omega_i(s)$ denotes the Cote coefficients given by

$$\omega_i(s) = 2\log\left|\frac{\sin 0.5(x_{i+1} - s)}{\sin 0.5(x_i - s)}\right| \tag{6}$$

and $E_n(f,s)$ the error functional.
We also define

$$k_s(x) = \begin{cases} (x-s)\cot\dfrac{x-s}{2}, & x \neq s, \\ 2, & x = s. \end{cases} \tag{7}$$

In the following analysis, C will denote a generic constant which is independent of h and s and it may have different values in different places.

Theorem 1 *Assume $f(x) \in C^{\alpha}[a,b], \alpha \in (0,1]$. For the middle rectangle rule $I_n(f,s)$ defined as Eq. 5. Assume that $s = x_m + (1+\tau)h/2$, there exist a positive constant C, independent of h and s, such that*

$$|E_n(f;s)| \leq C(|\ln h| + |\ln\gamma(\tau)|)h^{\alpha}, \quad \alpha \in (0,1], \tag{8}$$

where

$$\gamma(\tau) = \min_{0\leq i\leq n} \frac{|s-x_i|}{h} = \frac{1-|\tau|}{2}. \tag{9}$$

Proof: By setting $R(x) = f(x) - f_C(x)$, as $f(x) \in C^{\alpha}[a,b]$, then we have $|R(x)| \leq Ch^{\alpha}$ and

$$\begin{aligned} E_n(f;s) &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \cot\frac{x-s}{2}\,[f(x) - f_C(x)]\,dx \\ &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \cot\frac{x-s}{2} R(x)dx \end{aligned} \tag{10}$$

By the definition of $k_s(x)$, we have

$$\begin{aligned} \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \cot\frac{x-s}{2} R(x)dx &= 2\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{R(x)}{x-s}dx \\ &+ \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{k_s(x)-2}{x-s} R(x)dx \end{aligned} \tag{11}$$

For the first part of Eq. 11,by the definition of

$$\int_a^b \frac{f(x)}{x-s}dx = \int_a^b \frac{f(x)-f(s)}{x-s}dx + f(s)\ln\left|\frac{b-s}{s-a}\right|, \tag{12}$$

then we have

$$\int_c^{c+2\pi} \frac{R(x)}{x-s}dx = \int_c^{c+2\pi} \frac{R(x)-R(s)}{x-s} + R(s)\ln\frac{x_{m+1}-s}{s-x_m}, \tag{13}$$

and

$$\begin{aligned} \left|\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{R(x)}{x-s}dx\right| &= \left|\int_{x_m}^{x_{m+1}} \frac{R(x)}{x-s}dx + \sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{R(x)}{x-s}dx\right| \\ &\leq \left|\int_c^{c+2\pi} \frac{Rx)-R(s)}{x-s}dx\right| + \left|R(s)\ln\frac{x_{m+1}-s}{s-x_m}\right| \\ &+ Ch^{\alpha} \sum_{i=0,i\neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{|x-s|}dx \\ &\leq Ch^{\alpha}|\ln\gamma(\tau)|. \end{aligned} \tag{14}$$

For the second part of Eq. 11,we have

$$\begin{aligned}\left|\sum_{i=0}^{n-1} \fint_{x_i}^{x_{i+1}} \frac{k_s(x)-2}{x-s} R(x)dx\right| &= \left|\fint_{c}^{c+2\pi} \frac{k_s(x)-2}{x-s} R(x)dx\right| \\ &\leq Ch^{\alpha} \fint_{c}^{c+2\pi} \left|\frac{k_s(x)-2}{x-s}\right| dx \\ &= Ch^{\alpha} \fint_{c}^{c+2\pi} \left|\cot\frac{x-s}{2}\right| dx + \fint_{c}^{c+2\pi} \left|\frac{2}{x-s}\right| dx \\ &\leq Ch^{\alpha} |\ln\gamma(\tau)|.\end{aligned} \tag{15}$$

Combining Eq. 14 and Eq. 15 together, we get Eq. 8 and the proof of Theorem 1 is completed.
Setting

$$I_{n,i}(s) = \begin{cases} \fint_{x_m}^{x_{m+1}} (x-\hat{x}_m)\cot\frac{x-s}{2} dx, & i = m, \\ \int_{x_i}^{x_{i+1}} (x-\hat{x}_i)\cot\frac{x-s}{2} dx, & i \neq m. \end{cases} \tag{16}$$

Lemma 1 *Assume $s = x_m + (\tau+1)h/2$ with $\tau \in (-1,1)$. Let $I_{n,i}(s)$ be defined by Eq. 16, then there holds that*

$$\begin{aligned} I_{n,i}(s) &= h\sum_{k=1}^{\infty} \frac{1}{k} \left[\cos k(x_{i+1}-s) + \cos k(x_i - s)\right] \\ &+ \sum_{k=1}^{\infty} \frac{1}{k^2} \left[\sin k(x_{i+1}-s) - \sin k(x_i - s)\right]. \end{aligned} \tag{17}$$

