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Approximate Analytical Solution of Time-fractional order Cauchy-Reaction Diffusion equation

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Abstract: The objective of this article is to carry out an approximate analytical solution of the time fractional order Cauchy-reaction diffusion equation by using a semi analytical method referred as the fractional-order reduced differential transform method (FRDTM). The fractional derivative is illustrated in the Caputo sense. The FRDTM is very efficient and effective powerful mathematical tool for solving wide range of real world physical problems by providing an exact or a closed approximate solution of any differential equation arising in engineering and allied sciences. Four test numerical examples are provided to validate and illustrate the efficiency of FRDTM.

Keywords: Cauchy-reaction diffusion equation, Caputo time Fractional derivatives, Mittag-Leffler function, Fractional reduced differential transform method, exact solution.

1 Introduction

The fractional calculus theory has a great attention in engineering and allied sciences [Hilfer (2000); Carpinteri, and Mainardi (1997); Miller, and Ross (1993); Oldham, and Spanier (1974); Podlubny (1999)]. For instance, there are several physical phenomena which can be explained successfully by developing the models using fractional calculus theory. Fractional differential equations have achieved much more attention because of the fractional order systems converges to the integer order equations. In the recent years, the fractional differentiation has a wide range of application in the mathematical modeling of real world physical problems, for instance: in earthquake modeling, measurement of viscoelastic material properties, the traffic flow model, fluid flow model with fractional derivatives etc.

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In this paper, we consider the one dimensional time-fractional Cauchy reaction-diffusion equation (Kumar (2013)) given by

$$D_t^{\alpha}u(x,t) = \upsilon D_x^2 u(x,t) + p(x,t)\,u(x,t)\,,\ x \in \mathrm{R},\ t > 0, 0 < \alpha \leq 1, \tag{1}$$

subject to the initial condition (IC): $u(x,0) = u_0(x)$, where $D_t^{\alpha}u = \frac{\partial^{\alpha}u}{\partial t^{\alpha}}$, $D_x^2 u = \frac{\partial^2 u}{\partial x^2}$, $\upsilon > 0$ is the diffusion coefficient, u and p denote the concentration and the reaction parameter, respectively.

The classical Cauchy-reaction diffusion equations (i. e., Eq. (1) with $\alpha = 1$) describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [(Britton (1998); Cantrell, and Cosner (2003); Grindrod (1996); Smoller (1994)]. The approximate series solutions of classical Cauchy reaction-diffusion equation were obtained by using several analytical approaches, namely, Adomian decomposition method (ADM) by [Lesnic (2007)], and [Lesnic (2005)], Variational Iteration Method (VIM) by [Dehghan, and Shakeri (2008)], Homotopy Analysis Method (HAM) by [Bataineh, Noorani, and Hashim (2008)], Homotopy Perturbation Method (HPM) by [Yildirim (2009)]. RDTM by [Sohail, and Mohyud-Din (2012)]. [Wang, and He (2008)] applied VIM for a nonlinear reaction-diffusion process.

There was no scheme available for analytical solutions for linear or nonlinear fractional order differential equations, before the nineteenth century. Recently, the fractional order multi-dimensional diffusion equation was solved using a Modified Homotopy Perturbation Method (M-HPM) by [Kumar (2013)]. The major disadvantage of aforesaid approaches is that they require a very complicated and huge calculation. To overcome from such type of the drawbacks, the fractional reduced differential transform method (FRDTM) given by [Keskin, and Oturanc (2010)] has been employed. The FRDTM is the most easily implemented analytical method which provides the exact solution for both linear and nonlinear fractional differential equations, is very effective, reliable and efficient, and very powerful analytical approach, refer [Gupta (2011); Srivastava, Awasthi, and Tamsir (2013); Srivastava, Awasthi, and Kumar (2014); Srivastava, Kumar, Awasthi, and Singh (2014)]. In this paper, our main aim is to present approximate analytical solutions of time-fractional model of Cauchy-reaction diffusion equations of order $\alpha(0 < \alpha \leq 1)$ in series form converges to the exact solution rapidly, using FRDTM. Some other applications of fractional derivatives can be seen in [Chen, Liu, Li, and Sun (2014); Pang, Chen, and Sze (2014); Chen, Han, and Liu (2014); Li (2014)].

The rest of the paper is organized as follows: in Section 2, basic preliminaries and notations on fractional calculus theory are revisited that are used for further study. Section 3 presents the basic of FRDTM are what we use to find the exact solution of the time-fractional Cauchy-reaction diffusion equation. In Section 4, exact solutions of four test problems time-fractional Cauchy reaction-diffusion problems are presented and compared with the exact solutions available in the literature. Section 5 is the conclusion of the article.

2 Fractional Calculus Theory

In this section, the basic definitions and notations are revisited that will be used for further ongoing study. In fractional integrals and derivatives, several definitions are proposed but the first major contribution to give a proper and most meaningful definition goes to Liouville [Millar and Ross (1993)].

Definition 2.1 A real valued function $f(\mathrm{x}) \in \mathbb{R}$, x>0 is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number q $(>\mu)$ such that $f(\mathrm{x})=\mathrm{x}^q \mathrm{g}(\mathrm{x})$, where $\mathrm{g}(\mathrm{x}) \in \mathrm{C}[0,\infty)$, and is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $\mathrm{m} \in \mathbb{N}$.

