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Eshelby Stress Tensor T: a Variety of Conservation Laws for T in Finite Deformation Anisotropic Hyperelastic Solid & Defect Mechanics, and the MLPG-Eshelby Method in Computational Finite Deformation Solid Mechanics-Part I

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Abstract: The concept of a stress tensor [for instance, the Cauchy stress $\boldsymbol{\sigma}$, Cauchy (1789-1857); the first Piola-Kirchhoff stress **P**, Piola (1794-1850), and Kirchhoff (1824-1889); and the second Piola-Kirchhoff stress, **S**] plays a central role in Newtonian continuum mechanics, through a physical approach based on the conservation laws for linear and angular momenta. The pioneering work of Noether (1882-1935), and the extraordinarily seminal work of Eshelby (1916-1981), lead to the concept of an "energy-momentum tensor" [Eshelby (1951)]. An alternate form of the "energy-momentum tensor" was also given by Eshelby (1975) by taking the two-point deformation gradient tensor as an independent field variable; and this leads to a stress measure **T** (which may be named as the Eshelby Stress Tensor). The corresponding conservation laws for **T** in terms of the path-independent integrals, given by Eshelby (1975), were obtained through a sequence of imagined operations to "cut the stress states" in the current configuration. These imagined operations can not conceptually be extended to nonlinear steady state or transient dynamic problems [Eshelby (1975)]. To the authors' knowledge, these path-independent integrals for dynamic finite-deformations of inhomogeneous materials were first derived by Atluri (1982) by examining the various internal and external work quantities during finite elasto-visco-plastic dynamic deformations, to derive the energy conservation laws, in the undeformed configuration [ref. to Eq. (18) in Atluri (1982)]. The stress tensor **T** was derived, independently, in its path-independent integral form for computational purposes [ref. to Eq. (30) in Atluri (1982)]. The corresponding integrals were successfully applied to nonlinear dynamic fracture analysis to determine "the energy change rate", denoted as $\mathbf{T}^*$.

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A similar analytical work for elasto-statics was reported by Hill (1986). With the use of the stress measure **T** for finite-deformation solid and defect mechanics, the concept of "the strength of the singularities", labeled in this paper as the vector $\mathbf{T}^*$, is formulated for a defective hyperelastic anisotropic solid undergoing finite deformations, in its various path-independent integral forms.
We first derive a vector balance law for the Eshelby stress tensor **T**, and show that it involves a mathematically "weak-form" of a vector momentum balance law for **P**. In small deformation linear elasticity (where **P**, **S** and $\boldsymbol{\sigma}$ are all equivalent), the stress tensor $\boldsymbol{\sigma}$ is linear in the deformation gradient **F**. Even in small deformation linear elasticity, the Eshelby Stress Tensor **T** is quadratic in **F**. By considering the various weak-forms of the balance law for **T** itself, we derive a variety of "conservation laws" for **T** in Section 2. We derive four important "path-independent" integrals, T_K^*, $T_L^{*(L)}$, $T^{*(M)}$, $T_{IJ}^{*(G)}$, in addition to many others. We show the relation of T_K^*, $T_L^{*(L)}$, $T^{*(M)}$ integrals to the $J-$, $L-$ and $M-$ integrals given in Knowles and Sternberg (1972). The four laws derived in this paper are, however, valid for finite-deformation anisotropic hyperelastic solid- and defect-mechanics. Some discussions related to the use of **T** in general computational solid mechanics of finitely deformed solids are given in Section 3. The application of the Eshelby stress tensor in computing the deformation of a one-dimensional bar is formulated in Section 4 for illustration purposes. We present two computational approaches: the Primal Meshless Local Petrov Galerkin (MLPG)-Eshelby Method, and the Mixed MLPG-Eshelby Method, as applications of the original MLPG method proposed by Atluri (1998,2004). More general applications of **T** directly, in computational solid mechanics of finitely deformed solids, will be reported in our forthcoming papers, for mechanical problems, in their explicitly-linearized forms, through the Primal MLPG-Eshelby and the Mixed MLPG-Eshelby Methods.

Keywords: Energy Momentum Tensor, Eshelby Stress Tensor, Meshless Local Petrov Galerkin, MLPG

1 Balance laws for the Cauchy Stress σ, the first Piola-Kirchhoff Stress P, the second Piola-Kirchhoff Stress S, and the Eshelby Stress T

We consider the finite deformation of a solid, wherein a material particle initially at **X**, moves to a location **x**. We use a fixed Cartesian coordinate system with base vectors $\mathbf{e}_i$, such that ($\mathbf{X} = X_I\mathbf{e}_i$ and $\mathbf{x} = x_i\mathbf{e}_i$). The displacement of the material particle is $\mathbf{u} = \mathbf{x} - \mathbf{X}$, and $\mathbf{v} = -\mathbf{u}$ [$u_i = (x_i - X_I)\mathbf{e}_i$; $\mathbf{u} = \mathbf{u}(\mathbf{X})$]. The deformation gradient tensor is **F** [$F_{iJ} = \frac{\partial x_i}{\partial X_J} \equiv x_{i,J} = u_{i,J} + \delta_{iJ}$].
There are infinitely many possible definitions of a stress-tensor in a finitely deformed solid [see, for instance Atluri(1984)]. Among the more commonly used

ones are: the Cauchy stress tensor $\boldsymbol{\sigma}$; the first Piola-Kirchhoff stress tensor $\mathbf{P}$; and the second Piola-Kirchhoff stress $\mathbf{S}$, which are related to each other, thus [see Atluri (1984)]:

$$\mathbf{P} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} = \mathbf{S} \cdot \mathbf{F}^t \tag{1}$$

$$\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-t} = \mathbf{P} \cdot \mathbf{F}^{-t} \tag{2}$$

where J is $\|\mathbf{F}\|$, and $(\cdot)^t$ denotes a transpose.