Proof For $i = m$,by the definition of Cauchy principal value integral, we have

$$\begin{aligned} I_{n,m}(s) &= \lim_{\varepsilon\to 0}\left(\int_{x_m}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_{m+1}}\right)(x-\hat{x}_m)\cot\frac{x-s}{2}dx \\ &= -h\ln\left|2\sin\frac{x_m-s}{2}\right| - h\ln\left|2\sin\frac{x_{m+1}-s}{2}\right| \\ &+ 2\int_{x_m}^{x_{m+1}} \ln\left|2\sin\frac{x-s}{2}\right| dx. \end{aligned} \tag{18}$$

For $i \neq m$, taking integration by parts on the correspondent Riemann integral, we have

$$
\begin{aligned}
I_{n,i}(s) &= -h\ln\left|2\sin\frac{x_i - s}{2}\right| - h\ln\left|2\sin\frac{x_{i+1}-s}{2}\right| \\
&+ 2\int_{x_i}^{x_{i+1}} \ln\left|2\sin\frac{x-s}{2}\right| dx. \qquad (19)
\end{aligned}
$$

Now, by using the well-known identity

$$
\ln\left|2\sin\frac{x}{2}\right| = -\sum_{n=1}^{\infty}\frac{1}{n}\cos nx. \qquad (20)
$$

The proof of Lemma1 is completed.

Lemma 2 *Under the same assumptions of Lemma 1, there holds that*

$$
\sum_{i=0}^{n-1} I_{n,i}(s) = -2h\ln 2\cos\frac{\tau\pi}{2}. \qquad (21)
$$

Proof ByEq. 17, we have

$$
\begin{aligned}
\sum_{i=0}^{n-1} I_{n,i}(s) &= h\sum_{i=0}^{n-1}(\sum_{k=1}^{\infty}\frac{1}{k}\left(\cos k(x_{i+1}-s)+\cos k(x_i - s)\right) \\
&+ \sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sin k(x_{i+1}-s) - \sin k(x_i - s)\right)) \\
&= 2h\sum_{k=1}^{\infty}\sum_{i=0}^{n-1}\frac{1}{k}\cos k(x_i - s) \\
&= 2h\sum_{k=1}^{\infty}\frac{n\cos k(x_i - s)}{k} \\
&= 2h\sum_{k=1}^{\infty}\frac{\cos j(1+\tau)\pi}{j} \\
&= -2h\ln 2\sin\frac{(1+\tau)\pi}{2} \\
&= -2h\ln 2\cos\frac{\tau\pi}{2} \qquad (22)
\end{aligned}
$$

where we have used

$$\sum_{i=0}^{n-1} \cos k(x_i - s) = \begin{cases} n\cos k(x_1 - s), & k = nj, \\ 0, & k \neq nj. \end{cases} \tag{23}$$

The proof of Lemma 2 is completed.

Theorem 2 *Assume $f(x) \in C^2[a,b]$. For the middle rectangle rule $I_n(f,s)$ defined as Eq. 5. Assume that $s = x_m + (1+\tau)h/2$, there exist a positive constant C, independent of h and s, such that*

$$E_n(f;s) = -2hf'(s)\ln 2\cos\frac{\tau\pi}{2} + \mathscr{R}_n(s), \tag{24}$$

where

$$|\mathscr{R}_n(s)| \leq C\max\{|k_s(x)|\}(|\ln h| + |\ln\gamma(\tau)|)h^2 \tag{25}$$

and $\gamma(\tau)$ is defined as Eq. 9.

It is known that the global convergence rate of the composite middle rectangle rule is lower than Riemann integral.
For $\ln 2\cos\frac{\tau\pi}{2} = 0$, which means $\tau = \pm\frac{2}{3}$, then we have

Corollary 1 *Under the same assumption of Theorem 2,we have*

$$|E_n(f,s)| \leq C|\ln h|h^2. \tag{26}$$

Based on the Theorem 2, we present the modify rectangle rule

$$\tilde{I}_n(f;s) = I_n(f;s) - 2hf'(s)\ln 2\cos\frac{\tau\pi}{2}, \tag{27}$$

and

$$\tilde{E}_n(f;s) = I(f;s) - \tilde{I}_n(f;s) \tag{28}$$

then we have

Corollary 2 *Under the same assumption of Theorem 2,we have*

$$\tilde{E}_n(f;s) \leq C\max\{|k_s(x)|\}(|\ln h| + |\ln\gamma(\tau)|)h^2. \tag{29}$$

where $\gamma(\tau)$ is defined as Eq. 9.

3 Proof of the Theorem 2

In this section, we study the superconvergence of the composite rectangle rule for Hilbert singular integrals.

3.1 Preliminaries

Lemma 3 *Under the same assumptions of Theorem 2, it holds that*

$$f(x) - f_C(x) = f'(s)(x-\hat{x}_i) + R_f^1(x) + R_f^2(x) \tag{30}$$

where

$$R_f^1(x) = f''(\eta_i)(x-s)(x-\hat{x}_i) \tag{31}$$

$$R_f^2(x) = -\frac{f''(\xi_i)}{2}(x-\hat{x}_i)^2 \tag{32}$$

and $\eta_i, \xi_i \in (x_i, x_{i+1})$.

