A High-Order Finite-Difference Scheme with a Linearization Technique for Solving of Three-Dimensional Burgers Equation

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Abstract: The objective of this paper aims to present a numerical solution of high accuracy and low computational cost for the three-dimensional Burgers equations. It is a well-known problem and studied the form for one and two-dimensional, but still little explored numerically for three-dimensional problems. Here, by using the High-Order Finite Difference Method for spatial discretization, the Crank-Nicolson method for time discretization and an efficient linearization technique with low computational cost, two numerical applications are used to validate the proposed formulation. In order to analyze the numerical error of the proposed formulation, an unpublished exact solution was used.

Keywords: Burgers equations, High-order Finite Difference Method, linearization technique.

1   Introduction

In recent decades, many authors have been developing researches looking for the numerical solution of partial differential equations and their applications, particularly in the solution of the Burgers equations. In [Radwan (1999)], the present authors have solved the two-dimensional unsteady Burgers equations using the fourth-order accurate two-point compact alternating direction implicit scheme and the fourth order Du Fort Frankel scheme. Comparisons were made between the present schemes in terms of accuracy and computational efficiency for solving problems with severe internal and boundary gradients. The fourth-order compact alternating direction implicit scheme is stable and efficient and with better resolution of steep gradients related to other scheme.

In recent contributions, the high-order finite difference method has been widely used by several authors to solve the nonlinear convection-diffusion equations or

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Burgers equation. Accordingly, [Bahadir (2003)] proposed a fully implicit finite-difference scheme to solve two-dimensional nonlinear Burgers equations in which accuracy was checked with analytical and numerical results and indicated that the method was well suited. In [Radwan (2005)] the two-dimensional unsteady Burgers equation was solved using the fourth-order accurate two-point compact scheme and the fourth-order accurate Du Fort Frankel scheme. In conclusion, the fourth-order two-point compact scheme is highly stable and efficient related to the fourth-order accurate Du Fort Frankel scheme. [Young, Fan, Hu and Atluri, (2008)] demonstrated the accuracy and simplicity of the Eulerian–Lagrangian method to solve two-dimensional unsteady Burgers equations and compared the numerical results with others analytical and numerical results.

Liu (2009) employed the fictitious time integration method to solve the backward in time and forward in time Burgers equation. Because the Fictitious Time Integration Method is integrated in a new direction of fictitious time, which is independent to the real time, the ill-posedness and noised disturbance for the backward in time Burgers equation can be handled rather well. This method developed is very effective to find the numerical solutions of backward in time problems involving partial differential equations .

Recently, several authors have presented results for the numerical solution of the Burgers equations, among them are noteworthy [Srivastava, Tamsir, Bhardwaj and Sanyasiraju (2011); Srivastava, Awasthi and Tamsir (2013); Srivastava, Singh, Awasthi and Tamsir (2013), Zheng, Fan and Li (2014)].

However, there are few papers for the numerical treatment of the solutions of three-dimensional Burgers equation. In order to contribute to this topic as well as extend the problems already solved in [Campos, Romão and Moura (2014); Cruz, Campos, Martins and Romão (2014)], in this paper the high order finite difference method with an efficient technique of linearization and low computational cost were implemented for the solution the following system of equations: given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

where $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ are the velocity field in the x,y,z-directions, respectively, and ν is the kinematic viscosity. These equations coincide with the three-dimensional momentum equations for incompressible laminar flows

if the pressure terms are neglected [Lewis et al. (2004)] This system of equations was chosen because it is a non-linear three-dimensional problem which allows the testing of a finite difference method of high-order jointly to linearization method proposed in this paper.

2 Formulation – High-Order Finite Difference Method

The numerical formulation proposed in this paper to solve the three-dimensional Burgers equation according with Eq. 1-3 begins with a discretization in time from the Crank-Nicolson method, as follows: the following system of equations: given by

$$
\left(\frac{u^{n+1} - u^n}{\Delta t} \right)
$$
$$
= 0.5 \left(\nu \frac{\partial^2 u^{n+1}}{\partial x^2} + \nu \frac{\partial^2 u^{n+1}}{\partial y^2} + \nu \frac{\partial^2 u^{n+1}}{\partial z^2} - u^{n+1} \frac{\partial u^{n+1}}{\partial x} - v^{n+1} \frac{\partial u^{n+1}}{\partial y} - w^{n+1} \frac{\partial u^{n+1}}{\partial z} \right)
$$
$$
+ 0.5 \left(\nu \frac{\partial^2 u^n}{\partial x^2} + \nu \frac{\partial^2 u^n}{\partial y^2} + \nu \frac{\partial^2 u^n}{\partial z^2} - u^n \frac{\partial u^n}{\partial x} - v^n \frac{\partial u^n}{\partial y} - w^n \frac{\partial u^n}{\partial z} \right)
$$
$$
\tag{4}
$$

$$
\left(\frac{v^{n+1} - v^n}{\Delta t} \right)
$$
$$
= 0.5 \left(\nu \frac{\partial^2 v^{n+1}}{\partial x^2} + \nu \frac{\partial^2 v^{n+1}}{\partial y^2} + \nu \frac{\partial^2 v^{n+1}}{\partial z^2} - u^{n+1} \frac{\partial v^{n+1}}{\partial x} - v^{n+1} \frac{\partial v^{n+1}}{\partial y} - w^{n+1} \frac{\partial v^{n+1}}{\partial z} \right)
$$
$$
+ 0.5 \left(\nu \frac{\partial^2 u^n}{\partial x^2} + \nu \frac{\partial^2 u^n}{\partial y^2} + \nu \frac{\partial^2 u^n}{\partial z^2} - u^n \frac{\partial v^n}{\partial x} - v^n \frac{\partial v^n}{\partial y} - w^n \frac{\partial v^n}{\partial z} \right)
$$
$$
\tag{5}
$$

