

A Wavelet Method for Solving Bagley-Torvik Equation

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Abstract: In this paper, an efficient and robust wavelet Laplace inversion method of solving the fractional differential equations is proposed. Such an inverse function can be applied to any reasonable function categories and it is not necessary to know the properties of original function in advance. As an example, we have applied the proposed method to the solution of the Bagley–Torvik equations and Numerical examples are given to demonstrate the efficiency and accuracy of the proposed.

Keywords: Fractional differential equations Laplace transform, Bagley–Torvik equation wavelet.

1 Introduction

Techniques of fractional calculus has raised its importance and been employed at the modeling of many different phenomena in engineering and physics, such as, viscoelasticity, heat conduction, diffusion wave, control theory and so on [Torvik and Bagley (1984); Mainardi (1996); Podlubny (1999); Beyer (1995)]. Many investigations on the solving of fractional equations can be seen in [Wang et al. (2013); Ma, Wang, and Meng (2013); Li (2014); Wei, Chen, and Sun (2014); Pang, Chen, and Sze (2014)]. We mention the important example: The Bagley–Torvik equation, which arises, for example, in the modeling of the motion of a rigid plate immersed in a Newtonian fluid is originally formulated in the studies on behavior of real material by use of fractional calculus [Torvik and Bagley (1984)]. And the interest of solving the Bagley–Torvik equations has been stimulated by their widely applications.

In the literature, a number of methods have also been developed for the numerical or analytical solutions for the Bagley–Torvik equations The analytical solution has been given [Caputo (1969); Podlubny (1999)], but its expression is a singular integral kernel function with Mittag–Leffler type functions Diethelm and Ford [Diethelm and Ford (2002)] contrast the fractional linear multistep methods and

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a predictor-corrector method of Adams type and note that the analytical solution involves the evaluation of a convolution integral, containing a Green's function expressed as an infinite sum of derivatives of Mittag-Leffler functions, and the expressions arise more complicated for general functions and inhomogeneous initial conditions. The analytical solution obtained by Shen et al. [Shen and Liu (2004)] is proposed also by the corresponding Green's function. The Adomian Decomposition Method (ADM) have been used by Ray and Bera [Ray and Bera (2005)] and Hu et al. [Hu, Luo, and Lu (2008)] to solve the Bagley-Torvik equations. At each iterative step, this method needs to calculate fractional derivatives of different functions, and have to be expressed in terms of the Mittag-Leffler type functions, which are usually slow to converge and can lead to divergence due to the finite precision of scientific computation, as has been pointed out by Welch et al. [Welch, Ropper, and Duren Jr (1999)]. Wang et al. [Wang, Wu, and Baleanu (2013)] solved the Bagley-Torvik equations by the Variational Iteration Method (VIM). Although the VIM does not involve the calculation of the so-called Adomian polynomials, it is crucial to give a good guess to the initial approximation of the solution. Moreover, both the ADM and VIM need a large number of iterative calculations. Muhammad et al. [Muhammad, Junaid, and Qureshi (2011)] apply the stochastic numerical solvers to find the solution of this equation. However, it is a bit tricky to decide the appropriate number for steps in optimization of weights by the method. Recently, Ray [Ray (2012)] gave the solution of the Bagley-Torvik equation by operational matrix of Haar wavelet method. In addition, other numerical techniques have also been developed to solve the Bagley-Torvik equations. Like the Taylor collocation method [Enesiz, Keskin, and Kurnaz (2010)], the shifted Jacobi tau methods [Kazem (2013)], and the finite difference method [Li (2012)].