Proof: Performing Taylor expansion of $f_C(x)$ at the point x, we have

$$f_C(x) = f(x) + f'(x)(\hat{x}_i - x) + \frac{f''(\eta_i)}{2}(\hat{x}_i - x)^2. \tag{33}$$

Similarly, we have

$$f'(x) = f'(s) + f''(\xi_i)(x-s). \tag{34}$$

Combining Eq. 33 and Eq. 34 together complete the proof.
Setting

$$E_m(x) = f(x) - f_C(x) - f'(s)(x-\hat{x}_m). \tag{35}$$

Lemma 4 *Let $f(x) \in C^2[a,b]$, denote $E_m(x)$ to be the error functional for the composite rectangle rule, assume $s \neq x_i$ for any $i = 1,2,\cdots,n$, then there holds*

$$\left| \int_{x_m}^{x_{m+1}} \cot\frac{x-s}{2} E_m(x)dx \right| \leq Ch^2 |\ln\gamma(\tau)|. \tag{36}$$

Proof: As $f(x) \in C^2[a,b]$, we get $E_i(x) \in C^2[a,b]$. Then we have

$$\int_{x_m}^{x_{m+1}} \cot\frac{x-s}{2} E_m(x)dx \quad = 2\int_{x_m}^{x_{m+1}} \frac{E_m(x)}{x-s}dx + \int_{x_m}^{x_{m+1}} \frac{k_s(x)-2}{x-s}E_m(x)dx. \tag{37}$$

For the first part of Eq. 37, following the identity

$$\fint_a^b \frac{f(x)}{x-s}dx = \int_a^b \frac{f(x)-f(s)}{x-s}dx + f(s)\ln\left|\frac{b-s}{s-a}\right|, \tag{38}$$

we have

$$\fint_{x_m}^{x_{m+1}} \frac{E_m(x)}{x-s}dx = \int_{x_m}^{x_{m+1}} \frac{E_m(x)-E_m(s)}{x-s}dx + E_m(s)\ln\frac{x_{m+1}-s}{s-x_m}, \tag{39}$$

then we get

$$\begin{aligned}\left|\fint_{x_m}^{x_{m+1}} \frac{E_m(x)}{x-s}dx\right| &\le \left|\int_{x_m}^{x_{m+1}} \frac{E_m(x)-E_m(s)}{x-s}dx\right| + \left|E_m(s)\ln\frac{x_{m+1}-s}{s-x_m}\right| \\ &\le Ch^2|\ln\gamma(\tau)|.\end{aligned} \tag{40}$$

For the second part of Eq. 37, we have

$$\begin{aligned}&\left|\fint_{x_m}^{x_{m+1}} \frac{k_s(x)-2}{x-s}E_m(x)dx\right| \\ &\le Ch^2\int_{x_m}^{x_{m+1}}\left|\frac{k_s(x)-2}{x-s}\right|dx \\ &= Ch^2\left(\fint_{x_m}^{x_{m+1}}\left|\cot\frac{x-s}{2}\right|dx + \fint_{x_m}^{x_{m+1}}\frac{2}{|x-s|}dx\right) \\ &\le Ch^2|\ln\gamma(\tau)|.\end{aligned} \tag{41}$$

Combining Eq. 40 and Eq. 41 together, we get Eq. 36 and the proof of Lemma 4 is completed.

Proof of Theorem 2: By Lemma 3, we have

$$\begin{aligned}&\left(\int_c^{x_m} + \int_{x_{m+1}}^{c+2\pi}\right)\cot\frac{x-s}{2}(f(x)-f_C(x))dx \\ &= \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\cot\frac{x-s}{2}(f(x)-f_C(x))dx \\ &= f'(s)\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}(x-\hat{x}_i)\cot\frac{x-s}{2}dx \\ &+ \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}R_f^1(x)\cot\frac{x-s}{2}dx \\ &+ \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}R_f^2(x)\cot\frac{x-s}{2}dx.\end{aligned} \tag{42}$$

By the definition of $E_m(x)$, we have

$$\begin{aligned} \fint_{x_m}^{x_{m+1}} (f(x)-f_C(x))\cot\frac{x-s}{2}\,dx &= \fint_{x_m}^{x_{m+1}} E_m(x)\cot\frac{x-s}{2}\,dx \\ &+ f'(s)\fint_{x_m}^{x_{m+1}} (x-\hat{x}_m)\cot\frac{x-s}{2}\,dx. \end{aligned} \tag{43}$$

Putting Eq. 42 and Eq. 43 together yields

$$\begin{aligned} \fint_{c}^{c+2\pi} (f(x)-f_C(x))\cot\frac{x-s}{2}dx &= \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} (f(x)-f_C(x))\cot\frac{x-s}{2}dx \\ &+ \fint_{x_m}^{x_{m+1}} (f(x)-f_C(x))\cot\frac{x-s}{2}dx \\ &= -2hf'(s)\ln 2\cos\frac{\tau\pi}{2} + \mathscr{R}_n(s) \end{aligned} \tag{44}$$

where

$$\mathscr{R}_n(s) = R_1 + R_2$$

and

$$R_1 = \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} R_f^1(x)\cot\frac{x-s}{2}dx + \sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} R_f^2(x)\cot\frac{x-s}{2}dx \tag{45}$$

$$R_2 = \fint_{x_m}^{x_{m+1}} E_m(x)\cot\frac{x-s}{2}\,dx \tag{46}$$

Now we estimate $\mathscr{R}_n(s)$ term by term. For the first part of R_1,we have

$$\begin{aligned} &\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} R_f^2(x)\cot\frac{x-s}{2}dx\right| \\ &= \left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} k_s(x)\frac{R_f^2(x)}{x-s}dx\right| \\ &\leq C\max\{|k_s(x)|\}h^2\left(\int_c^{x_m}\frac{1}{s-x}dx + \int_{x_{m+1}}^{c+2\pi}\frac{1}{x-s}dx\right) \\ &= C\max\{|k_s(x)|\}h^2\ln\frac{(c+2\pi-s)(s-c)}{(x_{m+1}-s)(s-x_m)} \\ &\leq C\max\{|k_s(x)|\}(|\ln h| + |\ln\gamma(\tau)|)h^2. \end{aligned} \tag{47}$$