Considering a general *anisotropic hyperelastic* solid, with the strain energy per unit initial volume being denoted as W, the constitutive relation for $\mathbf{P}$ may be written as [see Atluri (1984)]:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}^t} \tag{3}$$

If W is a frame-indifferent function of $\mathbf{F}$, it should be a function only of $\mathbf{F}^t \cdot \mathbf{F}$. Thus [see Atluri (1984)],

$$P_{Ij} = \frac{\partial W}{\partial F_{jI}} = \frac{\partial W}{\partial E_{MN}} \frac{\partial E_{MN}}{\partial F_{jI}} = S_{IN}F_{jN}; \quad \mathbf{P} = \mathbf{S} \cdot \mathbf{F}^t \tag{4}$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^t \cdot \mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{C} - \mathbf{I}); \quad \mathbf{C} = \mathbf{F}^t \cdot \mathbf{F} = 2\mathbf{E} + \mathbf{I} \tag{5}$$

as $\mathbf{C}$ is the right Cauchy-Green deformation tensor, $\mathbf{E}$ being the Green-Lagrange Strain tensor.

The equations of Linear Momentum Balance (LMB) and Angular Momentum Balance (AMB) can be written equivalently in terms of $\boldsymbol{\sigma}$, $\mathbf{P}$, and $\mathbf{S}$ [see Atluri (1984)], as:

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j = 0 \quad (LMB); \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^t \quad (AMB) \tag{6a}$$

$$\frac{\partial P_{Ij}}{\partial X_I} + \rho_0 f_j = 0 \quad (LMB); \quad \mathbf{F} \cdot \mathbf{P} = \mathbf{P}^t \cdot \mathbf{F}^t \quad (AMB) \tag{6b}$$

$$\frac{\partial}{\partial X_I}[S_{IK}F_{jK}] + \rho_0 f_j = 0 \quad (LMB); \quad \mathbf{S} = \mathbf{S}^t \quad (AMB) \tag{6c}$$

where ρ_0 is the mass density (per unit initial volume). For a homogeneous solid, ρ_0 is not a function of $\mathbf{X}$, but is a constant.

The equivalence of Eqs. (6)a and (6)b can be seen from the following [using Eq. (1)],

$$P_{Ij,I} = \frac{\partial}{\partial X_I}\left(J\frac{\partial X_I}{\partial x_r}\sigma_{rj}\right) = \frac{\partial}{\partial X_I}\left(J\frac{\partial X_I}{\partial x_r}\right)\sigma_{rj} + J\frac{\partial \sigma_{rj}}{\partial X_I}\frac{\partial X_I}{\partial x_r} \tag{6)d}$$

However, as noted by Shield (1967) and Ogden (1975), we have the purely geometric identity for any finite deformation, that:

$$\frac{\partial}{\partial X_I}\left(J\frac{\partial X_I}{\partial x_r}\right) = 0 \tag{6)e}$$

Thus, from Eqs. (6)(d,e), we have:

$$\frac{\partial P_{Ij}}{\partial X_I} = J\frac{\partial \sigma_{rj}}{\partial x_r} \tag{6)f}$$

Thus, the geometric identity (6)e guarantees the equivalence of Eqs. (6)a and (6)b, in any exact solution. However, *in any computational solution*, such an equivalence may not always be assured.

When $\mathbf{F}$ is derived from an objective W, as in Eq. (4), $\mathbf{P} \equiv \mathbf{S}\cdot\mathbf{F}^t$ by definition [$\mathbf{S}$ is symmetric], and hence the AMB for $\mathbf{F}$ in Eq. (6)b is inherently embedded in the structure of W, viz, that W is a function of $\mathbf{E}$ only.

Following the seminal work of Noether (1918), and the extraordinarily important contributions of Eshelby (1957,1975), the Eshelby stress tensor is defined, for finite elasto-static deformations, as

$$\mathbf{T} = W\mathbf{I} - \mathbf{P}\cdot\mathbf{F} \tag{7)a}$$

or

$$T_{IJ} = W\delta_{IJ} - P_{Ik}F_{kJ} = W\,\delta_{IJ} - P_{Ik}(u_{k,J} + \delta_{kJ}) \tag{7)b}$$

where $\mathbf{I}$ is an identity tensor , W is the strain energy density (per unit initial volume), P_{ik} is the first Piola-Kirchhoff stress tensor, and $u_{k,J} = \partial u_k/\partial X_J$. While $\mathbf{T}$ is often referred to as the "Energy-Momentum Tensor", it clearly has the dimensions of "Stress" and was also independently derived by Atluri (1982) [see Eq. (29) in Atluri (1982)]. It was also discussed in Atluri (1984) that any function of $\mathbf{P}$ and $\mathbf{F}$ is a stress-measure (in finite deformation solid mechanics), such as $\mathbf{T}$ which is also a function of $\mathbf{P}$ and $\mathbf{F}$. We may also write $\mathbf{T}$, equivalently, as [see Eq. (27) in Atluri (1982)]:

$$\mathbf{T} = W\mathbf{I} - \mathbf{P}\cdot\mathbf{F} \tag{7)a}$$

$$= W\mathbf{I} - \mathbf{S}\cdot\mathbf{F}^t\cdot\mathbf{F} \quad (8)\text{a}$$

$$= W\mathbf{I} - \mathbf{S}\cdot\mathbf{C} \quad (8)\text{b}$$

$$= W\mathbf{I} - J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F} \quad (8)\text{c}$$