Definition 2.2 For any given function $f \in \mathbb{R}$, the Riemann-Liouville fractional integeral operator [Grindrod (1996)] of order $\alpha \geq 0$, is defined by

$$\begin{cases} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x (\mathrm{x\text{-}t})^{\alpha-1} f(t)\,\mathrm{dt}, \ \alpha > 0, x > 0, \\ J^0 f(x) = f(x). \end{cases} \tag{2}$$

In his work, [Caputo, and Mainardi (1971)] proposed a modified fractional differentiation operator D^α on the theory of visco-elasticity by overcoming the discrepancy of Riemann-Liouville derivative [Millar, and Ross (1993)] while modeling the real world problems using the fractional differential equations. They further, demonstrated that their proposed Caputo fractional derivative allow the utilization of initial and boundary conditions involving integer order derivatives, a straightforward physical interpretations.

Definition 2.3 The fractional derivative of $f(x) \in \mathbb{R}$, in the Caputo sense [Grindrod (1996)] is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int\limits_0^x (\mathrm{x\text{-}t})^{m-\alpha-1} f^{(m)}(t)\,\mathrm{dt}, \tag{3}$$

for $m-1 < \alpha \leq m,\ m \in \mathbb{N}, x > 0, f \in C_{-1}^m$.

The basic properties of the Caputo fractional derivative can be given by the following

Lemma 2.1 If $m-1 < \alpha \leq m,\ m \in \mathbb{N}$ and $f \in C_\mu^m,\ \mu \geq$ -1, then we have

$$\begin{cases} D^\alpha J^\alpha f(x) = f(x),\ \mathrm{x}>0, \\ J^\alpha D^\alpha f(x) = f(x) - \sum\limits_{k=0}^{m} f^{(k)}(0^+) \frac{x^k}{k!},\ \mathrm{x}>0, \end{cases} \tag{4}$$

In the present work, the Caputo fractional derivative is considered because it allows the traditional initial and boundary conditions to be included in the formulation of the physical problems. For further important characteristics of fractional derivatives, one can refer [Hilfer (2000); Carpinteri, and Mainardi (1997); Miller, and Ross (1993); Oldham, and Spanier (1974); Podlubny (1999)].

3 Fractional Reduced Differential Transform Method (FRDTM)

In this section, the basic properties of the fractional reduced differential transform method are described. Let $w(x,t)$ be a function of two variables, which can be represented as a product of two single-variable functions, that is $w(x,t) = F(x)G(t)$. Using the properties of the one-dimensional differential transform (RDT) method, $w(x,t)$ can be written as

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i,j)x^i t^j, \tag{5}$$

where $W(i,j) = F(i)G(j)$ is referred to as the spectrum of $w(x,t)$.

Let R_D and R_D^{-1} denotes operators for fractional reduced differential transform (FRDT) and inverse FRDT, respectively. The basic definition and properties of the FRDTM is described below.

Definition 3.1 If $w(x,t)$ is analytic and continuously differentiable with respect to space variable x and time variable t in the domain of interest, then the t-dimensional spectrum function

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[D_t^k \left(w(x,t) \right) \right]_{t=t_0} \tag{6}$$

is referred to as the FRDT function of $w(x,t)$, where α is a parameter which describes the order of time-fractional derivative. Throughout the paper, $w(x,t)$ (lowercase) is used for the original function and $W_k(x)$ (uppercase) stands for the fractional reduced transformed function.

The inverse FRDT of $W_k(x)$ is defined by

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^{k\alpha}. \tag{7}$$

From Eq. (6) and (7), it can be found that

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[D_t^k \left(w(x,t) \right) \right]_{t=t_0} (t-t_0)^{k\alpha}. \tag{8}$$

In particular, for $t=0$, Eq. (8) reduces to

$$w(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[D_t^k\left(w(x,t)\right)\right]_{t=0}t^{k\alpha}. \tag{9}$$

From the above discussion, it is found that the FRDTM is a special case of the power series expansion of a function.

Lemma 3.1 Let $u(x,t)=R_D^{-1}\left[U_k(x)\right]$, $v(x,t)=R_D^{-1}\left[V_k(x)\right]$ and the convolution $\otimes$ denotes the fractional reduced differential transform version of the multiplication, then the fundamental operations of the FRDT are illustrated in Table **I**, where Γ is the well known **Gama function** defined by $\Gamma(z):=\int\limits_0^\infty e^{-t}t^{z-1}dt, z\in \mathrm{C}$, is the continuous extension to the factorial function [Srivastava, Awasthi, and Tamsir (2013)].

Table 1: Basic properties of the FRDTM.