For the second part of R_1,there are no singularity and we have

$$
\begin{aligned}
&\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} R_f^1(x)\cot\frac{x-s}{2}dx\right| \\
&=\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} \frac{k_s(x)R_f^1(x)}{x-s}dx\right| \\
&=\left|\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}} k_s(x)f''(\eta_i)(x-\hat{x}_i)dx\right| \\
&\leq \sum_{i=0,i\neq m}^{n-1} |k_s(x)f''(\eta_i)|\int_{x_i}^{x_{i+1}}|x-\hat{x}_i|dx \\
&=\sum_{i=0,i\neq m}^{n-1} |k_s(x)f''(\eta_i)|\left(\int_{x_i}^{\hat{x}_i}(\hat{x}_i-x)dx+\int_{\hat{x}_i}^{x_{i+1}}(x-\hat{x}_i)dx\right) \\
&\leq C\max\{|k_s(x)|\}h^2.
\end{aligned}
\tag{48}
$$

For $\mathscr{R}_n(s)$, with Lemma 4, the we get

$$
|\mathscr{R}_n(s)|\leq|R_1|+|R_2|\leq C\max\{|k_s(x)|\}(|\ln h|+|\ln\gamma(\tau)|)h^2 \tag{49}
$$

and the proof is completed.

4 Collocation scheme for Hilbert singular integral equation of first kind

In this section, we consider the integral equation

$$
\frac{1}{2\pi}\int_0^{2\pi} f(x)\cot\frac{x-s}{2}\,dx=g(s),\ \ s\in(0,2\pi), \tag{50}
$$

with the compatibility condition

$$
\int_0^{2\pi} g(x)\,dx=0. \tag{51}
$$

As in [Yu (2002)], under the condition Eq. 51, there exists a unique solution for the integral equation Eq. 50. In order to arrive at a unique solution, we adopt following condition

$$
\int_0^{2\pi} f(x)\,dx=0. \tag{52}
$$

By choosing the middle points $\hat{x}_k = x_{k-1} + h/2 (k = 1, 2, \cdots, n)$ of each subintervals, we get the composite rectangle rule $I_n(f;s)$ to approximate the Hilbert singular integral in Eq. 50, then we have the following linear system

$$\frac{1}{\pi}\sum_{m=1}^{n}\left[\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| - \log\left|\sin\frac{(\hat{x}_k - x_{m-1})}{2}\right|\right] f_m = g(\hat{x}_k), \quad k = 1, 2, \cdots, n, \tag{53}$$

and written as the matrix expression as

$$A_n \mathbf{F}_n^a = \mathbf{G}_n^e, \tag{54}$$

where

$$A_n = (a_{km})_{n\times n},$$

$$a_{km} = \frac{1}{\pi}\left[\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| - \log\left|\sin\frac{(\hat{x}_k - x_{m-1})}{2}\right|\right], k, m = 1, 2, \cdots, n, \tag{55}$$

$$\mathbf{F}_n^a = (f_1, f_2, \cdots, f_n)^T, \mathbf{G}_n^e = (g(\hat{x}_1), g(\hat{x}_2), \cdots, g(\hat{x}_n))^T,$$

here $f_k (k = 1, 2, \cdots, n)$ denote the numerical solution of f at $\hat{x}_k$. By directly calculation, we get A_n is not only a symmetric Toeplitz matrix but also a circulant matrix. As for any $k = 1, 2, \cdots, n$,

$$\sum_{m=1}^{n} a_{km} = \frac{1}{\pi}\sum_{m=1}^{n}\left[\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| - \log\left|\sin\frac{(\hat{x}_k - x_{m-1})}{2}\right|\right] = 0, \tag{56}$$

from Eq. 56, we know that A_n is singular matrix, then we can not use system Eq. 53 or Eq. 54 to solve the integral equation Eq. 50.

In order to get a well-conditioned definite system, we introduce a regularizing factor γ_{0n} in Eq. 53, which leads to linear system

$$\begin{cases} \gamma_{0n} + \dfrac{1}{\pi}\displaystyle\sum_{m=1}^{n}\left[\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| - \log\left|\sin\frac{(\hat{x}_k - x_{m-1})}{2}\right|\right] f_m = g(\hat{x}_k), \\ \displaystyle\sum_{m=1}^{n} f_m = 0, \end{cases} \tag{57}$$

where γ_{0n} defined by

$$\gamma_{0n} = \frac{1}{2\pi}\sum_{k=1}^{n} g(\hat{x}_k) h. \tag{58}$$

Then the matrix form of system Eq. 57 can be presented as

$$A_{n+1}\mathbf{F}^a_{n+1} = \mathbf{G}^e_{n+1}, \tag{59}$$

where

$$\begin{aligned} A_{n+1} &= \begin{pmatrix} 0 & e_n^T \\ e_n & A_n \end{pmatrix}, \\ \mathbf{F}^a_{n+1} &= \begin{pmatrix} \gamma_{0n} \\ \mathbf{F}^a_n \end{pmatrix}, \mathbf{G}^e_{n+1} = \begin{pmatrix} 0 \\ \mathbf{G}^e_n \end{pmatrix}, \end{aligned} \tag{60}$$