It can be seen from Eqs. (7)a and (8) that **T** *is in general an unsymmetric tensor*, unless **S** and **C** are co-axial, which is the case only for isotropic materials. We note thus, that the stress tensor **T** *is symmetric for isotropic materials*. We note that **P** is a two-point tensor field such that

$$\begin{aligned}(\mathbf{N}dA)\cdot\mathbf{P} &= d\mathbf{f}\\ \mathbf{P} = P^{Lj}\mathbf{G}_L\mathbf{g}_j &= P_{Lj}\mathbf{G}^L\mathbf{g}^j\end{aligned} \quad (9)$$

where $(\mathbf{N}dA)$ is a vector of an oriented-area in the undeformed configuration and $d\mathbf{f}$ is a vector of force acting on an oriented area $(\mathbf{n}da)$ (the image of $(\mathbf{N}dA)$) in the deformed configuration. Likewise, **F** is a two-point tensor field such that

$$\begin{aligned}d\mathbf{x} &= \mathbf{F}\cdot d\mathbf{X}\\ \mathbf{F} = F^{kL}\mathbf{g}_k\mathbf{G}_L &= F_{kL}\mathbf{g}^k\mathbf{G}^L\end{aligned} \quad (10)$$

where $\mathbf{G}_L$ are covariant base vectors in the undeformed configuration, and $\mathbf{g}_k$ are covariant base vectors in the deformed configuration.

Hence it follows that $\mathbf{P}\cdot\mathbf{F}$ is a tensor defined in the undeformed configuration. Since W is the strain energy density per unit volume in the undeformed configuration, it is seen that the tensor **T** in Eq (7) is *a tensor defined entirely in the undeformed configuration*, similar to the second Piola-Kirchhoff stress tensor **S**, except that **S** is symmetric, and **T** is unsymmetric.

If we consider infinitesimal deformations, then the difference between **P**, **S** and $\boldsymbol{\sigma}$ disappears. Further, if we consider linear elastic behaviors, along with infinitesimal deformation, **P**(or **S** or $\boldsymbol{\sigma}$) become linear in **F**. Even for linear elastic infinitesimal deformations, the Eshelby Stress Tensor **T** is a quadratic function of **F**. This poses some difficulties in the computational mechanics of infinitesimal deformations of even linear elastic solids, if the Eshelby stress tensor is used directly as a variable.

The "traction vector" $\mathbf{t}^*$, corresponding to the stress-tensor **T**, at a boundary with a unit normal **N** in the initial configuration, is written as:

$$\mathbf{t}^* = \mathbf{N}\cdot\mathbf{T} = W\mathbf{N} - \mathbf{N}\cdot\mathbf{P}\cdot\mathbf{F} \quad (11)\text{a}$$

or

$$t^*_K = N_I T_{iK} = W N_K - N_I P_{Ij} F_{jK} \quad (11)\text{b}$$

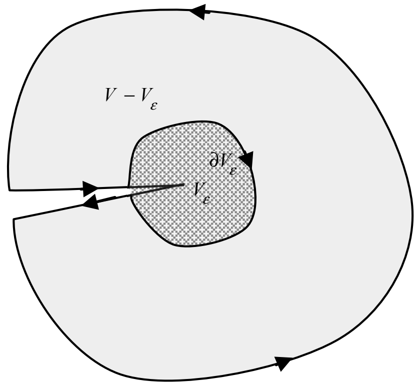

Figure 1: a close volume V includes a crack tip within V_ε

such that, the integral of the traction vector $\mathbf{t}^*$ over the small volume V_ε enclosing a crack-tip in 2-D, or a crack-segment in 3-D [see Fig. 1], may be denoted as the vector $\mathbf{T}^*$, and represented thus:

$$\begin{aligned}\mathbf{T}^* &= \int_{\partial V_\varepsilon} \mathbf{t}^* dS = \int_{\partial V_\varepsilon} \mathbf{N}\cdot\mathbf{T} dS \\ &\equiv \int_{\partial V} \mathbf{N}\cdot\mathbf{T} dS - \int_{V-V_\varepsilon} \nabla_X \cdot \mathbf{T} dV\end{aligned} \tag{12a}$$

or

$$\begin{aligned}T_K^* &= \int_{\partial V_\varepsilon} N_I T_{IK} dS = \int_{\partial V} N_I T_{IK} dS - \int_{V-V_\varepsilon} T_{IK,I} dV \\ &= \int_{\partial V} (W N_K - N_I P_{Ij} F_{jK}) dS - \int_{V-V_\varepsilon} (W\delta_{IK} - P_{Ij}F_{jK})_{,I} dV\end{aligned} \tag{12b}$$

Following the celebrated works of Eshelby (1957, 1975), we call $\mathbf{T}^*$ the vector of "Force on the Defect" in finite-deformation elasto-statics of general (anisotropic) hyperelastic solids. More precisely, in the case of cracks, it is a vector that quantifies the singular nature of stress and strain fields in V_ε. We note that $V - V_\varepsilon$ is free from any defects and singularities, and we may write:

$$\begin{aligned}&\int_{V-V_\varepsilon} (W\delta_{IK} - P_{Ij}F_{jK})_{,I} dV \\ &\quad = \int_{V-V_\varepsilon} [W_{,K}\big|_{\text{exp.}} + (P_{Ij}F_{jK} - P_{Ik} - P_{Ij}F_{jK})_{,I}] dV \\ &\quad = \int_{V-V_\varepsilon} (W_{,K}\big|_{\text{exp.}} + \rho_0 f_j F_{jK}) dV\end{aligned} \tag{13}$$

in which the following equations are used:

$$\frac{\partial P_{Ik}}{\partial X_I} + \rho_0 f_k = 0 \tag{14}$$

If the material is homogeneous, [i.e., W does not explicitly depend on $\mathbf{X}$], then $W_{,K}\big|_{\text{exp.}} = 0$.