Despite the progresses outlined above, yet there are still few methods with high accuracy and easy to implement numerical techniques that are suitable for solving the Bagley-Torvik equation. Wavelets represent a newly developed powerful mathematical tool, which has been broadly applied to signal decompositions and reconstructions, Laplace inversions [Wang, Zhou, and Gao (2003)], differential equation solutions [Liu et al. (2014); Zhou et al. (2011)]. Motivated by the recent paper by Ray [Ray (2012)], we present further study of the wavelet Laplace inversion method (WLIM). It has the same simplicity as the general Laplace inversions approximation, and is much more accurate. In this paper, we will first use the Laplace transform which can transform the fractional equations with initial values into non-fractional equations in the transform domain. With the help of the inverse Laplace transform based on the wavelets, we can obtain the numerical solutions. In order to numerically obtain the solutions to the time fractional Bagley-Torvik equation in time-space domain, we adopt a special method of the Laplace inversion suggested

by Wang et al. [Wang, Zhou, and Gao (2003)]. Finally, examples are given to demonstrate the high efficiency and accuracy of the proposed method. The results are compared with previous studies.

2 Fractional dynamic model of Bagley–Torvik equation

In this section, mathematical modeling of Bagley–Torvik equations is presented. The form of Bagley–Torvik equation can be written as [Torvik and Bagley (1984); Podlubny (1999)]

$$Ay''(t) + BD^{3/2}y(t) + Cy(t) = f(t), \quad t > 0 \tag{1}$$

with initial conditions given as

$$y(0) = y_0, \quad y'(0) = y_1 \tag{2}$$

where A , B , and C are constant coefficients, y_0 , y_1 are the constants. $y(t)$ is the solution of the equation, $D^{3/2}$ is the fractional derivative operator which is described in the Caputo sense as [Podlubny (1999)]

$$D_t^\alpha y(t) = \begin{cases} y^{(n)}(t), & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \end{cases} \tag{3}$$

where $y^{(n)}(t)$ denotes the ordinary derivative of order n and $\Gamma(x)$ is the Gamma function. By denoting the Laplace transform of $y(t)$ by $Y(s)$, i.e., $L[y(t)] = Y(s)$, the Laplace transform of $D_t^\alpha y(t)$ can be expressed as

$$\mathbf{L}[D_t^\alpha f(t)] = s^\alpha \mathbf{L}[f(t)] - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0), \quad n-1 < \alpha \leq n \tag{4}$$

Where An analytical solution is possible and can (for homogeneous initial conditions) be given in the form [Podlubny (1999)]

$$y(t) = \int_0^t G(t-u)f(u)du \tag{5}$$

with $G(t) = \frac{1}{A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\frac{C}{A})^k t^{2k+1} E_{1/2,2+3k/2}^{(k)}(-\frac{B}{A}\sqrt{t})$, and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler (M-L) function, defined as [Podlubny (1999)]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0), \tag{6}$$

and

$$E_{\alpha,\beta}^{(r)}(x) = \frac{d^r}{dx^r} E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{(j+r)!x^j}{j!\Gamma(\alpha j + \alpha r + \beta)}, \quad (r = 0, 1, 2, \dots). \tag{7}$$

For example, in the modeling of the motion of a rigid immersed plate in a Newtonian fluid. It was originally proposed by Bagley and Torvik in Ref. [Torvik and Bagley (1984)]. $A = m$, is the mass of thin rigid plate, $C = k$, is the stiffness of the spring, $B = 2A\sqrt{\mu\rho}$, μ is viscosity, and ρ is the fluid density, then Eqs. (1) and (2) represent the motion of a large thin plate in a Newtonian fluid [Podlubny (1999)].

3 Laplace inverse transform based on wavelet theory

Here, we consider the autocorrelation function $\theta(x)$ of the Daubechies scaling functions defined as [Donoho (1992); Xu et al. (1998)]

$$\theta(x) = \int_{-\infty}^{+\infty} \phi(y-x)\phi(y)dy \tag{8}$$

where $\phi(x)$ is the Daubechies scaling function. The compact support interval of $\theta(x)$ is $[1 - 2N, 2N - 1]$, when that of $\phi(x)$ is $[0, 2N - 1]$. According to the two-scale relation $\phi(x) = \sum_{k=0}^{2N-1} p_k\phi(2x-k)$, we can obtain [Donoho (1992); Xu et al. (1998)]

$$\theta(x) = \sum_{k=1-2N}^{2N-1} a_k\theta(2x-k) \tag{9}$$

in which $a_k = \frac{1}{2} \sum_{l=0}^{2N-1-k} p_l p_{l+k}$ or

$$\begin{cases} a_0 = 1 \\ a_{2n} = 0, n \neq 0 \\ a_{-n} = a_n \end{cases} \tag{10}$$