and $e_n = (\underbrace{1,1,\cdots,1}_{n})^T$. Then the linear system Eq. 57 can be written by

$$\begin{cases} \gamma_{0n} + \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n} -\dfrac{f_{m+1}-f_m}{h}\log\left|\sin\dfrac{(\hat{x}_k - x_m)}{2}\right| h = g(\hat{x}_k), \quad k = 1,2,\cdots,n, \\ -\dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n}\dfrac{f_{m+1}-f_m}{h}h = 0, \end{cases} \tag{61}$$

where we have used $f_1 = f_{n+1}$.
Let $v_m = -(f_{m+1} - f_m)/h$, we get

$$\begin{cases} \gamma_{0n} + \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n}\log\left|\sin\dfrac{(\hat{x}_k - x_m)}{2}\right| v_m h = g(\hat{x}_k), \quad k = 1,2,\cdots,n, \\ \dfrac{1}{2\pi}\displaystyle\sum_{m=1}^{n} v_m h = 0. \end{cases} \tag{62}$$

Lemma 5 *(Theorem 6.2.1, §6.2, [Lifanov and Poltavskii (2004)]) For the linear system Eq. 62, its solution has the following expression*

$$v_m = -\frac{h}{2\pi}\sum_{k=1}^{n}\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| g(\hat{x}_k). \tag{63}$$

Lemma 5 has been proved in [Lifanov and Poltavskii (2004)], which will be used in the proof of the following Theorem.

Lemma 6 *Let the inverse matrix of A_{n+1} to be $\mathscr{B}_{n+1} = (b_{ik})_{(n+1)\times(n+1)}$, defined in Eq. 59. Then we vave,* *(1) B_{n+1} has an expression of the form*

$$B_{n+1} = \begin{pmatrix} b_{00} & B_1 \\ B_2 & B_n \end{pmatrix}, \tag{64}$$

where

$$B_1 = (b_{01}, b_{02}, \cdots, b_{0n}), B_2 = (b_{10}, b_{20}, \cdots, b_{n0})^T, \tag{65}$$

$$b_{i0} = b_{0i} = \frac{1}{n}, 1 \le i \le n, \tag{66}$$

$$b_{ik} = \frac{h^2}{2\pi}\left[\sum_{m=i}^{n-1} \log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| - \frac{1}{n}\sum_{m=1}^{n-1} m \log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right|\right], \quad 1 \le i \le n-1, 1 \le k \le n, \tag{67}$$

$$b_{nk} = -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1} m \log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right|, 1 \le k \le n. \tag{68}$$

(2) B_n is both Toeplitz matrix and circulant matrix.

(3) There exist a positive constant C such that

$$\sum_{k=1}^{n} |b_{ik}| \le C, \tag{69}$$

for $i = 1, 2, \cdots, n$.

Proof: (1) Based on the last equation of Eq. 57, we can easily get

$$0 = \sum_{m=1}^{n} f_m = -h\sum_{m=1}^{n-1} m\frac{f_{m+1} - f_m}{h} + nf_n = h\sum_{m=1}^{n-1} mv_m + nf_n,$$

combining with Eq. 63, so we have

$$f_n = -\frac{h}{n}\sum_{m=1}^{n-1} mv_m = -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1}\sum_{k=1}^{n} m\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| g(\hat{x}_k), \tag{70}$$

which yields Eq. 68. Then, by using Eq. 63 once more, we can get

$$\begin{aligned} f_i &= h\sum_{m=i}^{n-1} v_m + f_n \\ &= \frac{h^2}{2\pi}\sum_{m=i}^{n-1}\sum_{k=1}^{n} \log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| g(\hat{x}_k) - \frac{h^2}{2n\pi}\sum_{m=1}^{n-1}\sum_{k=1}^{n} m\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| g(\hat{x}_k) \\ &= \frac{h^2}{2\pi}\sum_{k=1}^{n}\left[\sum_{m=i}^{n-1} \log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right| - \frac{1}{n}\sum_{m=1}^{n-1} m\log\left|\sin\frac{(\hat{x}_k - x_m)}{2}\right|\right] g(\hat{x}_k), \end{aligned}$$