Thus, for the problem of a hyperelastic solid, containing a crack-like defect[1], the vector of "concentrated Force on the Defect" $\mathbf{T}^*$ is given in Eq. (12). For an elasto-static problem of a defective homogenous solid, with no body forces or crack face tractions, we have a "path-independent" representation for $\mathbf{T}^*$:

$$T_K^* = \int_{\partial V_\varepsilon} N_I T_{IK})dS = \int_{\partial V_\varepsilon} (WN_K - N_I P_{Ij} F_{jK})dS = \int_{\partial V} (WN_K - N_I P_{Ij} F_{jK})dS \tag{15}$$

This force $\mathbf{T}^*$ in Eq. (12) has been called the "Force on the Defect" by Eshelby (1951,1957,1975).

In as much as the Stress Tensor $\mathbf{T}$ [Eq. (7)] is defined as:

$$T_{IJ} = W\,\delta_{IJ} - P_{Ik} F_{kJ} \tag{16}$$

We see that in [Eq. (13)], even for finite deformations,

$$\frac{\partial T_{IJ}}{\partial X_I} = W_{,J}\big|_{\text{exp.}} - \left(\frac{\partial P_{Ik}}{\partial X_I} + \rho_0 f_k\right) F_{kJ} + \rho_0 f_k F_{kJ} \quad \text{in } V - V_\varepsilon \tag{17}$$

If the material is homogeneous, [i.e., W does not explicitly depend on $\mathbf{X}$], and the body forces do not exist, and if the deformation is static, the stress tensor $\mathbf{T}$ is divergence free in a volume $V - V_\varepsilon$, which is free of singularities:

$$\frac{\partial T_{IJ}}{\partial X_I} = -\frac{\partial P_{Ik}}{\partial X_I} F_{kJ} = 0 \quad \text{in elasto-statics and no body forces in } V - V_\varepsilon \tag{18}$$

Otherwise, in general, we may write the generalized balance law for $\mathbf{T}$:

$$\frac{\partial T_{IJ}}{\partial X_I} - W_{,J}\big|_{\text{exp.}} - \rho_0 f_k F_{kJ} = -\left(\frac{\partial P_{Ik}}{\partial X_I} + \rho_0 f_k\right) F_{kJ} = 0 \quad \text{in } V - V_\varepsilon \tag{19a}$$

Eq. (19)a implies that the balance law for $\mathbf{T}$ *inherently involves the weak-form of the linear momentum balance law for* $\mathbf{P}$, *multiplied by the "test function"* $\mathbf{F}$.

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[1] In the case of a sharp-crack-like defect, in the integrand in $\mathbf{T}^*$ in ∂V_ε, on the right hand side of Eq. (14)a can have singularities of the type $r^{-1/2}$, such that, in a 2-D problem, $ds = r d\theta$, and $\mathbf{T}^*$ has a finite value even when $r \to 0$ in V_ε.

Eq. (19)a may be written more conveniently as,

$$\frac{\partial T_{IJ}}{\partial X_I} - \rho_0\rho_0 b_J = -\left(\frac{\partial P_{Ik}}{\partial X_I} + \rho_0 f_k\right) F_{kJ} = 0 \quad \text{in } V - V_\varepsilon \tag{19)b}$$

$$b_J \equiv \frac{1}{\rho_0} W_{,J}\Big|_{\text{exp.}} + f_k F_{kJ}$$

where the vector **b** is "the distributed force on the Defect". Thus, each of the 3 balance laws **T** is equal to a combination of the 3 balance laws for **P**. If the material is hypo-elastic or elastic-plastic, we may consider an objective rate of the stress-tensor **T**, and an incremental "Strength of the Singularity", $\Delta\mathbf{T}^*$, as contemplated in Atluri (1982).

It is interesting to observe that, the "generalized weak-form",

$$\int_V -\left(\frac{\partial P_{Ik}}{\partial X_I} + \rho_0 f_k\right) F^*_{kJ}\, dV = 0 = \int_V \left(\frac{\partial T_{IJ}}{\partial X_I} - \rho_0 b_J\right) dV \tag{20}$$

where V is a volume free of singularities and defects, and F^*_{kJ} is the virtual deformation gradient of a comparison state with displacement field $u^*_k(X_J)$, has been used by [Okada, Rajiyah and Atluri (1989a,b) , Han and Atluri (2003), and Qian, Han and Atluri (2004)] in deriving very novel non-hyper-singular integral equations for stresses in solid-mechanics, and gradient fields in acoustics, etc. These novel non-hyper-singular integral equations lead to extremely convenient algorithms for traction boundary value problems, and 3-dimensional fracture and fatigue mechanics problems [Han and Atluri (2002)].