$$
\left(\frac{w^{n+1} - w^n}{\Delta t} \right)
$$
$$
= 0.5 \left(\nu \frac{\partial^2 w^{n+1}}{\partial x^2} + \nu \frac{\partial^2 w^{n+1}}{\partial y^2} + \nu \frac{\partial^2 w^{n+1}}{\partial z^2} - u^{n+1} \frac{\partial w^{n+1}}{\partial x} - v^{n+1} \frac{\partial w^{n+1}}{\partial y} - w^{n+1} \frac{\partial w^{n+1}}{\partial z} \right)
$$
$$
+ 0.5 \left(\nu \frac{\partial^2 w^n}{\partial x^2} + \nu \frac{\partial^2 w^n}{\partial y^2} + \nu \frac{\partial^2 w^n}{\partial z^2} - u^n \frac{\partial w^n}{\partial x} - v^n \frac{\partial w^n}{\partial y} - w^n \frac{\partial w^n}{\partial z} \right)
$$
$$
\tag{6}
$$

Note that in Eq. 4-6 the existence of nonlinear convective terms which require special treatment. In the literature, several authors have presented procedures for

the linearization of the convective term, with emphasis [Galpin and Raithby (1986), Ozisik (1994), Deblois (1997), Smith (1998), Sheu and Lin, (2004), Sheu and Lin (2005)]. In this work the linearization technique proposed by [Jiang (1998), Jiang and Chang (1990)] considering a sufficiently small time step for the convective terms. Considering $F = f\frac{\partial f}{\partial x}$, which, for simplicity of notation, will be denoted by $F = st$, we can expand it in a Taylor series about the current value and terminate the series expansion after the first-derivative terms. The result is as follows:

$$s^{n+1}t^{n+1} \approx s^n t^n + \left[\frac{\partial}{\partial s}(s^n t^n)\right](s^{n+1} - s^n) + \left[\frac{\partial}{\partial t}(s^n t^n)\right](t^{n+1} - t^n)$$

$$\Rightarrow s^{n+1}t^{n+1} \approx s^n t^n + s^{n+1}t^n - s^n t^n + s^n t^{n+1} - s^n t^n \Rightarrow s^{n+1}t^{n+1} \approx s^{n+1}t^n + s^n t^{n+1} - s^n t^n$$

Replacing, we obtain:

$$f^{n+1}\frac{\partial f^{n+1}}{\partial x} \approx f^n \frac{\partial f^{n+1}}{\partial x} + f^{n+1}\frac{\partial f^n}{\partial x} - f^n \frac{\partial f^n}{\partial x} \tag{7}$$

This technique is referred to as Newton's method because it propitiates a quadratic convergence [Dennis and Schmabel (1983)]. Note that this technique does not require an iterative linearization at each time step, making quicker the computation of f.

Writing Eq. 7 for the term uu_x, for example, we have:

$$u^{n+1}\frac{\partial u^{n+1}}{\partial x} \approx u^n \frac{\partial u^{n+1}}{\partial x} + u^{n+1}\frac{\partial u^n}{\partial x} - u^n \frac{\partial u^n}{\partial x} \tag{8}$$

A similar procedure will be used in other nonlinear terms of Eq. 4-6.

In this manner, replacing the Eq. 7 in Eq. 4:

$$0.5\left(-v\frac{\partial^2 u^{n+1}}{\partial x^2} - v\frac{\partial^2 u^{n+1}}{\partial y^2} - v\frac{\partial^2 u^{n+1}}{\partial z^2} + u^n \frac{\partial u^{n+1}}{\partial x} + v^n \frac{\partial u^{n+1}}{\partial y} + w^n \frac{\partial u^{n+1}}{\partial z}\right.$$

$$\left. -u^{n+1}\frac{\partial u^n}{\partial x} - v^{n+1}\frac{\partial u^n}{\partial y} - w^{n+1}\frac{\partial u^n}{\partial z}\right) + \frac{u^{n+1}}{\Delta t} = F_1 \tag{9}$$

where $F_1 = \frac{u^n}{\Delta t} + 0.5\left(v\frac{\partial^2 u^n}{\partial x^2} + v\frac{\partial^2 u^n}{\partial y^2} + v\frac{\partial^2 u^n}{\partial z^2}\right)$.

Now, replacing the Eq. 7 in Eq. 5, it yields

$$0.5\left(-v\frac{\partial^2 v^{n+1}}{\partial x^2} - v\frac{\partial^2 v^{n+1}}{\partial y^2} - v\frac{\partial^2 v^{n+1}}{\partial z^2} + u^n \frac{\partial v^{n+1}}{\partial x} + v^n \frac{\partial v^{n+1}}{\partial y} + w^n \frac{\partial v^{n+1}}{\partial z}\right.$$

$$-u^{n+1}\frac{\partial v^n}{\partial x} - v^{n+1}\frac{\partial v^n}{\partial y} - w^{n+1}\frac{\partial v^n}{\partial z}\Bigg) + \frac{v^{n+1}}{\Delta t} = F_2 \tag{10}$$

where $F_2 = \frac{v^n}{\Delta t} + 0.5\left(v\frac{\partial^2 v^n}{\partial x^2} + v\frac{\partial^2 v^n}{\partial y^2} + v\frac{\partial^2 v^n}{\partial z^2}\right)$.