Table 1 [Liu et al. (2014)] shows the coefficients $a_k, k = 1, 3, 5, \dots, 2N - 1$ for $N=3, 4$ and 5 .

Applying the Fourier transform to Eq. (9), we have

$$\hat{\theta}(\omega) = P(e^{-i\omega/2})\hat{\theta}\left(\frac{\omega}{2}\right), \tag{11}$$

where P is defined by

$$P(z) = \frac{1}{2} \sum_{k=1-2N}^{2N-1} a_k z^k. \tag{12}$$

Table 1: Coefficients a_k for $N=3, 4$ and 5 .

k	$a_k(N=3)$	$a_k(N=4)$	$a_k(N=5)$
1	0.58593750000333	0.59814453124935	0.60562133789140
3	-0.09765625000216	-0.11962890624957	-0.13458251953155
5	0.011171874999883	0.02392578125020	0.03460693359369
7		-0.00244140624998	-0.00617980957065
9			5.340576171116696e-004

Noting that $\hat{\theta}(0) = 1$, we can therefore applying Eq. (11) recursively, and obtain a product formula [Daubechies (1988)]

$$\hat{\theta}(\omega) = \prod_{k=1}^{+\infty} P(e^{-i\omega/2^k}). \tag{13}$$

The convergence of this product has been discussed in References [Daubechies (1988)].

It is known that $\hat{\theta}(\omega)$ is a low pass filter. Fig. 1(a) illustrates the energy spectrums $|\hat{\theta}(\omega)|^2$ for $N=3, 4$ and 5 . As N increases, $|\hat{\theta}(\omega)|^2$ approaches the energy spectrum of the perfect low pass filter. Fig. 1(b) shows the difference between $|\hat{\theta}(\omega)|^2$ and the energy spectrum of the perfect low pass filter. This difference can be reduced to less than 0.52% within the frequency domain $[0, \pi/2]$ and less than 0.01% within the frequency domain $[0, 1]$ when $N=5$, indicating that the autocorrelation function $\theta(x)$ has good low pass characteristic for sufficiently large N .

For a function $f(x) \in L^2(\mathbb{R})$, we have the following multiresolution decomposition [Donoho (1992); Xu et al. (1998); Daubechies (1988)]

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{+\infty} c_{n,k} \theta_{n,k}(x) + \sum_{j=n}^{+\infty} \sum_{k=-\infty}^{+\infty} c_{j,k} \theta_{j,k}(x) \\ &= \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2^n}\right) \theta(2^n x - k) + \sum_{j \geq n} \sum_{k \in \mathbb{Z}} c_{j,k} \theta[2^{j+1} x - (2k + 1)] \end{aligned} \tag{14}$$

in which $c_{j,k} = f\left(\frac{2k+1}{2^{j+1}}\right) - \mathbf{I}_n[f\left(\frac{2k+1}{2^{j+1}}\right)]$ and According to the theory of multiresolution analysis [Daubechies (1988)], we have the following relations:

$$f(x) = \lim_{n \rightarrow \infty} \mathbf{I}_n[f(x)] = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2^n}\right) \theta(2^n x - k) \tag{15}$$