T is in general an *unsymmetric tensor defined in the undeformed configuration* for anisotropic materials, and the balance law for **T**, even in finite deformation as stated in Eq. (19) is a set of linear PDEs in the undeformed coordinates, X_I. It has been stated earlier that, *for isotropic materials*, **T** is symmetric and, similar to **S**, is a tensor entirely defined in the undeformed configuration. Thus, *for infinitesimal deformations in an isotropic solid*, **T** still satisfies the balance law:

$$\frac{\partial T_{IJ}}{\partial X_I} = \rho_0 b_J \tag{19)c}$$

For *a general anisotropic hyperelastic solid*, one may find the displacement-like quantities v_I. We may require v_K to satisfy the same boundary conditions as for u_i at $\partial(V - V_\varepsilon)$. Thus v_K may be thought of as "the variation of the undeformed body" which is compatible with the prescribed boundary conditions for u_i, and correspond to the compatible strains derived from the stress tensor **T**. If we assume the displacement $\mathbf{u} = \mathbf{u}(\mathbf{X})$ is one-to-one mapping, its inverse mapping can be taken as the displacements [Knowles and Sternberg (1972)], as,

$$\mathbf{v}(\mathbf{x}) = -\mathbf{u}(\mathbf{x}) = -\mathbf{u}\left(\mathbf{X} + \mathbf{u}(\mathbf{X})\right) \quad \text{and} \quad \mathbf{X} = \mathbf{x} + \mathbf{v}(\mathbf{x}) \tag{21}$$

Thus the deformed configuration has been mapped back to the undeformed configuration. In the other words, v_K can also be considered as "the displacements of the deformed body".

For finite deformations, the inverse deformation gradients, $\mathbf{F}^{-1}$, is defined as,

$$F_{Ij}^{-1} = \frac{\partial X_I}{\partial x_j} \equiv X_{I,j} = v_{I,j} + \delta_{ij} \tag{22}$$

One may define the strain energy in the deformed configuration, denoted as $\tilde{W}$, as a frame-indifferent function of $\mathbf{F}^{-1}$. The corresponding first Piola-Kirchhoff stress tensor of the inverse deformation, denoted as $\tilde{\mathbf{P}}$, can be defined accordingly, as done in Eq. (3), as,

$$\tilde{\mathbf{P}} = \frac{\partial \tilde{W}}{\partial \mathbf{F}^{-t}} \tag{23}$$

The Eshelby stress tensor $\mathbf{T}$ can also be defined alternately [Eshelby (1975), Chadwick (1975)], as

$$\mathbf{T} = \frac{1}{J}\mathbf{F}^{t} \cdot \tilde{\mathbf{P}} \tag{24}$$

It shows the duality of the Eshelby Stress and the Cauchy Stress which is discussed in Section 3.

The corresponding left Cauchy-Green deformation tensor of the inverse deformation gradients can be written as,

$$\mathbf{b} \equiv \mathbf{F}^{-1} \cdot \mathbf{F}^{-t} = \mathbf{C}^{-1} \tag{25}$$

which is also the inverse of the right Cauchy-Green deformation tensor. It has been addressed as the "Finger Deformation Tensor" in the chemistry community for handling various physical fields in the current configuration, which is not common in applied mechanics. The corresponding strain tensor can be defined as

$$\tilde{\mathbf{e}} \equiv \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-1} \cdot \mathbf{F}^{-t}) = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1}) \tag{26}$$

One may observe that for a linear elastic material,

$$W(\mathbf{F}) = \frac{1}{2}\mathbf{S} : \mathbf{E} = \frac{1}{2}(W\mathbf{I} - \mathbf{T}) : \tilde{\mathbf{e}} \tag{27}$$

where $\mathbf{E}$ being the Green-Lagrange Strain tensor in Eq. (5).

2 A variety of weak-forms for the balance laws for T, in finite-deformation, anisotropic hyperelasticity

For finite deformations of a non-isotropic and non-homogeneous hyperelastic solid as shown in Fig. 1, the balance laws for the unsymmetric Eshelby stress tensor **T** are:

$$\frac{\partial T_{IJ}}{\partial X_I} = \rho_0 b_J \quad \text{in } V - V_\varepsilon \tag{28}$$

which are a set of strong-form linear partial differential equations in the undeformed coordinates X_I in the initial configuration in which the unsymmetic tensor **T** is entirely defined.

If we choose arbitrary but differentiable test functions $w_J(\mathbf{X})$, we may write the weak-form of Eq.(28), as:

$$\int_{V-V_\varepsilon} \frac{\partial T_{IJ}}{\partial X_I} w_J dV = \int_{V-V_\varepsilon} \rho_0 b_J w_J dV \tag{29}$$

Since $V - V_\varepsilon$ is free of any singularities, an application of the divergence theorem to Eq. (29) leads to:

$$\int_{\partial(V-V_\varepsilon)} t_J^* w_J dS - \int_{V-V_\varepsilon} (T_{IJ} w_{J,I} + \rho_0 b_J w_J) dV = 0 \tag{30}$$

On the other hand, if we choose the gradients $w_{J,K}(\mathbf{X})$ as the test functions, the weak-form of Eq. (28) may be written in a vector-from, as:

$$\int_{V-V_\varepsilon} \frac{\partial T_{IJ}}{\partial X_I} w_{J,K} dV = \int_{V-V_\varepsilon} \rho_0 b_J w_{J,K} dV \tag{31}$$

Or, equivalently, as:

$$\int_{\partial(V-V_\varepsilon)} t_J^* w_{J,K} dS - \int_{V-V_\varepsilon} (\rho_0 b_J w_{J,K} - T_{IJ} w_{J,KI}) dV = 0 \tag{32}$$

Now we consider a class of test functions in Eq. (30), such that[2],

$$w_J(\mathbf{X}) = B_{JK} X_K + C_J \tag{33}$$

[2] We already considered the general test function $F_{kJ}^* \left(\equiv \frac{\partial u_k^*}{\partial X_J} \right)$ corresponding to a virtual displacement field u_k^*, in deriving non-hyper-singular integral equations for stresses in solid mechanics, in Eq. (20).

where B_{JK} and C_J are constants.