Original Function $w(x,t)$	Fractional Reduced Differential Transformed Function $R_D\{w(x,t)\}=W_k(x)$
$u(x,t)\,v(x,t)$	$U_k(x)\otimes V_k(x)=\sum\limits_{r=0}^{k}U_r(x)\,V_{k-r}(x)$
$\alpha u(x,t)\pm\beta v(x,t)$	$\alpha U_k(x)\pm\beta V_k(x)$
$x^m t^n u(x,t)$	$x^m U_{k-n}(x), \forall k\geq n$; $0, \quad else,$
$e^{\lambda t}$	$\frac{\lambda^k}{k!}$
$\sin(wt+\alpha)$	$\frac{w^k}{k!}\sin\left(\frac{\pi k}{2!}+\alpha\right)$
$\cos(wt+\alpha)$	$\frac{w^k}{k!}\cos\left(\frac{\pi k}{2!}+\alpha\right)$
$D_x^l u(x,t)$	$D_x^l U_k(x)$
$D_t^{N\alpha}\left(u(x,t)\right)$	$\frac{\Gamma(1+(k+N)\alpha)}{\Gamma(1+k\alpha)}U_{k+N}(x)$

Definition 3.1 The Mittag-Leffler function $E_\alpha(z)$ with $\alpha>0$ is defined by the following series representation, is valid in the whole complex plane [Mainardi (1994)]

$$\mathrm{E}_\alpha(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}$$

Which is an advanced form of $\exp(z)$. In particular, $\exp(z)=\lim\limits_{\alpha\to 1}\mathrm{E}_\alpha(z)$.

4 Numerical experiments

This section describes FRDTM explained in Section 3 by giving four numerical examples to validate the reliability and efficiency of FRDTM for the time fractional-order Cauchy-reaction diffusion equation. The approximate analytical solutions of the four numerical examples are obtained by considering first twenty terms in the series and 40 grid points.

Example 4.1: Consider the time fractional-order Cauchy-reaction diffusion equation (1) with $\upsilon = 1$, $p(x,t) = -1$ as given in (Kumar (2013))

$$D_t^\alpha u(x,t) = D_x^2 u(x,t) - u(x,t),\ x \in [0,1],\ \ t > 0,\ \ 0 < \alpha \leq 1, \tag{10}$$

with the initial condition

$$u(x,0) = e^{\pm x} + x. \tag{11}$$

The following recurrence relation is obtained by implementing FRDTM in Eq. (10)

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x) = D_x^2 \mathrm{U}_k(\mathrm{x}) - \mathrm{U}_k(\mathrm{x}). \tag{12}$$

Next, applying FRDTM in the initial condition (11), we obtain

$$U_0(x) = e^{\pm x} + x. \tag{13}$$

Using Eq. (13) into Eq. (12), we get the values of $U_k(\mathrm{x})$ successively as follows

$$\begin{aligned} &U_1(x) = \frac{-x}{\Gamma(1+\alpha)}, \quad U_2(x) = \frac{x}{\Gamma(1+2\alpha)}, \\ &U_3(x) = \frac{-x}{\Gamma(1+3\alpha)}, \ldots, U_k(x) = \frac{(-1)^k x}{\Gamma(1+k\alpha)}, \ldots \end{aligned} \tag{14}$$

Applying inverse FRDTM of $U_k(\mathrm{x})$, we obtain

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(\mathrm{x}) t^{k\alpha} = U_0(\mathrm{x}) + \sum_{k=1}^{\infty} U_k(\mathrm{x}) t^{k\alpha} = e^{\pm x} + x\left(\sum_{k=0}^{\infty}\left(\tfrac{(-1)^k}{\Gamma(1+k\alpha)}\right) t^{k\alpha}\right) \\ &= e^{\pm x} + x\left(\sum_{k=0}^{\infty} \tfrac{(-t^\alpha)^k}{\Gamma(1+k\alpha)}\right) = e^{\pm x} + \mathrm{x}\mathrm{E}_\alpha(-t^\alpha). \end{aligned}$$

where $\mathrm{E}_\alpha(-t^\alpha)$ is the well known as Mittag-leffler function. Thus, it is demonstrated that the exact solutions for the Cauchy-reaction diffusion equation (10) subject to the initial condition $e^{-x} + x$ have a complete agreement with that of using M-HPM [Kumar (2013)]. In particular, for $\alpha \to 1$ in Eq. (10), we obtain

$$u(x,t) = e^{\pm x} + x \sum_{k=0}^{\infty} \frac{(t)^k}{\Gamma(1+k)} = e^{\pm x} + xe^{-t}, \tag{15}$$

Eq. (15) is the exact solution for the classical Cauchy-reaction diffusion equation (10) with $\alpha = 1$.

Fig. 1 depicts the comparison between the exact solutions and the approximate analytical solution at $t = 1$. Fig. 2 depicts the concentration profiles of u in three dimension ($3D$) and its contour form (b) at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$. Fig. 3 depicts the concentration profiles $u$in two dimension ($2D$) at $t \leq 1$ with the differential values of the fractional coefficients $\alpha \leq 1$ and $\upsilon = 1$

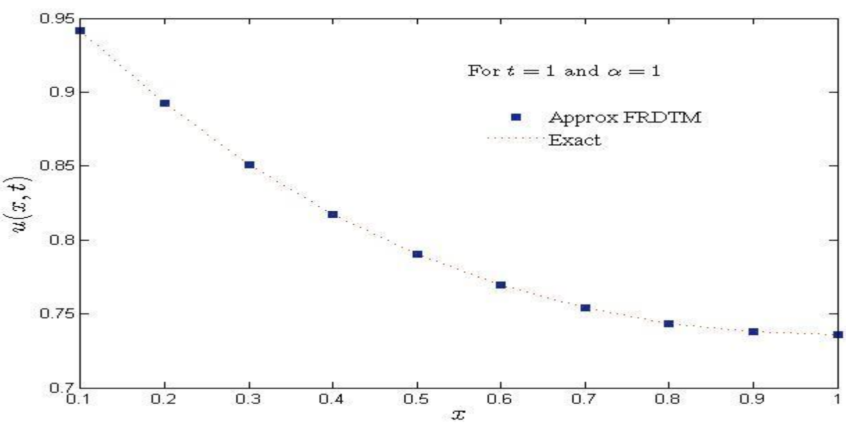

Figure 1: Comparison of the approximate concentration in Example 4.1 with the exact concentrations.