Letting a function $f(x)$ be defined over the interval $[0, +\infty)$, the Laplace transform

and inversion of this function can be defined as

$$\tilde{F}(s) = \mathbf{L}[f(x)] = \int_0^{+\infty} f(x)e^{-st} dt, \quad f(x) = \mathbf{L}^{-1}[\tilde{F}(s)] = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \tilde{F}(s)e^{st} ds \tag{16}$$

where $s = \beta + i\omega$, ω is a real variable and β is a real constant to insure $f(x)e^{-\beta t} \in L^2(R)$. Using the same procedure as developed earlier by Wang et al. [Wang, Zhou, and Gao (2003)], we obtain the following exact inversion formula

$$f(x) = \mathbf{L}^{-1}[\tilde{F}(s)] = f_n + \sum_{j \geq n} \frac{e^{\beta x}}{2^{j+2}\pi} \hat{\theta}\left(-\frac{x}{2^{j+1}}\right) \sum_{k \in \mathbb{Z}} \tilde{c}_{j,k} e^{i(2k+1)x/2^{j+1}} \tag{17}$$

and

$$f_n(x) = \frac{e^{\beta x}}{2^{n+1}\pi} \hat{\theta}\left(-\frac{x}{2^n}\right) \sum_{k \in \mathbb{Z}} \tilde{F}\left(\beta + i\frac{k}{2^n}\right) e^{ikx/2^n}, \tag{18}$$

where $\tilde{c}_{j,k} = \tilde{F}\left(\beta + i\frac{2k+1}{2^{j+1}}\right) - \mathbf{I}_n[\tilde{F}\left(\beta + i\frac{2k+1}{2^{j+1}}\right)]$, and $f(x) \approx f_n(x)$ for $x \in [0, 2^n]$ [Wang, Zhou, and Gao (2003)]. As has been demonstrated by Wang et al. [Wang, Zhou, and Gao (2003)], the numerical accuracy of Eq. (18) for the approximation of $f(x)$ depends on the difference between 1 and the energy spectrum $|\hat{\theta}(2^{-n}t)|^2$. It can be seen from Fig. 1 that such a difference is less than 0.01% within domain $[0, 1]$ for $N=5$ and resolution level $n=0$, implying that Eq. (18), ones can at least give a good approximation to $f(x)$ for $x \in [0, 2^n]$ by taking into account the dilation effect of resolution level n . For example, if the function $f(x)$ with an exponential growth law, denote a as the growth in order ($|f(x)| \leq Me^{ax}$, M is a constant), we can take a proper constant β ($\beta > a$), then, the numerical solution $f(x)$ with high accuracy is obtained by Eq. (17) at the interval $x \in [0, 2^{n-1}\pi]$ (seen in Ref. [Wang, Zhou, and Gao (2003); Wei (1998) and Wang (2001)]).

4 Applications of wavelet Laplace inversion method in solving Bagley–Torvik equation

In the following, we shall apply the wavelet Laplace method for the numerical solution of the Bagley–Torvik equation with initial conditions as shown in Eqs. (1) and (2).

First, Applying the Laplace transform to the Eq. (1) and taking the initial conditions into account to remove the fractional time derivative as

$$As^2Y(s) + Bs^{3/2}Y(s) + CsY(s) = F(s) + \sum_{k=0}^1 s^{1-k}y^{(k)}(0) + \sum_{k=0}^1 s^{1/2-k}y^{(k)}(0) \tag{19}$$