We now consider several simple cases of the constants B_{JK} and C_J.
Case (A): $B_{JK} = 0$ **and** $C_J \neq 0$**.**
In this case,

$$w_J(\mathbf{X}) = C_J \tag{34}$$

Use of Eq. (34) in Eq. (30) results in:

$$\int_{\partial(V-V_\varepsilon)} t_J^* C_J dS - \int_{V-V_\varepsilon} \rho_0 b_J C_J dV = 0 \tag{35}$$

For arbitrary C_J, Eq. (35) leads to:

$$\int_{\partial(V-V_\varepsilon)} t_J^* dS - \int_{V-V_\varepsilon} \rho_0 b_J dV = 0 \tag{36}$$

In Knowles and Sternberg (1972), and in Eshelby (1975), the notion of a "conservation law" is used only when the integral over the volume $V - V_\varepsilon$ is zero; thus in the view of Knowles and Sternberg (1972) and Eshelby (1975), "path-independent-integrals" are identities expressed as integrals over the surface $\partial(V - V_\varepsilon)$ only. However, in a computational sense, we allow here that a path-independent-integral may involve both surface and volume integrals; such that an integral over ∂V_ε may be, for computational purposes, expressed as an integral over ∂V plus another integral over $V - V_\varepsilon$ [See Nikishkhov and Atluri(1987)]. In the present sense of a path-independent-integral, for finite deformations of nonhomogeneous non-isotropic solids, the "path-independent" integrals have the representation:

$$T_K^* = \int_{\partial V_\varepsilon} t_K^* dS = \int_{\partial V_\varepsilon} N_L T_{LK} dS \quad \equiv \int_{\partial V} N_L T_{LK} dS - \int_{V-V_\varepsilon} \rho_0 b_K dV \tag{37a}$$

or

$$T_K^* = \int_{\partial V} (W N_K - N_L P_{Lj} F_{jK}] dS - \int_{V-V_\varepsilon} (W_{,K}\big|_{\text{exp.}} + \rho_0 f_j F_{jK}) dV \tag{37b}$$

For elasto-static problems and for homogeneous anisotropic materials with zero body forces, Eq. (37) reduces to:

$$T_K^* = \int_{\partial V_\varepsilon} (W N_K - N_L P_{Lj} F_{jK}) dS = \int_{\partial V} (W N_K - N_L P_{Lj} F_{jK}) dS \tag{38}$$

Case (B): $B_{JK} = -B_{KJ}$ **and** $C_J = 0$**.**

We consider the test function w_J such that B_{JK} is skew-symmetric, and can be expressed as:

$$B_{JK} = e_{JLK}\omega_L \tag{39)a}$$

where e_{JLK} is the permutation tensor, such that

$$w_J(\mathbf{X}) = e_{JLM}\,\omega_L X_M \tag{39)b}$$

The weak form of Eq. (30) can now be written as:

$$\int_{\partial(V-V_\varepsilon)} t_J^*\, e_{JLM}\,\omega_L X_M dS - \int_{V-V_\varepsilon} [\rho_0 b_J\, e_{JLM}\,\omega_L X_M + T_{MJ}\, e_{JLM}\,\omega_L] dV = 0 \tag{40)a}$$

Eq. (40)a may be written for arbitrary ω_L as:

$$\int_{\partial(V-V_\varepsilon)} N_L T_{LJ}\, e_{JLM} X_M dS - \int_{V-V_\varepsilon} (\rho_0 b_J\, e_{JLM} X_M + T_{MJ} e_{JLM}) dV = 0 \tag{40)b}$$

Thus, *for finite deformations of an anisotropic hyperelastic material, with body forces, the generalized L-Integral has the representation*:

$$\begin{aligned} T_L^{*(L)} &\equiv \int_{\partial V_\varepsilon} N_L T_{LJ} e_{JLM} X_M dS \\ &= \int_{\partial V} N_L T_{LJ} e_{JLM} X_M dS - \int_{V-V_\varepsilon} (\rho_0 b_J e_{JLM} X_M + T_{MJ} e_{JLM}) dV \end{aligned} \tag{41)a}$$

If one considers only an isotropic material, $T_{IJ} = T_{JI}$ and thus $T_{MJ} e_{JLM} = 0$. Further, if one restricts to infinitesimal elasto-statics, and zero body forces, the integrands in the volume integral over $V - V_\varepsilon$ in Eq. (41) vanishes, the generalized L-Integral can be reduced as,

$$T_L^{*(L)} = \int_{\partial V} N_L T_{LJ} e_{JLM} X_M dS \tag{41)b}$$

The L-Integral in Eq. (41) is equivalent to its alternative form in term of the displacements, as given by Knowles and Sternberg in 1972. By definition, Eq. (40)a can be written as,

$$\int_{\partial(V-V_\varepsilon)} N_L (W\delta_{LJ} - P_{Lk} F_{k,J}) e_{JLM} X_M dS - \int_{V-V_\varepsilon} (\rho_0 b_J e_{JLM} X_M + T_{MJ} e_{JLM}) dV = 0 \tag{40)c}$$

However, the angular momentum balance for the stress $\mathbf{P}$ [$\mathbf{N}dS\cdot\mathbf{P} = d\mathbf{f}$ in the deformed configuration] states that:

$$\int_{\partial(V-V_\varepsilon)} \mathbf{x}\times(\mathbf{N}\cdot\mathbf{P})dS + \int_{V-V_\varepsilon} \mathbf{x}\times\rho_0\mathbf{f}dV = 0 \tag{42)a}$$

or

$$\int_{\partial(V-V_\varepsilon)} (\mathbf{X}+\mathbf{u})\times(\mathbf{N}\cdot\mathbf{P})dS + \int_{V-V_\varepsilon} \mathbf{x}\times\rho_0\mathbf{f}dV = 0 \tag{42)b}$$