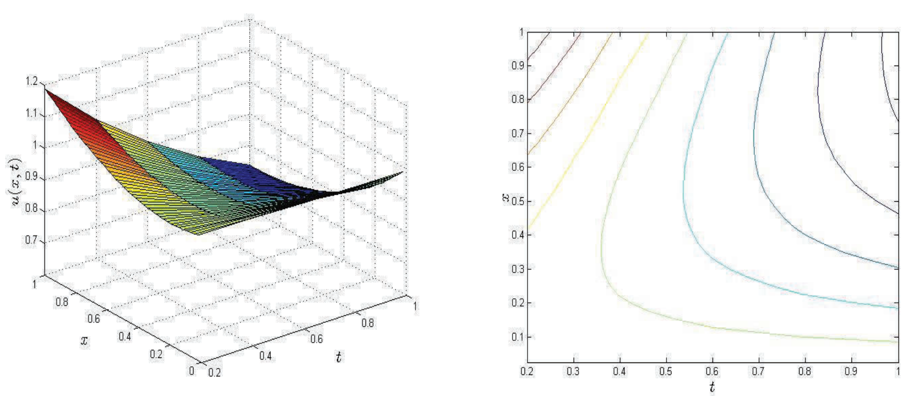

Figure 2: Concentration profiles of u in $3D$ (a) and contour form (b) of Example 4.1 at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$.

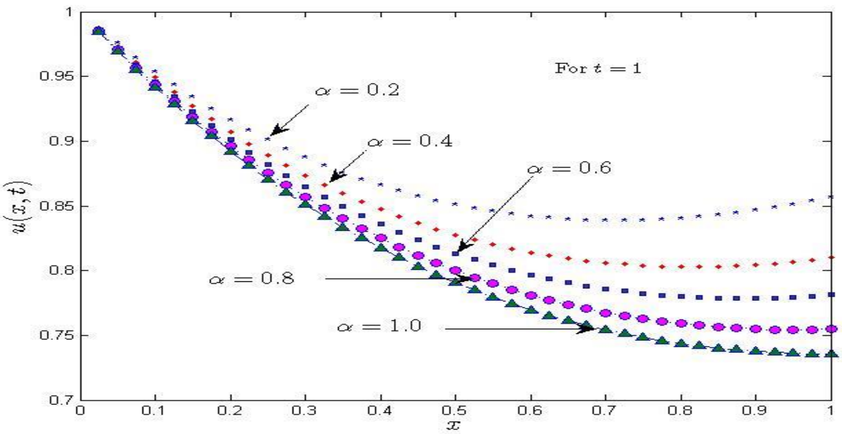

Figure 3: Concentration profiles of u in Example 4.1 in $2D$ at $t = 1$ for different values of α.

Example 4.2: Consider the time fractional-order Cauchy-reaction diffusion equation (1) with $\upsilon = 1,\ p(x,t) = -(1+4x^2)$, as given in (Kumar (2013))

$$D_t^{\alpha} u(x,t) = D_x^2 \mathrm{u}\,(\mathrm{x,t}) - (1+4x^2)u\,(x,t)\,, \quad x \in [0,1]\,,\ t > 0, 0 < \alpha \le 1, \tag{16}$$

subject to the initial condition

$$u\,(x,0) = e^{x^2}. \tag{17}$$

The following recurrence relation is obtained by implementing FRDTM in Eq. (16)

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}\,(x) = D_x^2 \mathrm{U}_k\,(\mathrm{x}) - \left(1+4x^2\right) \mathrm{U}_k\,(\mathrm{x})\,. \tag{18}$$

Applying the FRDTM on the initial condition (17), we have

$$U_0\,(x) = e^{x^2}. \tag{19}$$

Using Eq. (19) into Eq. (18), one can get the values of $U_k\,(\mathrm{x})$ successively

$$\begin{aligned} &U_1\,(x) = \frac{e^{x^2}}{\Gamma(1+\alpha)}, \quad U_2\,(x) = \frac{e^{x^2}}{\Gamma(1+2\alpha)}, \\ &U_3\,(x) = \frac{e^{x^2}}{\Gamma(1+3\alpha)}, \ldots, U_k\,(x) = \frac{e^{x^2}}{\Gamma(1+k\alpha)}, \ldots \end{aligned} \tag{20}$$