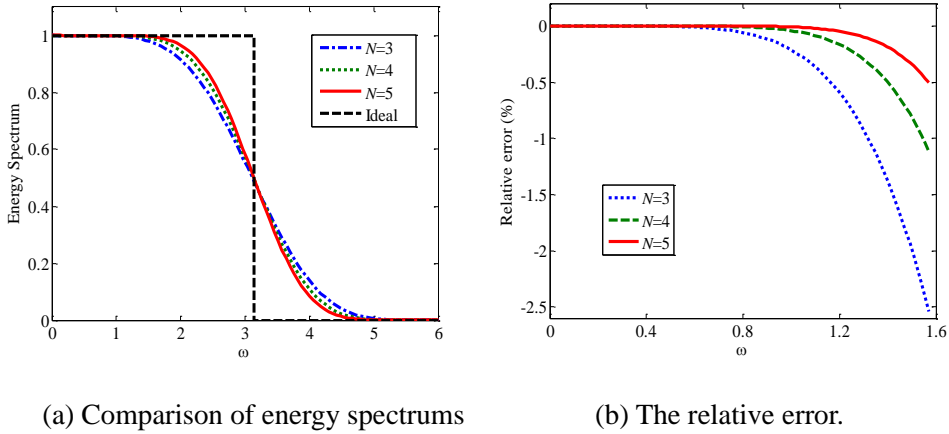


Figure 1: Comparison between energy spectrum of the ideal low pass filter and $|\hat{\theta}(\omega)|^2$ for $N=3, 4, 5$.

Denote $R(s) = F(s) + \sum_{k=0}^1 s^{1-k}y^{(k)}(0) + \sum_{k=0}^1 s^{1/2-k}y^{(k)}(0)$, Further

$$Y(s) = R(s) / (As^2 + Bs^{3/2} + Cs) \tag{20}$$

By the Eq. (17), we get the inverse Laplace transform based on wavelet to Eq. (20), we can obtain

$$y(t) = \lim_{n \rightarrow \infty} \left\{ \frac{e^{\beta t}}{2^{n+1}\pi} \hat{\theta}\left(-\frac{t}{2^n}\right) \sum_{k=-\infty}^{\infty} Y\left(\beta + i\frac{k}{2^n}\right) e^{itk/2^n} \right\}. \tag{21}$$

which is the numerical solution of Eq. (1) when Take a proper constant β , the numerical solution $y(t)$ with high accuracy is obtained by Eq. (21).

5 Numerical results and discussions

To have comparison of the present analysis through the wavelet Laplace inversion method with that of other available methods [Muhammad, Junaid, and Qureshi (2011); Ray (2012); Enesiz, Keskin, and Kurnaz (2010); Rawashdeh (2006)], examples are given as following:

Example 1: We consider the Bagley–Torvik equation [Rawashdeh (2006)]

$$y''(t) + D^{3/2}y(t) + y(t) = 2 + 4\sqrt{t}/\Gamma(1/2) + t^2, \quad t > 0 \tag{22}$$

$$y(0) = y'(0) = 0 \tag{23}$$

the exact solution of Eqs. (22) and (23) is $y(t) = t^2$.

Applying the Laplace transform to the Eqs. (22) and (23), and taking the initial conditions into account to remove the fractional time derivative as

$$s^2Y(s) + s^{3/2}Y(s) + Y(s) = \frac{2}{s} + \frac{2}{s^{3/2}} + \frac{2}{s^3} \quad (24)$$

Further

$$Y(s) = 2/s^3 \quad (25)$$

By using the Eq. (21), we get the numerical solution of Eqs. (22) and (23). Here take $\beta = 1$, Fig. 2 shows the comparison between the exact result of this problem and its numerical approximation according to Eqs. (22) and (23) when $N = 5$, $n = 4$ corresponding to a time step more than $1/4$. It can be seen from Fig. 2 that the approximate result almost coincides with the exact one. E.A. Rawashdeh [Rawashdeh (2006)] solved the same equation by using a method based on the Spline collocation methods. In Ref. [Rawashdeh (2006)], the absolute error of the approximate solution at $t = 2.5$ obtained by numerically solving 100 nonlinear algebraic equations is $0.66e-6$, which is much larger than the absolute error obtained by using Eq. (21) when $N = 5$, $n \geq 2$ as shown in Table 2. Moreover, one can see that the absolute error remains while the approximate solution in a very wider region.