which leads to Eq. 67. Proof of Eq. 66 will be given in next section.
(2) As

$$
\begin{aligned}
&-\frac{1}{n}\sum_{m=1}^{n-1} m\log\left|\sin\frac{(\hat{x}_{k+1}-x_m)}{2}\right| \\
&=-\frac{1}{n}\sum_{m=1}^{n-1} m\log\left|\sin\frac{\hat{x}_{k}-x_{m-1}}{2}\right| \\
&=-\frac{1}{n}\sum_{m=0}^{n-2} (m+1)\log\left|\sin\frac{\hat{x}_{k}-x_{m}}{2}\right| \\
&=-\frac{1}{n}\left[\sum_{m=1}^{n-2} m\log\left|\sin\frac{\hat{x}_{k}-x_{m}}{2}\right|+\sum_{m=0}^{n-2}\log\left|\sin\frac{\hat{x}_{k}-x_{m}}{2}\right|\right] \\
&=-\frac{1}{n}\left[\sum_{m=1}^{n-1} m\log\left|\sin\frac{\hat{x}_{k}-x_{m}}{2}\right|+\sum_{m=0}^{n-1}\log\left|\sin\frac{\hat{x}_{k}-x_{m}}{2}\right|-n\log\left|\sin\frac{\hat{x}_{k}-x_{n-1}}{2}\right|\right] \\
&=\log\left|\sin\frac{\hat{x}_{k}-x_{n-1}}{2}\right|-\frac{1}{n}\sum_{m=1}^{n-1} m\log\left|\sin\frac{\hat{x}_{k}-x_{m}}{2}\right|,
\end{aligned}
\tag{71}
$$

for $i=1,2,\cdots,n-2$, we have

$$
\begin{aligned}
&b_{i+1,k+1} \\
&=\frac{h^2}{2\pi}\left[\sum_{m=i+1}^{n-1}\log\left|\sin\frac{\hat{x}_{k+1}-x_m}{2}\right|-\frac{1}{n}\sum_{m=1}^{n-1} m\log\left|\sin\frac{\hat{x}_{k+1}-x_m}{2}\right|\right] \\
&=\frac{h^2}{2\pi}\left[\sum_{m=i}^{n-2}\log\left|\sin\frac{\hat{x}_{k}-x_m}{2}\right|+\log\left|\sin\frac{\hat{x}_{k}-x_{n-1}}{2}\right|-\frac{1}{n}\sum_{m=1}^{n-1} m\log\left|\sin\frac{\hat{x}_{k}-x_m}{2}\right|\right] \\
&=b_{ik},
\end{aligned}
\tag{72}
$$

where we have used

$$
\sum_{m=0}^{n-1}\log\left|\sin\frac{\hat{x}_k-x_m}{2}\right|=0. \tag{73}
$$

Moreover, from the equation Eq. 71, we get

$$
b_{n,k+1}=b_{n-1,k}, for\ k=1,2,\cdots,n-1. \tag{74}
$$

Combining Eq. 72 and Eq. 74 together, we show that B_n is a Toeplitz matrix. As

$$\begin{aligned} b_{1k} &= \frac{h^2}{2\pi}\left[\sum_{m=1}^{n-1}\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right| - \frac{1}{n}\sum_{m=1}^{n-1} m\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right|\right] \\ &= \frac{h^2}{2\pi}\left[\sum_{m=1}^{n}\cot\frac{\hat{x}_k - x_m}{2} - \frac{1}{n}\sum_{m=1}^{n} m\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right|\right] \\ &= -\frac{h^2}{2n\pi}\sum_{m=1}^{n} m\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right|, \end{aligned}$$

and

$$\begin{aligned} b_{n,k-1} &= -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1}\log\left|\sin\frac{\hat{x}_{k-1} - x_m}{2}\right| \\ &= -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1}\log\left|\sin\frac{\hat{x}_k - x_{m+1}}{2}\right| \\ &= -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1}(m+1)\log\left|\sin\frac{\hat{x}_k - x_{m+1}}{2}\right| + \frac{h^2}{2n\pi}\sum_{m=1}^{n-1}\log\left|\sin\frac{\hat{x}_k - x_{m+1}}{2}\right| \\ &= -\frac{h^2}{2n\pi}\sum_{m=2}^{n} m\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right| + \frac{h^2}{2n\pi}\sum_{m=2}^{n}\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right| \\ &= -\frac{h^2}{2n\pi}\sum_{m=1}^{n} m\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right| + \frac{h^2}{2n\pi}\sum_{m=1}^{n}\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right| \\ &= -\frac{h^2}{2n\pi}\sum_{m=1}^{n} m\log\left|\sin\frac{\hat{x}_k - x_m}{2}\right|, \end{aligned}$$

we have $b_{n,k-1} = b_{1k}$ for $k = 2,3,\cdots,n$, which show that B_n is also a circulant matrix by noting that B_n is a Toeplitz matrix.

Since B_{n+1} is the inverse matrix of A_{n+1}, and A_{n+1} is symmetric, we see that B_{n+1} is also symmetric, we have

$$b_{j0} = b_{0j}, \tag{75}$$

for $j = 1,\cdots,n$.

By multiplying the ith row of B_{n+1} with the ith column of A_{n+1} , we have

$$b_{i0} + \sum_{j=1}^{n} b_{ij}a_{ji} = 1, 1 \le i \le n,$$

which can be written as

$$b_{i0} = 1 - \sum_{j=1}^{n} b_{ij} a_{ji}, 1 \le i \le n. \tag{76}$$

The first row of matrix B_{n+1} to be multiply with the first column of matrix A_{n+1}, we obtain

$$\sum_{j=1}^{n} b_{0j} = 1. \tag{77}$$

Combining Eq. 75, Eq. 76 and Eq. 77 leads to $b_{i0} = b_{0k} = 1/n$.