Applying inverse FRDTM on $U_k\,(\mathrm{x})$, we obtain

$$
\begin{aligned}
u\,(x,t) &= \sum_{k=0}^{\infty} U_k\,(\mathrm{x})\,t^{k\alpha} = U_0\,(\mathrm{x}) + U_1\,(\mathrm{x})\,t^{\alpha} + U_2\,(\mathrm{x})\,t^{2\alpha} + U_3\,(\mathrm{x})\,t^{3\alpha} + \ldots \\
&= e^{x^2}\left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \ldots + \frac{t^{k\alpha}}{\Gamma(1+k\alpha)} + \ldots\right) \\
&= e^{x^2}\left[\sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}\right] = e^{x^2}\mathrm{E}_{\alpha}(t^{\alpha}).
\end{aligned}
\tag{21}
$$

The exact solution (21) have a complete agreement with the exact solution obtained using M-HPM [6]. In particular, for $\alpha \to 1$ in Eq. (16), we have $u\,(x,t) = e^{x^2+t}$. Thus, the exact solutions for the Cauchy-reaction diffusion equation (16) with $\alpha = 1$ have complete agreement with the exact solution obtained in [Kumar (2013)] using M-HPM.

The comparison between the exact solutions and the approximate solution at $t = 1$ is shown in Fig. 4 while Fig. 5 shows the concentration profiles of u in $3D$, and its contour form (b) at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$. Fig. 6 shows the concentration profiles u in $2D$ at $t \leq 1$ with the differential values of the fractional coefficients $\alpha \leq 1$ and $\upsilon = 1$.

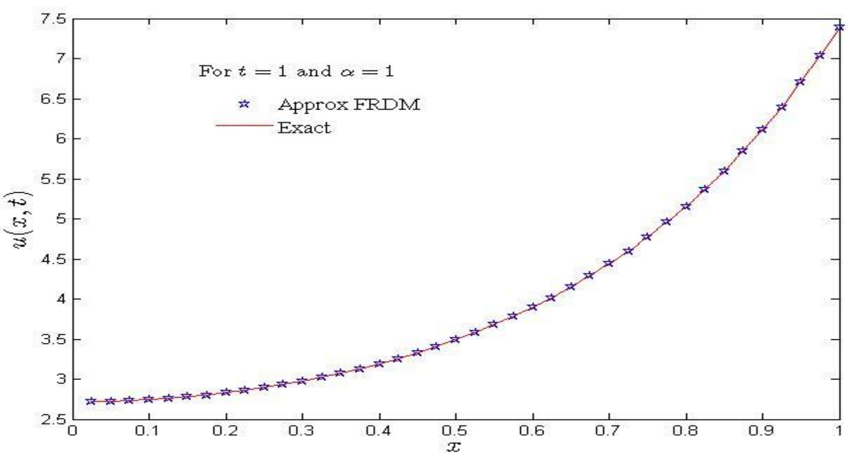

Figure 4: Comparison of the approximate concentration in Example 4.2 with the exact concentrations.

Example 4.3: Consider the time fractional-order Cauchy-reaction diffusion equation (1) with $\upsilon = 1,\ p(x,t) = 2t$, given [Kumar (2013)] as

$$
D_t^{\alpha} u\,(x,t) = D_x^2 u\,(x,t) + 2tu\,(x,t)\,, \quad x \in [0,1]\,, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{22}
$$

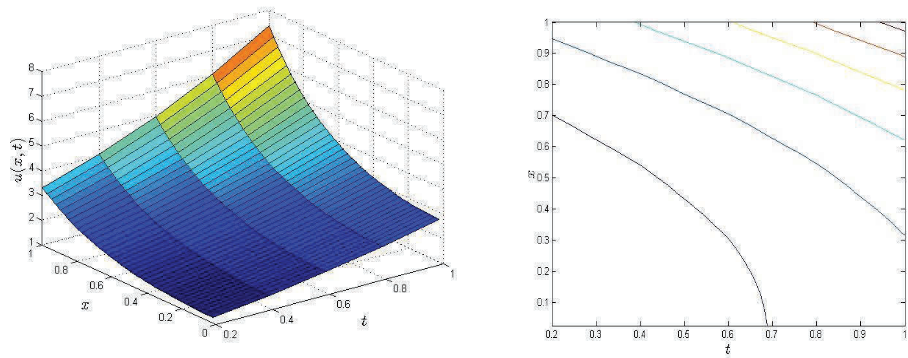

Figure 5: Concentration profiles of u in $3D$ (a) and contour form (b) of Example 4.2 at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$.

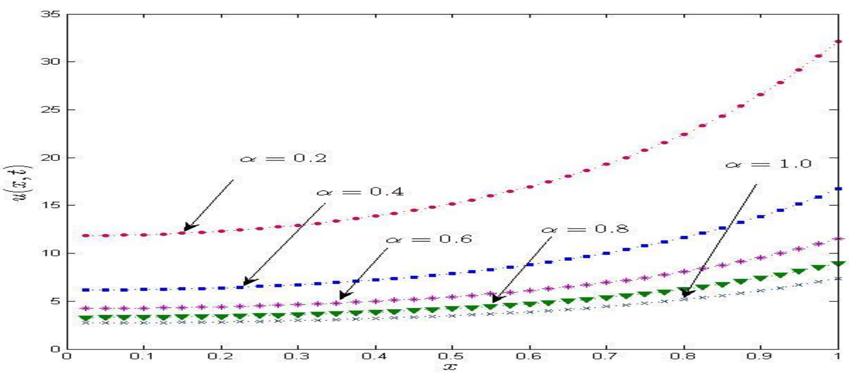

Figure 6: Concentration profiles of u in Example 4.2 in $2D$ at $t = 1$ for different values of α.