Finally, replacing the Eq. 7 in Eq. 6, we obtain

$$0.5\left(-v\frac{\partial^2 w^{n+1}}{\partial x^2} - v\frac{\partial^2 w^{n+1}}{\partial y^2} - v\frac{\partial^2 w^{n+1}}{\partial z^2} + u^n\frac{\partial w^{n+1}}{\partial x} + v^n\frac{\partial w^{n+1}}{\partial y} + w^n\frac{\partial w^{n+1}}{\partial z}\right.$$

$$-u^{n+1}\frac{\partial w^n}{\partial x} - v^{n+1}\frac{\partial w^n}{\partial y} - w^{n+1}\frac{\partial w^n}{\partial z}\Bigg) + \frac{w^{n+1}}{\Delta t} = F_3 \tag{11}$$

where $F_3 = \frac{w^n}{\Delta t} + 0.5\left(v\frac{\partial^2 w^n}{\partial x^2} + v\frac{\partial^2 w^n}{\partial y^2} + v\frac{\partial^2 w^n}{\partial z^2}\right)$.

Now, in order to carry out the spatial discretization of Eq. 9-11, the following procedure is used: considering nodes with $\Delta x, \Delta y$ or Δz distance from the boundary using the Central Difference Method with $O(\Delta x^2)$ (see Romão, Aguilar, Campos and Moura (2012)) to Eq. 9, we obtain:

$$\left(-\frac{0.5v}{\Delta z^2} - \frac{0.25w^n_{i,j,k}}{\Delta z}\right)u^{n+1}_{i,j,k-1} + \left(-\frac{0.5v}{\Delta y^2} - \frac{0.25v^n_{i,j,k}}{\Delta y}\right)u^{n+1}_{i,j-1,k}$$

$$+\left(-\frac{0.5v}{\Delta x^2} - \frac{0.25u^n_{i,j,k}}{\Delta x}\right)u^{n+1}_{i-1,j,k} + \left(\frac{v}{\Delta x^2} + \frac{v}{\Delta y^2} + \frac{v}{\Delta z^2} + \frac{1}{\Delta t} + 0.5\frac{\partial u^n_{i,j,k}}{\partial x}\right)u^{n+1}_{i,j,k}$$

$$+\left(0.5\frac{\partial u^n_{i,j,k}}{\partial y}\right)v^{n+1}_{i,j,k} + \left(0.5\frac{\partial u^n_{i,j,k}}{\partial z}\right)w^{n+1}_{i,j,k}$$

$$+\left(-\frac{0.5v}{\Delta x^2} + \frac{0.25u^n_{i,j,k}}{\Delta x}\right)u^{n+1}_{i+1,j,k} + \left(-\frac{0.5v}{\Delta y^2} + \frac{0.25v^n_{i,j,k}}{\Delta y}\right)u^{n+1}_{i,j+1,k}$$

$$+\left(-\frac{0.5v}{\Delta z^2} + \frac{0.25w^n_{i,j,k}}{\Delta z}\right)u^{n+1}_{i,j,k+1} = F_1 \tag{12}$$

where
$$F_1 = \frac{u^n_{i,j,k}}{\Delta t} + 0.5v\left(\frac{u^n_{i+1,j,k} - 2u^n_{i,j,k} + u^n_{i-1,j,k}}{\Delta x^2} + \frac{u^n_{i,j+1,k} - 2u^n_{i,j,k} + u^n_{i,j-1,k}}{\Delta y^2} + \frac{u^n_{i,j,k+1} - 2u^n_{i,j,k} + u^n_{i,j,k-1}}{\Delta z^2}\right).$$

Similarly to Eq. 10:

$$
\begin{aligned}
&\left(-\frac{0.5\nu}{\Delta z^2} - \frac{0.25w_{i,j,k}^n}{\Delta z}\right) v_{i,j,k-1}^{n+1} + \left(-\frac{0.5\nu}{\Delta y^2} - \frac{0.25v_{i,j,k}^n}{\Delta y}\right) v_{i,j-1,k}^{n+1} \\
&+ \left(-\frac{0.5\nu}{\Delta x^2} - \frac{0.25u_{i,j,k}^n}{\Delta x}\right) v_{i-1,j,k}^{n+1} + \left(0.5\frac{\partial v_{i,j,k}^n}{\partial x}\right) u_{i,j,k}^{n+1} \\
&+ \left(\frac{\nu}{\Delta x^2} + \frac{\nu}{\Delta y^2} + \frac{\nu}{\Delta z^2} + \frac{1}{\Delta t} + 0.5\frac{\partial v_{i,j,k}^n}{\partial y}\right) v_{i,j,k}^{n+1} + \left(0.5\frac{\partial v_{i,j,k}^n}{\partial z}\right) w_{i,j,k}^{n+1} \qquad (13) \\
&+ \left(-\frac{0.5\nu}{\Delta x^2} + \frac{0.25u_{i,j,k}^n}{\Delta x}\right) v_{i+1,j,k}^{n+1} + \left(-\frac{0.5\nu}{\Delta y^2} + \frac{0.25v_{i,j,k}^n}{\Delta y}\right) v_{i,j+1,k}^{n+1} \\
&+ \left(-\frac{0.5\nu}{\Delta z^2} + \frac{0.25w_{i,j,k}^n}{\Delta z}\right) v_{i,j,k+1}^{n+1} = F_2
\end{aligned}
$$

where
$$
F_2 = \frac{v_{i,j,k}^n}{\Delta t} + 0.5\nu \left(\frac{v_{i+1,j,k}^n - 2v_{i,j,k}^n + v_{i-1,j,k}^n}{\Delta x^2} + \frac{v_{i,j+1,k}^n - 2v_{i,j,k}^n + v_{i,j-1,k}^n}{\Delta y^2} + \frac{v_{i,j,k+1}^n - 2v_{i,j,k}^n + v_{i,j,k-1}^n}{\Delta z^2}\right).
$$
Finally, to Eq. 11:

$$
\begin{aligned}
&\left(-\frac{0.5\nu}{\Delta z^2} - \frac{0.25w_{i,j,k}^n}{\Delta z}\right) w_{i,j,k-1}^{n+1} + \left(-\frac{0.5\nu}{\Delta y^2} - \frac{0.25v_{i,j,k}^n}{\Delta y}\right) w_{i,j-1,k}^{n+1} \\
&+ \left(-\frac{0.5\nu}{\Delta x^2} - \frac{0.25u_{i,j,k}^n}{\Delta x}\right) w_{i-1,j,k}^{n+1} + \left(0.5\frac{\partial w_{i,j,k}^n}{\partial x}\right) u_{i,j,k}^{n+1} \\
&+ \left(0.5\frac{\partial w_{i,j,k}^n}{\partial y}\right) v_{i,j,k}^{n+1} + \left(\frac{\nu}{\Delta x^2} + \frac{\nu}{\Delta y^2} + \frac{\nu}{\Delta z^2} + \frac{1}{\Delta t} + 0.5\frac{\partial w_{i,j,k}^n}{\partial z}\right) w_{i,j,k}^{n+1} \qquad (14) \\
&+ \left(-\frac{0.5\nu}{\Delta x^2} + \frac{0.25u_{i,j,k}^n}{\Delta x}\right) w_{i+1,j,k}^{n+1} + \left(-\frac{0.5\nu}{\Delta y^2} + \frac{0.25v_{i,j,k}^n}{\Delta y}\right) w_{i,j+1,k}^{n+1} \\
&+ \left(-\frac{0.5\nu}{\Delta z^2} + \frac{0.25w_{i,j,k}^n}{\Delta z}\right) w_{i,j,k+1}^{n+1} = F_3
\end{aligned}
$$

where
$$
F_3 = \frac{w_{i,j,k}^n}{\Delta t} + 0.5\nu \left(\frac{w_{i+1,j,k}^n - 2w_{i,j,k}^n + w_{i-1,j,k}^n}{\Delta x^2} + \frac{w_{i,j+1,k}^n - 2w_{i,j,k}^n + w_{i,j-1,k}^n}{\Delta y^2} + \frac{w_{i,j,k+1}^n - 2w_{i,j,k}^n + w_{i,j,k-1}^n}{\Delta z^2}\right).
$$
Now, considering the internal nodes and using the Central Difference Method with

$O(\Delta x^4)$ (see Romão, Aguilar, Campos and Moura (2012)), to Eq. 9, we obtain:

$$
\begin{aligned}
&\left(\frac{\nu}{24\Delta z^2} + \frac{w_{i,j,k}^n}{24\Delta z} \right) u_{i,j,k-2}^{n+1} + \left(-\frac{2\nu}{3\Delta z^2} - \frac{w_{i,j,k}^n}{3\Delta z} \right) u_{i,j,k-1}^{n+1} \\
&+ \left(\frac{\nu}{24\Delta y^2} + \frac{v_{i,j,k}^n}{24\Delta y} \right) u_{i,j-2,k}^{n+1} + \left(-\frac{2\nu}{3\Delta y^2} - \frac{v_{i,j,k}^n}{3\Delta y} \right) u_{i,j-1,k}^{n+1} \\
&+ \left(\frac{\nu}{24\Delta x^2} + \frac{u_{i,j,k}^n}{24\Delta x} \right) u_{i-2,j,k}^{n+1} + \left(-\frac{2\nu}{3\Delta x^2} - \frac{u_{i,j,k}^n}{3\Delta x} \right) u_{i-1,j,k}^{n+1} \\
&+ \left(\frac{1,25\nu}{\Delta x^2} + \frac{1,25\nu}{\Delta y^2} + \frac{1,25\nu}{\Delta z^2} + \frac{1}{\Delta t} + 0.5 \frac{\partial u_{i,j,k}^n}{\partial x} \right) u_{i,j,k}^{n+1} \\
&+ \left(0.5 \frac{\partial u_{i,j,k}^n}{\partial y} \right) v_{i,j,k}^{n+1} + \left(0.5 \frac{\partial u_{i,j,k}^n}{\partial z} \right) w_{i,j,k}^{n+1} \\
&+ \left(-\frac{2\nu}{3\Delta x^2} + \frac{u_{i,j,k}^n}{3\Delta x} \right) u_{i+1,j,k}^{n+1} + \left(\frac{\nu}{24\Delta x^2} - \frac{u_{i,j,k}^n}{24\Delta x} \right) u_{i+2,j,k}^{n+1} \\
&+ \left(-\frac{2\nu}{3\Delta y^2} + \frac{v_{i,j,k}^n}{3\Delta y} \right) u_{i,j+1,k}^{n+1} + \left(\frac{\nu}{24\Delta y^2} - \frac{v_{i,j,k}^n}{24\Delta y} \right) u_{i,j+2,k}^{n+1} \\
&+ \left(-\frac{2\nu}{3\Delta z^2} + \frac{w_{i,j,k}^n}{3\Delta z} \right) u_{i,j,k+1}^{n+1} + \left(\frac{\nu}{24\Delta z^2} - \frac{w_{i,j,k}^n}{24\Delta z} \right) u_{i,j,k+2}^{n+1} = F_4
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
F_4 = &\frac{u_{i,j,k}^n}{\Delta t} + 0.5\nu \left(\frac{-u_{i+2,j,k}^n + 16u_{i+1,j,k}^n - 30u_{i,j,k}^n + 16u_{i-1,j,k}^n - u_{i-2,j,k}^n}{12\Delta x^2} + \right. \\
&+ \frac{-u_{i,j-2,k}^n + 16u_{i,j-1,k}^n - 30u_{i,j,k}^n + 16u_{i,j+1,k}^n - u_{i,j+2,k}^n}{12\Delta y^2} \\
&+ \left. \frac{-u_{i,j,k-2}^n + 16u_{i,j,k-1}^n - 30u_{i,j,k}^n + 16u_{i,j,k+1}^n - u_{i,j,k+2}^n}{12\Delta z^2} \right).
\end{aligned}
$$