(3) In order to prove Eq. 69, we just consider the case $k = n$ because of B_n is a circulant matrix. Then we have

$$b_{nk} = \frac{1}{2n\pi} \sum_{m=1}^{n} (\hat{x}_m - x_{k-1}) \log \left| \sin \frac{\hat{x}_m - x_{k-1}}{2} \right| h + \frac{h}{n} \log \left| \sin \frac{x_{k-1} - \hat{x}_n}{2} \right|. \tag{78}$$

We know that the first term in the righthand of Eq. 78 which can be considered as the middle rectangle quadrature of the integral

$$\frac{1}{2n\pi} \int_0^{2\pi} (x-s) \log \left| \sin \frac{x-s}{2} \right| dx = \frac{1}{2n\pi} \left[J_1(2\pi - s) + J_1(s) \right] - \frac{\log 2}{n}, \tag{79}$$

with $s = x_{k-1}$, where we have used the identity (See [Cvijović (2008)])

$$J_1(s) = \int_0^s t \log 2 \sin \frac{t}{2} dt. \tag{80}$$

The integrand function in Eq. 79 is continuous function except one point at s, from the error estimate of the middle rectangle for Riemann integrals, we have

$$-\frac{1}{2n\pi} \sum_{m=1}^{n} \hat{x}_m \log \left| \sin \frac{x_{k-1} - \hat{x}_m}{2} \right| h = \frac{1}{2n\pi} \left[J_1(2\pi - x_{k-1}) + J_1(x_{k-1}) \right] + O(h^2).$$

Based on Eq. 80, for any $k = 1, 2, \cdots, n$, which leads to

$$|b_{nk}| \le \frac{1}{2n\pi} \left[|J_1(2\pi - x_{k-1})| + |J_1(x_{k-1})| \right] + O(h^3) + \left| \frac{h}{n} \log \left| \sin \frac{h}{4} \right| \right| \le \frac{C}{n}, \tag{81}$$

where we has been used the inequality

$$\left| \log \left| \sin \frac{\hat{x}_n - x_{k-1}}{2} \right| \right| \le \left| \log \left| \sin \frac{h}{4} \right| \right|, k = 1, 2, \cdots, n.$$

Therefore, Eq. 69 can be obtained from Eq. 81.

Now we present our main result of this section.

Theorem 3 *Assume that $f(x)$, the solution of the Hilbert singular integral equation Eq. 50, belongs to $C^2[0,2\pi]$. Then, for the linear system Eq. 57 or Eq. 59, we get the error estimate as below*

$$\max_{1\le i\le n} |f(\hat{x}_i) - f_i| \le Ch^2[1 + |\ln h|]. \tag{82}$$

Proof: Let $\mathbf{F}^e_{n+1} = (0, f(\hat{x}_1), f(\hat{x}_2), \cdots, f(\hat{x}_n))^T$ be the exact vector. Then, from Eq. 59, we have

$$\mathbf{F}^e_{n+1} - \mathbf{F}^a_{n+1} = B_{n+1}(A_{n+1}\mathbf{F}^e_{n+1} - \mathbf{G}^e_{n+1}), \tag{83}$$

which implies

$$f(\hat{x}_i) - f_i = b_{i0}\sum_{m=1}^{n} f(\hat{x}_m) + \sum_{k=1}^{n} b_{ik}E_n(f;\hat{x}_k), i = 1, 2, \cdots, n, \tag{84}$$

where $\{b_{ik}\}$ are the entries of B_{n+1} and $E_n(f;\hat{x}_k)$ is defined in Eq. 24. By Eq. 66 and Eq. 69, we obtain

$$\begin{aligned} |f(\hat{x}_i) - f_i| &\le \frac{1}{2\pi}\left|\sum_{m=1}^{n} f(\hat{x}_m)h\right| + \sum_{k=1}^{n} |b_{ik}||E_n(f;\hat{x}_k)| \\ &\le Ch^2 + Ch^2|\ln h|\sum_{k=1}^{n} |b_{ik}| \le Ch^2[1 + |\ln h|]. \end{aligned}$$

where $\sum_{m=1}^{n} f(\hat{x}_m)h$ is the rectangle rule of the Rimemann integral Eq. 52 with accuracy $O(h^2)$ has been used. The proof of Theorem 3 is completed.

5 Numerical Examples

In this section, computational results are reported to confirm our theoretical analysis.

Example 1 *We consider the Hilbert integral with $f(x) = \sin x \quad c = 0$, the exact value is $g(s) = 2\pi\cos s$. We consider the dynamic singular points $s = x_{[n/3]} + (\tau + 1)h/2$ and $s = x_0 + (\tau + 1)h/2$ with $\tau = \pm\frac{2}{3}$ is the superconvergence point.*

In Table 1 and Table 2 show that the superconvergence rate is $O(h^2)$ when the local coordinate equal $\pm\frac{2}{3}$ with the singular point $s = x_{[n/3]} + (\tau + 1)h/2$ and $s = x_0 + +(\tau + 1)h/2$. In Table 3 and 4 show that the convergence rate of modify rectangle rule is $O(h^2)$ for both the superconvergence rate and the non-superconvergence rate which agree with our Corralary 2.