Thus, Eq. (42) leads to:

$$\int_{\partial(V-V_\varepsilon)} N_L P_{Lj} e_{jlm}(X_m+u_m)dS + \int_{V-V_\varepsilon} \rho_0 f_j e_{jlm} x_m dV = 0 \tag{43}$$

Using Eq. (43) into Eq. (40)c, we obtain:

$$\begin{aligned}&\int_{\partial(V-V_\varepsilon)} N_I(W\delta_{IJ}-P_{Ik}u_{k,J})e_{JLM}X_M dS + \int_{\partial(V-V_\varepsilon)} N_I P_{Ij} e_{jLm} u_m dS \\ &\quad - \int_{V-V_\varepsilon} (\rho_0 b_J e_{JLM} X_M + T_{MJ} e_{JLM})dV + \int_{V-V_\varepsilon} \rho_0 f_j e_{jLm} x_m dV = 0\end{aligned} \tag{44}$$

Eq. (44) *is the conservation law in terms of the displacements for finite deformations of a hyperelastic, anisotropic solid. Eq.* (44) *can be reduced to the original form identified by Knowles and Sternberg in 1972, for small deformation linear elasticity without body forces,*

$$\int_{\partial(V-V_\varepsilon)} N_I\left\{(W\delta_{IJ}-P_{Ik}u_{k,J})X_M + P_{Ij}u_m\right\} e_{jLm} dS = 0 \tag{45}$$

Thus, with Eq. (44), the generalized L-Integral in Eq. (41) has the alternative representation in terms of the displacements, as

$$\begin{aligned}T_L^{*(L)} &\equiv \int_{\partial V_\varepsilon} N_I\left\{(W\delta_{IJ}-P_{Ik}u_{k,J})X_M + P_{Ij}u_m\right\} e_{jLm} dS \\ &= \int_{\partial V} N_I\left\{(W\delta_{IJ}-P_{Ik}u_{k,J})X_M + P_{Ij}u_m\right\} e_{jLm} dS \\ &\quad - \int_{V-V_\varepsilon} (\rho_0 b_J X_M + T_{MJ} - \rho_0 f_j x_m) e_{jLm} dV\end{aligned} \tag{46}$$

Case (C): $B_{JK} = C\delta_{JK}$ **and** $C_J = 0$**.**
such that

$$w_J(\mathbf{X}) = CX_J \tag{47}$$

where C is a constant. Use of Eq. (47) in Eq. (30) results in a scalar identity:

$$\int_{\partial(V-V_\varepsilon)} t_J^* X_J dS - \int_{V-V_\varepsilon} (\rho_0 b_J X_J + T_{JJ}) dV = 0 \tag{48}$$

Thus, for finite deformations of an anisotropic hyperelastic material, with body forces, the generalized M-Integral has the representation:

$$T^{*(M)} \equiv \int_{\partial V_\varepsilon} N_L T_{LJ} X_J dS = \int_{\partial(V-V_\varepsilon)} t_J^* X_J dS - \int_{V-V_\varepsilon} (\rho_0 b_J X_J + T_{JJ}) dV \tag{49}$$

The conservation law in Eq. (48) and the generalized M-Integral in Eq. (49) may also have the representation in term of the displacements. Eq. (48) may be written as:

$$\begin{aligned} &\int_{\partial(V-V_\varepsilon)} N_L [W\delta_{LJ} - P_{Lk}(u_{k,J} + \delta_{kJ})] X_J dS - \int_{V-V_\varepsilon} (\rho_0 b_J X_J + T_{JJ}) dV \\ &= \int_{\partial(V-V_\varepsilon)} N_L [W\delta_{LJ} - P_{Ik} u_{k,J}] X_J dS - \int_{\partial(V-V_\varepsilon)} N_L P_{Lk} X_k dS \\ &\quad - \int_{V-V_\varepsilon} (\rho_0 b_J X_J + T_{JJ}) dV = 0 \end{aligned} \tag{50}$$

However,

$$\int_{\partial(V-V_\varepsilon)} N_L P_{Lj} X_J dS = \int_{V-V_\varepsilon} (P_{Lj,L} X_J + P_{Jj}) dV = \int_{V-V_\varepsilon} (-\rho_0 f_j X_J + P_{Jj}) dV \tag{51}$$

Thus, one is lead to the general conservation law:

$$\begin{aligned} &\int_{\partial(V-V_\varepsilon)} N_L (W\delta_{LJ} - P_{Lk} u_{k,J}) X_J dS \\ &\quad - \int_{V-V_\varepsilon} [(-\rho_0 f_j + \rho_0 b_J) X_J + T_{JJ} + P_{Jj}] dV = 0 \end{aligned} \tag{52a}$$

where

$$T_{JJ} + P_{Jj} = 3W - P_{Lk} F_{kL} + P_{JJ} = 3W - P_{Lk} u_{k,L} \tag{52b}$$

For semi-linear anisotropic hyperelastic materials, one may postulate W such that:

$$W = \frac{1}{2} P_{Lk} u_{k,L} \tag{53}$$

Thus, for semi-linear anisotropic hyperelastic materials, one may write:

$$T_{JJ} + P_{Jj} = \frac{1}{2} P_{Lk} u_{k,L} \tag{54}$$

For semi-linear anisotropic materials, one may write the conservation law for finite deformations:

$$\begin{aligned}&\int_{\partial(V-V_\varepsilon)} N_L(W\delta_{LJ} - P_{Lk}u_{k,J})X_J dS \\ &\quad - \int_{V-V_\varepsilon} [(-\rho_0 f_j + \rho_0 b_J)X_J + \frac{1}{2}P_{Lk}u_{k,L}]dV = 0\end{aligned} \tag{55)a$$