Similarly, to Eq. 10:

$$
\begin{aligned}
&\left(\frac{\nu}{24\Delta z^2} + \frac{w_{i,j,k}^n}{24\Delta z} \right) v_{i,j,k-2}^{n+1} + \left(-\frac{2\nu}{3\Delta z^2} - \frac{w_{i,j,k}^n}{3\Delta z} \right) v_{i,j,k-1}^{n+1} \\
&+ \left(\frac{\nu}{24\Delta y^2} + \frac{v_{i,j,k}^n}{24\Delta y} \right) v_{i,j-2,k}^{n+1} + \left(-\frac{2\nu}{3\Delta y^2} - \frac{v_{i,j,k}^n}{3\Delta y} \right) v_{i,j-1,k}^{n+1} \\
&+ \left(\frac{\nu}{24\Delta x^2} + \frac{u_{i,j,k}^n}{24\Delta x} \right) v_{i-2,j,k}^{n+1} + \left(-\frac{2\nu}{3\Delta x^2} - \frac{u_{i,j,k}^n}{3\Delta x} \right) v_{i-1,j,k}^{n+1}
\end{aligned}
$$

$$+ \left(0.5 \frac{\partial v_{i,j,k}^n}{\partial x} \right) u_{i,j,k}^{n+1} + \left(\frac{1.25\nu}{\Delta x^2} + \frac{1.25\nu}{\Delta y^2} + \frac{1.25\nu}{\Delta z^2} + \frac{1}{\Delta t} + 0.5 \frac{\partial v_{i,j,k}^n}{\partial y} \right) v_{i,j,k}^{n+1}$$

$$+ 0.5 \frac{\partial v_{i,j,k}^n}{\partial z} w_{i,j,k}^{n+1} + \left(-\frac{2\nu}{3\Delta x^2} + \frac{u_{i,j,k}^n}{3\Delta x} \right) v_{i+1,j,k}^{n+1} + \left(\frac{\nu}{24\Delta x^2} - \frac{u_{i,j,k}^n}{24\Delta x} \right) v_{i+2,j,k}^{n+1}$$

$$+ \left(-\frac{2\nu}{3\Delta y^2} + \frac{v_{i,j,k}^n}{3\Delta y} \right) v_{i,j+1,k}^{n+1} + \left(\frac{\nu}{24\Delta y^2} - \frac{v_{i,j,k}^n}{24\Delta y} \right) v_{i,j+2,k}^{n+1} \qquad (16)$$

$$+ \left(-\frac{2\nu}{3\Delta z^2} + \frac{w_{i,j,k}^n}{3\Delta z} \right) v_{i,j,k+1}^{n+1} + \left(\frac{\nu}{24\Delta z^2} - \frac{w_{i,j,k}^n}{24\Delta z} \right) v_{i,j,k+2}^{n+1} = F_5$$

where

$$F_5 = \frac{v_{i,j,k}^n}{\Delta t} + 0.5\nu \left(\frac{-v_{i+2,j,k}^n + 16 v_{i+1,j,k}^n - 30 v_{i,j,k}^n + 16 v_{i-1,j,k}^n - v_{i-2,j,k}^n}{12\Delta x^2} \right.$$

$$+ \frac{-v_{i,j-2,k}^n + 16 v_{i,j-1,k}^n - 30 v_{i,j,k}^n + 16 v_{i,j+1,k}^n - v_{i,j+2,k}^n}{12\Delta y^2}$$

$$\left. + \frac{-v_{i,j,k-2}^n + 16 v_{i,j,k-1}^n - 30 v_{i,j,k}^n + 16 v_{i,j,k+1}^n - v_{i,j,k+2}^n}{12\Delta z^2} \right).$$

Finally, to Eq. 11:

$$\left(\frac{\nu}{24\Delta z^2} + \frac{w_{i,j,k}^n}{24\Delta z} \right) w_{i,j,k-2}^{n+1} + \left(-\frac{2\nu}{3\Delta z^2} - \frac{w_{i,j,k}^n}{3\Delta z} \right) w_{i,j,k-1}^{n+1}$$

$$+ \left(\frac{\nu}{24\Delta y^2} + \frac{v_{i,j,k}^n}{24\Delta y} \right) w_{i,j-2,k}^{n+1} + \left(-\frac{2\nu}{3\Delta y^2} - \frac{v_{i,j,k}^n}{3\Delta y} \right) w_{i,j-1,k}^{n+1}$$

$$+ \left(\frac{\nu}{24\Delta x^2} + \frac{u_{i,j,k}^n}{24\Delta x} \right) w_{i-2,j,k}^{n+1} + \left(-\frac{2\nu}{3\Delta x^2} - \frac{u_{i,j,k}^n}{3\Delta x} \right) w_{i-1,j,k}^{n+1}$$

$$+ \left(0.5 \frac{\partial w_{i,j,k}^n}{\partial x} \right) u_{i,j,k}^{n+1} + \left(0.5 \frac{\partial w_{i,j,k}^n}{\partial y} \right) v_{i,j,k}^{n+1}$$

$$+ \left(\frac{1.25\nu}{\Delta x^2} + \frac{1.25\nu}{\Delta y^2} + \frac{1.25\nu}{\Delta z^2} + \frac{1}{\Delta t} + 0.5 \frac{\partial w_{i,j,k}^n}{\partial z} \right) w_{i,j,k}^{n+1} \qquad (17)$$

$$+ \left(-\frac{2\nu}{3\Delta x^2} + \frac{u_{i,j,k}^n}{3\Delta x} \right) w_{i+1,j,k}^{n+1} + \left(\frac{\nu}{24\Delta x^2} - \frac{u_{i,j,k}^n}{24\Delta x} \right) w_{i+2,j,k}^{n+1}$$