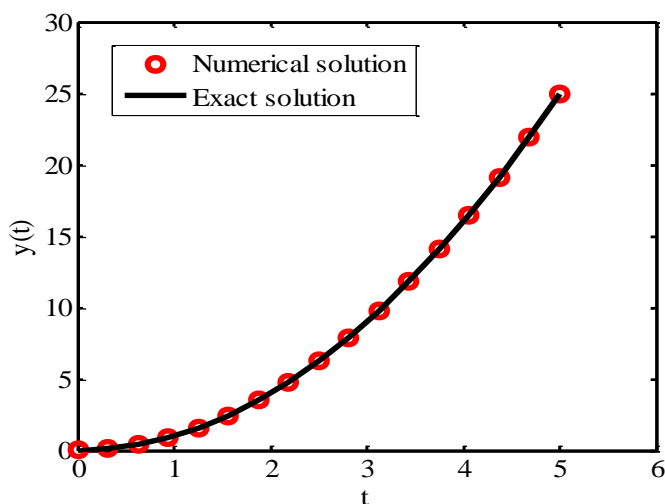


Figure 2: Comparison between the numerical and exact solutions of Example 1 when $n = 4$.

Table 2: Absolute error of numerical solutions under different resolution level n .

t	$n = 2$	$n = 3$	$n = 4$
1.25	2.1248e-011	2.1248e-011	2.1248e-011
2.50	8.3489e-013	8.3489e-013	8.3489e-013
3.75	8.3537e-011	8.3537e-011	8.3537e-011
5.00	3.1816e-010	3.1816e-010	3.1816e-010

Example 2: Let us consider the Bagley–Torvik equation, which has assumed $A = 1$, $B = 1/2$ and $C = 1/2$, as is taken in [Podlubny (1999); Ray and Bera (2005); Muhammad, Junaid, and Qureshi (2011); Ray (2012); Enesiz, Keskin, and Kurnaz (2010)].

$$y''(t) + \frac{1}{2}D^{3/2}y(t) + \frac{1}{2}y(t) = f(t), \quad t > 0, \tag{26}$$

$$\text{where } f(t) = \begin{cases} 8, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases},$$

subject to $y(0) = y'(0) = 0$.

Applying the Laplace transform to the Eq. (26) and taking the initial conditions into account, to remove the fractional time derivative as

$$s^2Y(s) + \frac{1}{2}s^{3/2}Y(s) + \frac{1}{2}Y(s) = 8\left(\frac{1}{s} - \frac{e^{-s}}{s}\right) \tag{27}$$

Further

$$Y(s) = 8(1 - e^{-s}) / (s^3 + s^{5/2} / 2 + s^2 / 2) \tag{28}$$

By using the Eq. (21), we get the inverse Laplace transform based on wavelet to Eq. (28), which is the numerical solution of Eq. (26).

Now, the analytical solution of Eq. (26) is [Podlubny (1999)]

$$y(t) = 8[y_u(t) - y_u(t - 1)], \quad \text{if } f(t) = 8[u(t) - u(t - 1)] \tag{29}$$

where

$$\begin{aligned} y_u(t) &= u(t) \left[\frac{1}{A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{C}{A}\right)^k t^{2(k+1)} E_{1/2, 3+3k/2}^{(k)} \left(-\frac{B}{A} \sqrt{t}\right) \right] \\ &= u(t) \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{1}{2}\right)^k t^{2(k+1)} E_{1/2, 3+3k/2}^{(k)} \left(-\frac{1}{2} \sqrt{t}\right) \right], \end{aligned} \tag{30}$$

in which $E_{\alpha,\beta}^{(r)}(x) = \frac{d^r}{dx^r} E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{(j+r)!x^j}{j!\Gamma(\alpha j + \alpha r + \beta)}$, ($r = 0, 1, 2, \dots$)

and $u(t)$ is the Heaviside Step function [Ray and Bera (2005)].