subject to the initial condition

$$u(x,0) = e^x. \tag{23}$$

The following recurrence relation is obtained by implementing FRDTM in Eq. (22)

$$\frac{\Gamma(1+(k+1)\alpha)}{\Gamma(1+k\alpha)} U_{k+1}(x) = D_x^2 \mathrm{U}_k(\mathrm{x}) + 2\ \mathrm{U}_{k-1}(\mathrm{x}). \tag{24}$$

Next, on taking FRDTM of the initial condition (23), we obtain

$$U_0(x) = e^x. \tag{25}$$

Using Eq. (25) into Eq. (24), we obtain

$$U_k(\mathrm{x}) = a_k e^x, \forall k = 1,2,3,4,\cdots \tag{26}$$

where the coefficients $a_k's$ are obtained by solving the following recurrence relation

$$\Gamma(1+(k+1)\alpha)\, a_{k+1} = \Gamma(1+k\alpha)(a_k + 2a_{k-1}),\ k > 1,\ a_0 = 1,\ a_1 = \frac{1}{\Gamma(1+\alpha)}. \tag{27}$$

Applying the inverse FRDTM on $U_k(\mathrm{x})$, we obtain

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(\mathrm{x})\, t^{k\alpha} = \mathrm{e}^x \sum_{k=0}^{\infty} a_k t^{k\alpha} \\ &= \mathrm{e}^x \left[1 + \tfrac{t^\alpha}{\Gamma(1+\alpha)} + \left(\tfrac{1+2\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right) t^{2\alpha} + \left(\tfrac{(1+2\Gamma(1+\alpha))\Gamma(1+\alpha)+2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right) t^{3\alpha} + \cdots\right]. \end{aligned} \tag{28}$$

In particular, for $\alpha = 1$ in Eq. (22), the exact solution given in Eq. (28) becomes $u(x,t) = e^{x+t+t^2}$, which is in complete agreement to the exact solutions of the given Cauchy-reaction diffusion Eq. (24) for $\alpha = 1$ with those obtained by ADM [Lesnic, D. (2005, 2007)], RDTM [Lesnic, D. (2005)], M-HPM [Kumar (2013)], and RDTM [Sohail, M.; Mohyud-Din, S.T. (2012)].

Fig. 7 shows the comparison between the exact solutions and the approximate analytical solution at $t = 1$. Fig. 8 depicts the concentration profiles of u in $3D$, and its contour form (b) at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$, whereas Fig. 9 depicts the concentration profiles u in $2D$ at $t \leq 1$ with $\alpha = 0.8$ and $\upsilon = 1$.

Example 4.4: Consider the time fractional-order Cauchy-reaction diffusion equation (1) with $\upsilon = 1,\ p(x,t) = -4x^2 + 2t - 2$, given as in [6]

$$D_t^\alpha u(x,t) = D_x^2 u(x,t) - \left(4x^2 - 2t + 2\right) u(x,t),\ x \in [0,1], t > 0, 0 < \alpha \leq 1, \tag{29}$$

with the initial condition

$$u(x,0) = e^{x^2}. \tag{30}$$

The following recurrence relation is obtained by implementing FRDTM in Eq. (29)

$$\frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)} U_{k+1}(x) = D_x^2 \mathrm{U}_k(\mathrm{x}) \text{-}2(1\text{+}2\mathrm{x}^2)\ \mathrm{U}_k(\mathrm{x}) \text{+}2\ \mathrm{U}_{k-1}(\mathrm{x}),\ \text{k=0,1,2,3,...} \tag{31}$$

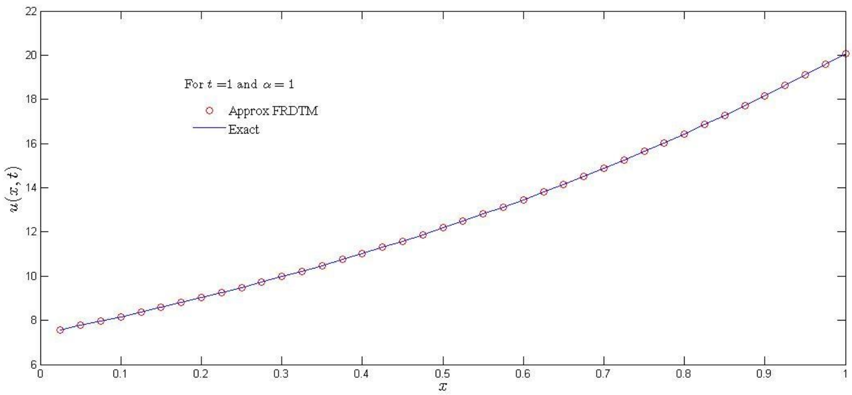

Figure 7: Comparison of the approximate concentration in Example 4.3 with the exact concentrations.