$$+ \left(-\frac{2\nu}{3\Delta y^2} + \frac{v_{i,j,k}^n}{3\Delta y} \right) w_{i,j+1,k}^{n+1} + \left(\frac{\nu}{24\Delta y^2} - \frac{v_{i,j,k}^n}{24\Delta y} \right) w_{i,j+2,k}^{n+1}$$

$$+ \left(-\frac{2\nu}{3\Delta z^2} + \frac{w_{i,j,k}^n}{3\Delta z} \right) w_{i,j,k+1}^{n+1} + \left(\frac{\nu}{24\Delta z^2} - \frac{w_{i,j,k}^n}{24\Delta z} \right) w_{i,j,k+2}^{n+1} = F_6$$

where

$$F_6 = \frac{w_{i,j,k}^n}{\Delta t} + 0.5\nu \left(\frac{-w_{i+2,j,k}^n + 16w_{i+1,j,k}^n - 30w_{i,j,k}^n + 16w_{i-1,j,k}^n - w_{i-2,j,k}^n}{12\Delta x^2} \right.$$
$$+ \frac{-w_{i,j-2,k}^n + 16w_{i,j-1,k}^n - 30w_{i,j,k}^n + 16w_{i,j+1,k}^n - w_{i,j+2,k}^n}{12\Delta y^2}$$
$$\left. + \frac{-w_{i,j,k-2}^n + 16w_{i,j,k-1}^n - 30w_{i,j,k}^n + 16w_{i,j,k+1}^n - w_{i,j,k+2}^n}{12\Delta z^2} \right).$$

3   Numerical Applications

A linear system was generated from the Eq. 12-17 to solve the three-dimensional Burgers equation. Gauss-Seidel method was implemented to solve the linear system and in order to save computational time the matrix generated has only non-zero coefficients. The numerical implementation was performed in FORTRAN.

In order to evaluate the efficiency of the proposed formulation, two numerical applications are proposed and compared to the exact solution, providing the analysis of the error from L_∞ and L_2 norms [Romão, Campos and Moura (2011)].

Case 1: Here, in order to validate the numerical code, it was adopted the following exact solution: $u(x,y,z,t) = \frac{-0.5x+y+z-2.25xt}{1-2.25t^2}$, $v(x,y,z,t) = \frac{x-0.5y+z-2.25yt}{1-2.25t^2}$ and $w(x,y,z,t) = \frac{x+y-0.5z-2.25zt}{1-2.25t^2}$, using the same principle as used in Romão (2014).

Taking $L_x = L_y = L_z = 1$, $L_t = 0.1$ (end instant), $\Delta x = \Delta y = \Delta z = L_x/20$, $\Delta t = L_t/20$, the numerical results were compared with the exact solution considering the maximum error for stopping criterion for the Gauss-Seidel on the order of 10^{-14}.

Table 1 shows the accuracy of the numerical solutions of u, v and w according to L_∞ and L_2 norms. It was figured it out that the accuracy for the L_2 norm is in the same order of Gauss-Seidel method truncation error.

Table 1: Analysis of numerical accuracy of the solution u, v and w according to the L_∞ and L_2 norms.

	L_∞ *norm*	L_2 *norm*
u	1.48E-13	4.46E-14
v	1.51E-13	4.55E-14
w	1.54E-13	4.64E-14

Figures 1-3 show the velocity profiles of u, v and w in the *XY* plane, respectively, for z= 0,5. It was noted, for example, in Fig. (1), for $x = y$=1 the velocity profile

reaches approximately $u \approx 0.8$, which approaches the value given by the exact solution $u \approx 0.7928$. Now, Fig. 2, for $x = 0$ and $y = 0.15$, the velocity profile v reaches the value of approximately 0.4, which coincides with the value obtained via exact solution ($v \approx 0.40025$). Finally, in Fig. 3, for $x = 0$ and $y = 0.75$, we have $w(0;0.75;0.5;0.1) \approx 0.4$, which value approaches the value of $w \approx 0.3964$, obtained by exact solution.
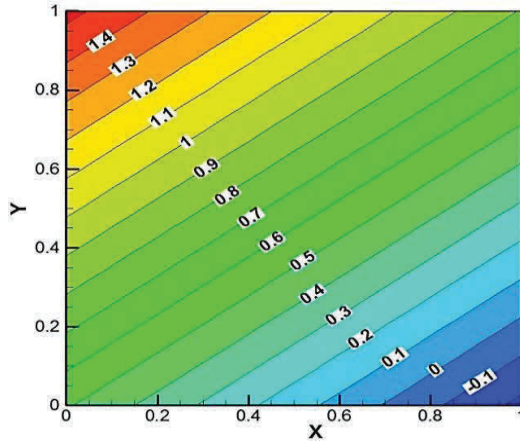

Figure 1: Two-dimensional velocity profile of u in the XY-plane with $z = 0.5$.
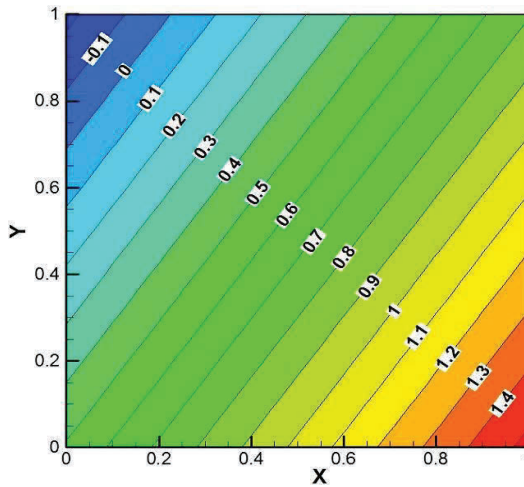