Fig. 3 shows the comparison between the exact result of this problem and its numerical approximation according to Eq. (26) for $N = 5, n = 5$. A good agreement between the exact and approximate results can be observed from Fig. 3. And Table 3 gives a quantitative comparison between the exact results and some existing approximate results. From Table 3, we can see that the absolute errors given in this paper have higher accuracy than the other results obtained by, e.g., Y. C, Enesiz et al. [Enesiz, Keskin, and Kurnaz (2010)] and Muhammad et al. [Muhammad, Junaid, and Qureshi (2011)]. While, Saha Ray [Ray and Bera (2005); Ray (2012)] also solved Eq. (26) by using the modified Adomain Decomposition Method (ADM) and Haar wavelet method, and the value of absolute error is at least $1.0e-003$, however, the method proposed in this paper is going $1.0e-010$ at time steps of $1/16$.

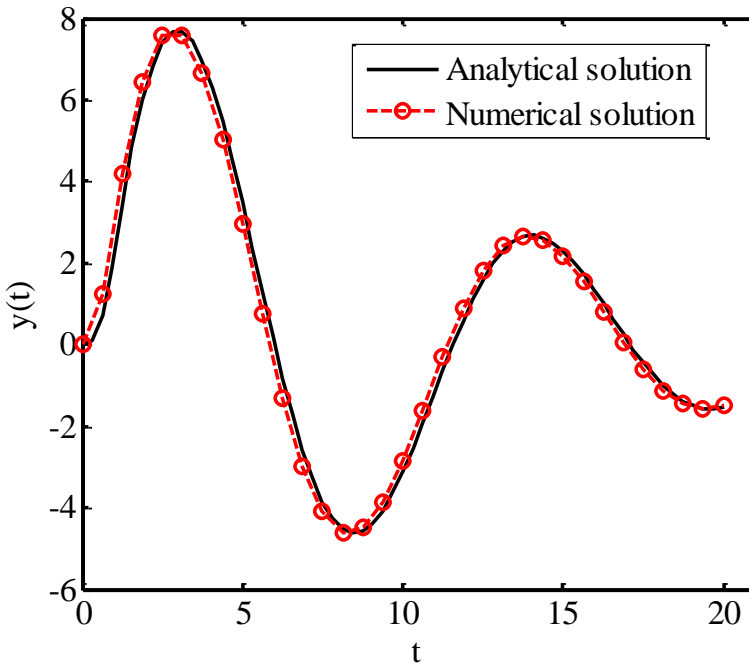


Figure 3: Comparison between the numerical and exact solutions of Example 2 when $n = 5$.

Table 3: Comparison of absolute error between numerical solutions.

t	GTCM by Ref. [13]	PMA by Ref. [11]	GS-PA by Ref. [11]	WLIM by This paper
0	0	0	0.001437	4.166e-010
0.1	0.000001932	0.003753	0.0003855	2.620e-010
0.2	0.000004905	0.00734	0.007797	2.153e-010
0.3	0.000008398	0.01083	0.01492	1.311e-010
0.4	0.00001282	0.01426	0.01287	1.056e-010
0.5	0.00002134	0.01765	0.01347	1.125e-010
0.6	0.00003162	0.02102	0.02191	4.645e-011
0.7	0.00004418	0.02437	0.02769	1.310e-010
0.8	0.00005431	0.02771	0.03479	1.140e-010
0.9	0.0001219	0.03106	0.0546	2.463e-010
1.0	0.001486	0.03441	0.04235	4.520e-010

6 Conclusions

In this paper, a numerical method based on the Daubechies wavelet operational method is applied to solve the fractional differential equations. We proposed an approximation scheme for a L^2 -function defined on a bounded interval $[0, +\infty)$ by combining techniques of the Laplace method and Daubechies autocorrelation-type wavelet expansion. By using the Bagley–Torvik equation as an example, we find that this wavelet algorithm has a convergence rate, and shows a very high precision comparing with many other existing numerical methods. The examples shows simplicity and effectiveness of this method. Moreover, such an approach can be applied to any reasonable function categories and it is not necessary to know the properties of original function in advance.

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