Table 1: Errors of the rectangle rule with $s = x_{[n/3]} + (\tau+1)h/2$

n	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 1/2$
36	1.4331e-001	-2.8376e-003	1.2500e-002	7.1381e-002
72	6.6068e-002	-9.2338e-004	3.1271e-003	3.2859e-002
144	3.1638e-002	-2.5728e-004	7.8189e-004	1.5762e-002
288	1.5470e-002	-6.7600e-005	1.9547e-004	7.7192e-003
576	7.6481e-003	-1.7308e-005	4.8868e-005	3.8199e-003
1152	3.8023e-003	-4.3781e-006	1.2217e-005	1.9001e-003
h^α	1.0472	1.8680	1.9998	1.0463

Table 2: Errors of the rectangle rule with $s = x_0 + (1+\tau)h/2$

n	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 1/2$
36	-2.4891e-001	-9.2064e-003	-7.5746e-003	-1.2888e-001
72	-1.2285e-001	-2.1535e-003	-1.9441e-003	-6.2478e-002
144	-6.0971e-002	-5.1876e-004	-4.9225e-004	-3.0742e-002
288	-3.0367e-002	-1.2717e-004	-1.2384e-004	-1.5247e-002
576	-1.5153e-002	-3.1474e-005	-3.1055e-005	-7.5922e-003
1152	-7.5688e-003	-7.8283e-006	-7.7759e-006	-3.7883e-003
h^α	1.0079	2.0399	1.9856	1.0177

Table 3: Error estimate of the modify rectangle rule with $s = x_{[n/3]} + (\tau+1)h/2$

n	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 1/2$
36	-4.5356e-003	-2.8376e-003	1.2500e-002	2.2651e-003
72	-1.0668e-003	-9.2338e-004	3.1271e-003	7.4697e-004
144	-2.5804e-004	-2.5728e-004	7.8189e-004	2.0919e-004
288	-6.3413e-005	-6.7600e-005	1.9547e-004	5.5092e-005
576	-1.5715e-005	-1.7308e-005	4.8868e-005	1.4121e-005
1152	-3.9114e-006	-4.3781e-006	1.2217e-005	3.5738e-006
h^α	2.0359	1.8680	1.9998	1.8616

Table 4: Error estimate of the modify rectangle rule with $s = x_0 + +(\tau+1)h/2$

n	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 1/2$
36	7.8775e-003	-9.2064e-003	-7.5746e-003	8.9403e-003
72	1.9836e-003	-2.1535e-003	-1.9441e-003	2.1191e-003
144	4.9729e-004	-5.1876e-004	-4.9225e-004	5.1439e-004
288	1.2447e-004	-1.2717e-004	-1.2384e-004	1.2662e-004
576	3.1136e-005	-3.1474e-005	-3.1055e-005	3.1404e-005
1152	7.7860e-006	-7.8283e-006	-7.7759e-006	7.8196e-006
h^α	1.9965	2.0399	1.9856	2.0318

Example 2 *Now we consider an example of solving Hilbert integral equation Eq. 50 by collocation scheme Eq. 57. Let* $g(s) = \cos s - \sin s$*, the exact solution is* $f(x) = \cos x + \sin x$.

We examine the maximal nodal error and the maximal truncation error, defined by

$$e_\infty = \max_{1 \le i \le n} |f(x_i) - f_i|, \text{trunc} - e_\infty = \max_{1 \le i \le n} |E_n(f; \hat{x}_k)|, \tag{85}$$

respectively, where $f_i (i = 1, 2, \cdots, n)$ denotes the approximation of $f(x)$ at $\hat{x}_i$ and $E_n(f; \hat{x}_k)$ is defined in Eq. 5. Numerical results presented in Table 5 show that both the maximal nodal error and the maximal truncation error are $O(h^2)$, which is in good agreement with the result in Theorem 3. For the case with the local coordinate with $\tau = 0$, numerical results presented in Table 6 show that both the maximal nodal error and the maximal truncation error are $O(h)$ when the collocation point does not take the superconvergence point.

Table 5: Errors for the solution of the Hilbert integral equation of first kind with $\tau = 2/3$

n	e_∞	$\text{trunc} - e_\infty$
32	3.5896e-003	2.2491e-002
64	8.9891e-004	5.6480e-003
128	2.2516e-004	1.4146e-003
256	5.6338e-005	3.5395e-004
512	1.4090e-005	8.8527e-005
h^α	1.998	1.997

Table 6: Errors for the solution of the Hilbert integral equation of first kind with $\tau = 0$

n	e_∞	$\text{trunc} - e_\infty$
32	1.7734e-001	3.9716e-001
64	1.0535e-001	1.9579e-001
128	4.5149e-002	9.7097e-002
256	2.7684e-002	4.8337e-002
512	4.2344e-003	2.4114e-002
h^α	1.35	1.01

6 Conclusion

In this paper, we study the composite rectangle rule for numerical evaluation integrals defined on a circle with a Hilbert kernel and numerical solution of corresponding Hilbert integral equation. Based on the superconvergence phenomenon in each subinterval, a collocation scheme is presented by choosing the superconvergence point in each subinterval as the collocation points and an error estimate of the Hilbert integral equation is obtained.

This kind of Hilbert integral and integral equation is widely used in many engineering area [Yu (2002)]. The results in this paper show a possible way to improve the accuracy of the collocation method for singular integral equations by choosing the superconvergence points to be the collocation points. The local coordinate with superconvergence phenomenon of the middle rectangle rule are $\pm\frac{2}{3}$. Moreover, the inverse of the coefficient matrix has an explicit expression, then an optimal error estimate is established. Both the theoretical analysis and numerical results show that the method is of higher-order accuracy.

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