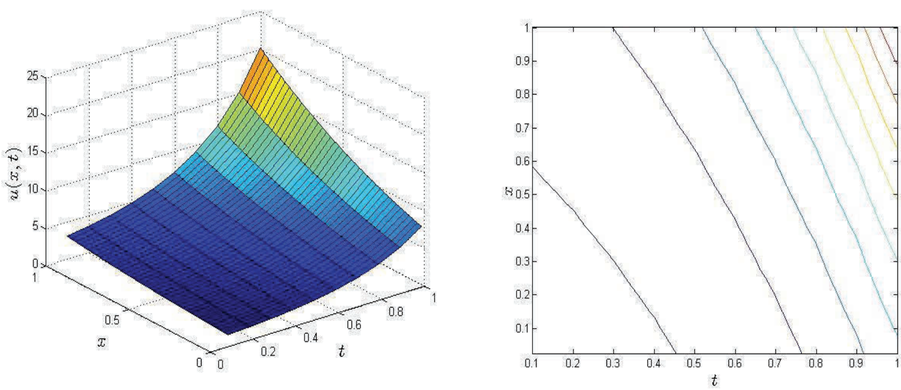

Figure 8: Concentration profiles of u in $3D$ (a) and contour form (b) of Example 4.3 at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$.

Next, applying FRDTM on the initial condition (30), we obtain

$$U_0(x) = e^{x^2}. \tag{32}$$

Using Eq. (32) into Eq. (31), one can obtain the values of $U_k(\mathrm{x})$ successively as

$$\left.\begin{array}{l} U_{2k-1}(x) = 0, \ \textit{and} \\ U_{2k}(x) = 2^k e^{x^2} \prod\limits_{i=1}^{k} \left(\frac{\Gamma(1+(2i-1)\alpha)}{\Gamma(1+2i\alpha)}\right) \end{array}\right\} \ \forall k = 1, 2, 3, \cdots \tag{33}$$

Where $\prod\limits_{i=1}^{k} x_i = x_1 x_2 x_3 \ldots x_k$. Next, on applying the inverse FRDTM on $U_k(\mathrm{x})$, we

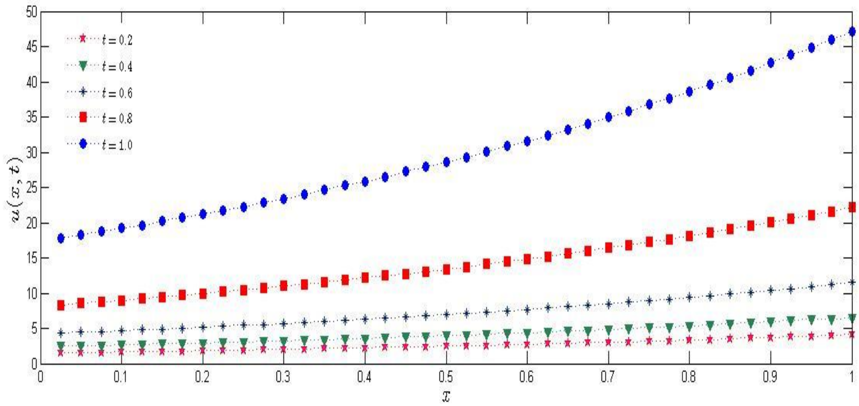

Figure 9: Concentration profiles of u in Example 4.3 in $2D$ at different time levels $t \leq 1$ for $\alpha = 0.8$.

obtain

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} U_k(\mathrm{x})\, t^{k\alpha} = U_0(\mathrm{x}) + \sum_{k=1}^{\infty} U_{2k-1}(\mathrm{x})\, t^{(2k-1)\alpha} + \sum_{k=1}^{\infty} U_{2k}(\mathrm{x})\, t^{2k\alpha} \\ &= e^{x^2}\left[1 + \sum_{k=1}^{\infty} 2^k \left(\prod_{i=1}^{k}\left(\frac{\Gamma(1+(2i-1)\alpha)}{\Gamma(1+2i\alpha)}\right)\right) t^{2k\alpha}\right]. \end{aligned} \tag{34}$$

In particular, for $\alpha = 1$ in Eq. (29), the exact solutions (35) becomes

$$\begin{aligned} u(x,t) &= e^{x^2}\left[1 + \sum_{k=1}^{\infty} 2^k \left(\prod_{i=1}^{k}\left(\frac{\Gamma(1+(2i-1))}{\Gamma(1+2i)}\right)\right) t^{2k}\right] \\ &= e^{x^2}\left[1 + \sum_{k=1}^{\infty} 2^k \left(\prod_{i=1}^{k}\left(\frac{(2i-1)!}{2i!}\right)\right) t^{2k}\right] \\ &= e^{x^2}\left[1 + \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k}\left(\frac{1}{i}\right)\right) t^{2k}\right] = e^{x^2}\left[\sum_{k=0}^{\infty} \frac{(t^2)^k}{k!}\right] = e^{x^2+t^2}. \end{aligned} \tag{35}$$

The similar exact solution was obtained by [Kumar (2013)] using a modified HPM. Further, it is found that the exact solutions of the given Cauchy-reaction diffusion Eq. (24) for $\alpha = 1$ have complete agreement to with that of obtained by using [Lesnic, D. (2005, 2007)], RDTM [Lesnic, D. (2005)], M-HPM [Kumar (2013)], and RDTM [Sohail, M.; Mohyud-Din, S.T. (2012)]. Fig. 10 gives the comparison between the exact solutions and the approximate analytical solution at $t = 1$. Fig. 11 depicts the concentration profiles of u in $3D$, and its contour form (b) at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$. Fig. 12 shows the concentration profiles u in $2D$ at $t \leq 1$ with the differential values of the fractional coefficients $\alpha \leq 1$ and $\upsilon = 1$.