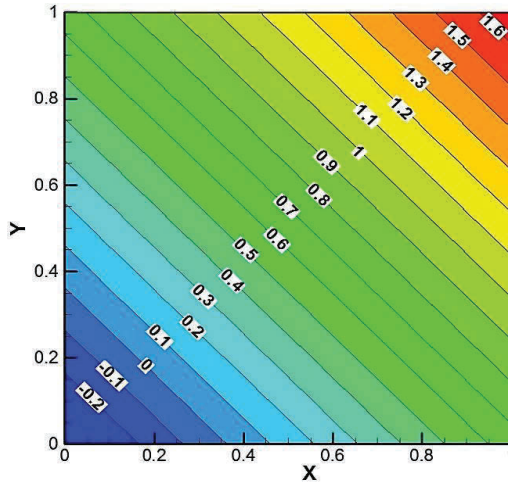Figure 2: Two-dimensional velocity profile of v in the XY-plane with $z = 0.5$.

Figure 3: Two-dimensional velocity profile of w in the *XY* -plane with $z = 0.5$.

Case 2: Considering the governing equations given by Eq. (1-3) with the following analytical solution proposed by Srivastava and Ashutosh (2013):

$$u(x,y,z,t) = -\frac{2}{Re}\frac{a_2 + a_5 y + a_7 z + a_8 yz + \gamma A(B\cos\gamma x - C\sin\gamma x)(D\sin\delta y + E\cos\delta y)(F\sin\mu z + G\cos\mu z)e^{\left(\frac{-\alpha^2}{Re}t\right)}}{H},$$

$$v(x,y,z,t) = -\frac{2}{Re}\frac{a_3 + a_5 x + a_6 z + a_8 xz + \delta A(B\sin\gamma x + C\cos\gamma x)(D\cos\delta y - E\sin\delta y)(F\sin\mu z + G\cos\mu z)e^{\left(\frac{-\alpha^2}{Re}t\right)}}{H},$$

$$w(x,y,z,t) = -\frac{2}{Re}\frac{a_3 + a_6 y + a_7 x + a_8 xy + \mu A(B\sin\gamma x + C\cos\gamma x)(D\sin\delta y + E\cos\delta y)(F\cos\mu z - G\sin\mu z)e^{\left(\frac{-\alpha^2}{Re}t\right)}}{H}$$

where $H = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 xz + a_8 xyz + A(B\sin\gamma x + C\cos\gamma y)(D\sin\delta x + E\cos\delta y)(F\sin\mu x + G\cos\mu y)e^{\left(\frac{-\alpha^2}{Re}t\right)}$, with α, γ, δ, μ, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_3, A, B, C, D, E, F and G arbitrary constants.

Taking $a_1 = a_2 = 1$; $a_3 = a_4 = ... = a_8 = 0$; $A = B = D = F = 1$; $C = E = G = 0$; $\gamma = \delta = \mu = 1$ and $\alpha = \sqrt{Re}$, the analytical solution analytical solution is employed of the form: $u(x,y,z,t) = -\frac{2}{Re}\left(\frac{1 + \cos(x)\sin(y)\sin(z)e^{-t}}{1 + x + \sin(x)\sin(y)\sin(z)e^{-t}}\right)$, $v(x,y,z,t) = -\frac{2}{Re}\left(\frac{\sin(x)\cos(y)\sin(z)e^{-t}}{1 + x + \sin(x)\sin(y)\sin(z)e^{-t}}\right)$ and $w(x,y,z,t) = -\frac{2}{Re}\left(\frac{\sin(x)\sin(y)\cos(z)e^{-t}}{1 + x + \sin(x)\sin(y)\sin(z)e^{-t}}\right)$.

It was considered, then, $h = \Delta x = \Delta y = \Delta z$, $L_x = L_y = L_z = L_t = 0.1$ and the maximum error for stopping criterion for the Gauss–Seidel on the order of 10^{-10}.

Table 2 shows the analysis of the error in terms of u, showing that is similar to the ones found to v and w, considering $h = \Delta t = 0.005$ and varying the kinematic viscosity. Several computational tests were performed by fixing the kinematic viscosity

and refining the mesh spatially or temporally, and no differences in the solution accuracy were visualized.

Figures 4-6 show, respectively, speed profiles of u, v and w in the XY-plane with $z = 0.5$, considering the kinematic viscosity at 10^0 and 10^{-2}. Considering, for example, in the Fig. 4, $x = 0.02$ and $y = 0$, the velocity profile reaches, approximately, $u \approx -1.96$, which approaches the value of the analytical solution ($u \approx -1.9607$). Similarly, in Fig. 5, for $x = 0.075$ and $y = 0$, the profile of the velocity u reaches the value of

-0.01860, coinciding with the value obtained via analytical solution ($u \approx -0.01860$).

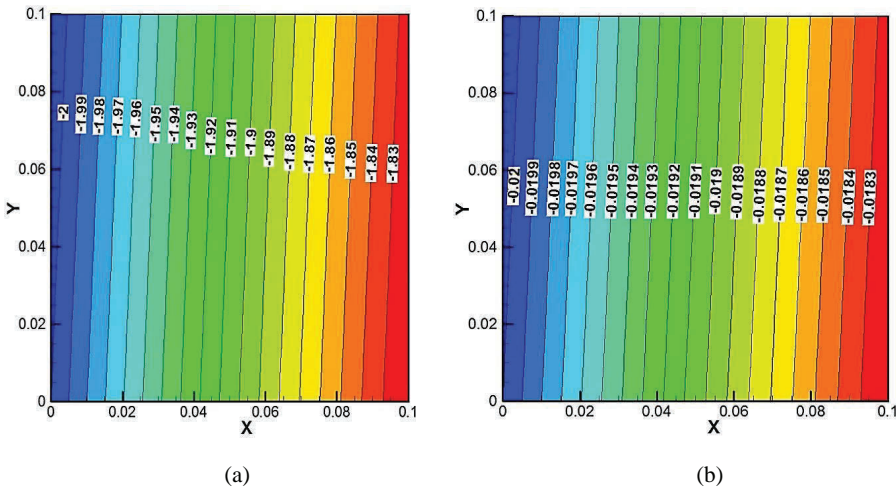(a)   (b)

Figure 4: Two-dimensional velocity profile of u in the XY-plane with $z = 0.5$ considering (a) $v = 10^0$ and (b) $v = 10^{-2}$.