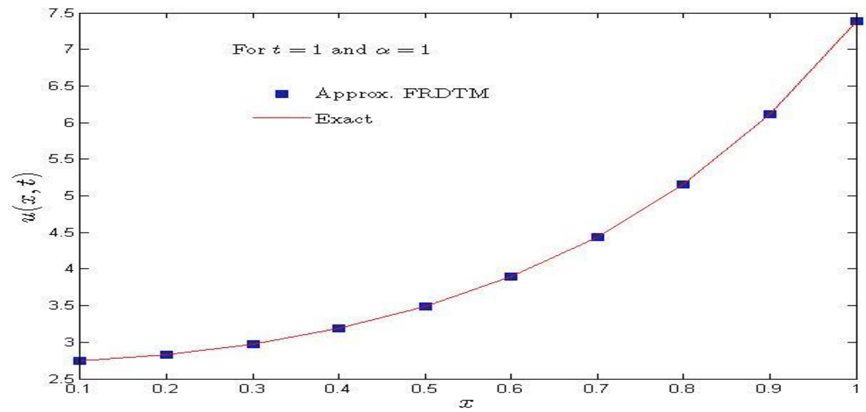

Figure 10: Comparison of the approximate concentration in Example 4.4 with the exact concentrations.

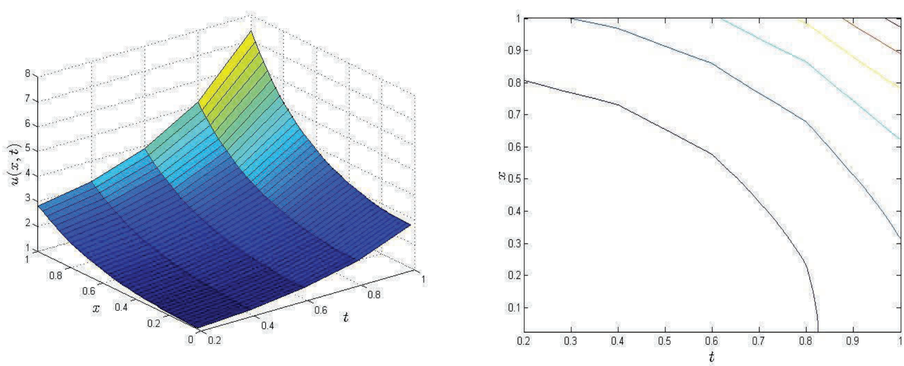

Figure 11: Concentration profiles of u in $3D$ (a) and contour form (b) of Example 4.2 at different time levels $t \leq 1$ with the diffusion coefficient $\upsilon = 1$.

5 Conclusions

In this study, the FRDTM has been implemented successfully to find out the analytical solution of the time-fractional order Cauchy-reaction diffusion equation. The obtained solutions by FRDTM is an infinite power series for appropriate initial condition, and provides the approximate solution without any transformation, perturbation, discretization, or any other restrictive conditions. Four examples are carried out to study the accurateness and effectiveness of the technique. The computed solutions by the method are in excellent agreement with those obtained [Kumar [2013)] using M-HPM. However, the performed computations depicts that the

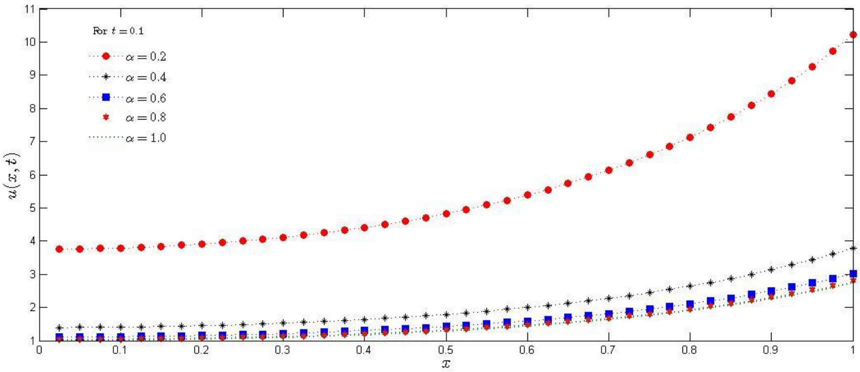

Figure 12: Concentration profiles of u in Example 4.4 in $2D$ at $t = 0.1$ for different values of α.

implemented method is very easy to use to solve the problems as compared to M-HPM. The advantage of this technique is that it needs small size of calculation contrary to the modified homotopy perturbation method. Further, in particular, for the associated classical Cauchy reaction-diffusion problems of the aforesaid examples (that is, for $\alpha = 1$) the exact solutions have a complete agreement with the solutions obtained by using M-HPM, ADM, VIM, HAM, HPM, RDTM available in the literature.

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