Considering the domain and the time the proposed in this application, some variations of the h and Δt were performed and the results showed no significant changes (see Table 3) in order to allow an a study of the convergence rate.

4   Conclusions

The objective of this study was to present a numerical solution of high accuracy and low computational cost for the three-dimensional nonlinear Burgers equations. Using a numerical code in FORTRAN, it was possible to obtain excellent results in both applications, even when using coarse meshes. It is noteworthy that the proposed linearization technique has shown good results for a small number of time steps and there is no need to generate some iterative code each time step.
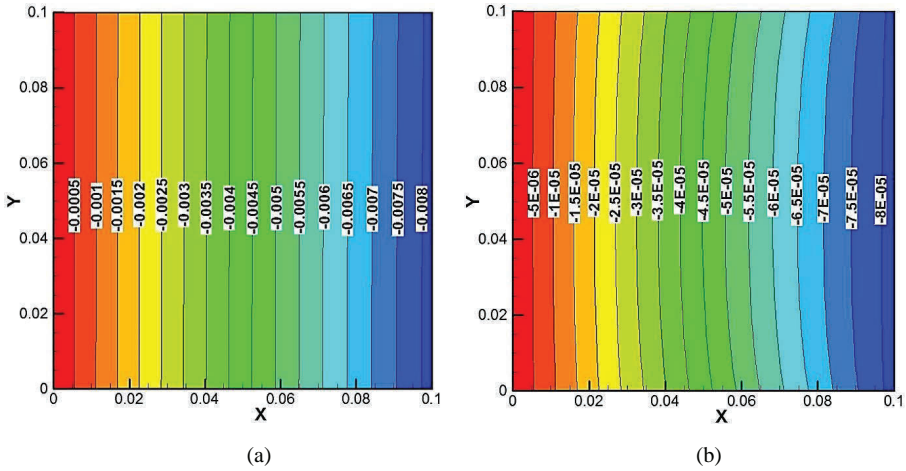
(a)   (b)

Figure 5: Two-dimensional velocity profile of *v* in the *XY*-plane with *z* = 0.5 considering (a) $v = 10^0$ and (b) $v = 10^{-2}$.
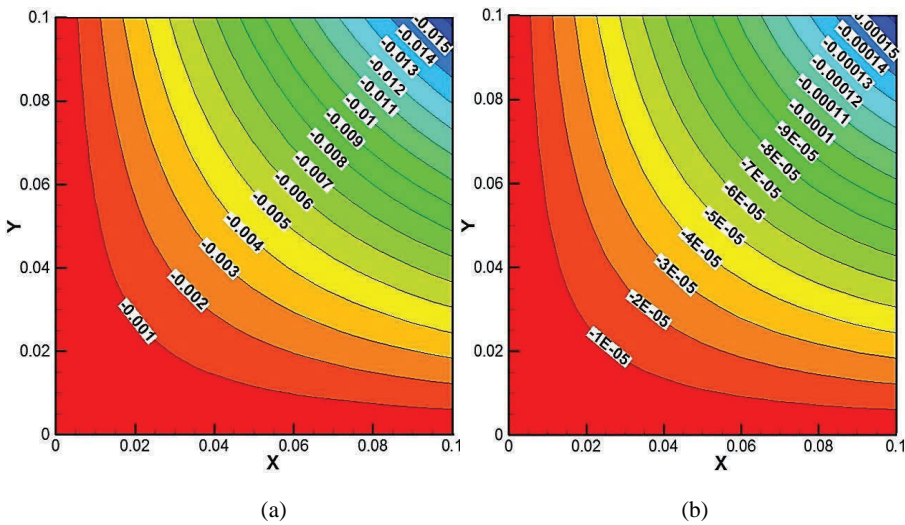

(a)   (b)

Figure 6: Two-dimensional velocity profile of *w* in the *XY*-plane with *z* = 0.5 considering (a) $v = 10^0$ and (b) $v = 10^{-2}$.

Table 2: Analysis of numerical accuracy of the solution u for $h = \Delta t = 0.005$ varying the kinematic viscosity.

ν	L_∞ *norm*	L_2 *norm*
10^1	2.95E-05	9.80E-06
10^0	5.19E-06	1.98E-06
10^{-1}	1.98E-06	7.12E-07
10^{-2}	2.70E-06	9.66E-07
10^{-3}	9.76E-07	3.29E-07
10^{-4}	1.45E-07	4.51E-08
10^{-5}	1.60E-08	4.78E-09

Table 3: Variation of some mesh for case 2 considering $\Delta t = 0.025$.

	$\nu = 10^1$	$\nu = 10^0$	$\nu = 10^{-1}$
$L_x/4$	5.57E-06	6.48E-07	1.44E-06
$L_x/5$	5.86E-06	6.81E-07	1.64E-06
$L_x/6$	6.36E-06	6.73E-07	1.66E-06
$L_x/7$	6.48E-06	6.62E-07	1.70E-06
$L_x/8$	6.30E-06	6.81E-07	1.75E-06

Thus, an important contribution of this work is the fact that the linearization technique can be applied for other numerical formulations that make use of Finite Element Method or Finite Volume Method.

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