However,

$$\frac{1}{2}\int_{V-V_\varepsilon} P_{Lk}u_{k,L}dV = \frac{1}{2}\int_{\partial(V-V_\varepsilon)} N_L P_{Lk}u_k dS - \frac{1}{2}\int_{V-V_\varepsilon} P_{Lk,L}u_k dV \tag{55)b$$

Thus, Eq. (55)a may be written, for finite deformations, of semi-linear anisotropic hyperelastic materials, as:

$$\begin{aligned}&\int_{\partial(V-V_\varepsilon)} N_L \left\{ (W\delta_{LJ} - P_{Lk}u_{k,J})X_J - \frac{1}{2}P_{Lk}u_k \right\} dS \\ &\quad - \int_{V-V_\varepsilon} [(-\rho_0 f_j + \rho_0 b_J)X_J - \frac{1}{2}(-\rho_0 f_j)u_j]dV = 0\end{aligned} \tag{56}$$

If we consider only infinitesimal deformations of anisotropic linear-elastic materials, and when deformations are independent of time and body forces are absent, Eq. (56) reduces to:

$$\int_{\partial(V-V_\varepsilon)} N_L \left\{ (W\delta_{LJ} - P_{Lk}u_{k,J})X_J - \frac{1}{2}P_{Lk}u_k \right\} dS = 0 \tag{57}$$

where, for infinitesimal deformations, P_{Lk} becomes synonymous with the Cauchy stress tensor, i.e., σ_{ik}. Eq. (57) has been identified as a conservation law, leading *to the now so-called M integral given by Knowles and Sternberg (1972) for infinitesimal deformations of linear elastic anisotropic materials.*

Thus, the generalized M-Integral in Eq. (49), for finite deformations of an anisotropic hyperelastic material, with body forces, has the representation in term of the displacements.

$$\begin{aligned}T^{*(M)} &= \int_{\partial V_\varepsilon} N_L \left\{ (W\delta_{LJ} - P_{Lk}u_{k,J})X_J - \frac{1}{2}P_{Lk}u_k \right\} dS \\ &= \int_{\partial V} N_L \left\{ (W\delta_{LJ} - P_{Lk}u_{k,J})X_J - \frac{1}{2}P_{Lk}u_k \right\} dS \\ &\quad - \int_{V-V_\varepsilon} [(-\rho_0 f_j + \rho_0 b_J)X_J + \frac{1}{2}\rho_0 f_j u_j]dV\end{aligned} \tag{58}$$

Case (D): $B_{JK} = B_{KJ}$, $J = |B_{JK}| = 1$ **and** $C_J = 0$.

We now consider test functions $w_J = B_{JK}X_K$, where B_{JK} is a symmetric deformation matrix, with the constraint that $|B_{JK}| = 1$.

Thus,

$$w_J(\mathbf{X}) = B_{JK}X_K \quad \text{and} \quad w_{J,J} = 0 \tag{59}$$

With the polar decomposition, the Eshelby stress tensor T can be written as

$$T_{IJ} = H_{IJ} + e_{IMN}G_{NJ,M} \tag{60}$$

where $H_{IJ,I} = T_{IJ,I} = \rho_0 b_J$ is curl-free and $G_{IJ,I} = 0$ is divergence free.

With zero body forces and using Eq. (59) in Eq. (30), we may write:

$$\int_{\partial(V-V_\varepsilon)} N_I T_{IJ} B_{JK} X_K dS = 0 \tag{61}$$

Thus we obtain the generalized conservation law for finite deformation in an anisotropic hyperelastic solid, as:

$$\int_{\partial(V-V_\varepsilon)} N_I T_{IJ} X_K dS = 0 \tag{62}$$

By applying Stoke's Theorem to Eq. (61), the generalized conservation law can be written for the potential function of the Eshelby stress tensor, as

$$\int_{\partial(V-V_\varepsilon)} e_{IKL} N_K G_{LJ} dS = 0 \tag{63}$$

Thus Eq. (63) may be considered to lead to the G-Integral:

$$T_{IJ}^{*(G)} \equiv \int_{\partial V_\varepsilon} e_{IKL} N_K G_{LJ} dS = \int_{\partial(V-V_\varepsilon)} e_{IKL} N_K G_{LJ} dS \tag{64}$$

Note that by using other arbitrary test functions, which may be arbitrary polynomials in X_K, *we may obtain an arbitrary number of generalized conservation laws*. However, each of these four special cases (A-D) discussed above has its own physical meaning and is corresponding to its own conservation law, as

- Case A, T_K^*: the linear momentum conservation law;
- Case B, $T_L^{*(L)}$: the angular momentum conservation law;
- Case C, $T^{*(M)}$: the divergence theorem;

- Case D, $T_{IJ}^{*(G)}$: Stokes' theorem;

Note that the G-Integral is the fourth path-independent integral of Noether's type [Noether (1918)], besides the three integrals reported by [Knowles and Sternberg (1972) and Eshelby (1975)]. Here these four conservation laws provide 12 independent equations for 3-D problems, and 6 independent equations for 2-D problems. $T^{*(M)}$ has been widely applied to incompressible problems with the pressure as an independent variable. $T_{IJ}^{*(G)}$ can be applied for the independent shear stresses for problems with materials such as liquid crystals or meta-materials, or anti-plane problems assuming no shear strains.

In addition, the Eshelby stress tensor can also be extended to the gradient theory of solids if the strain energy function in Eq. (3) is also dependent on the second derivatives of the displacements, $u_{i,JK}$ [Eshelby (1975)]. The corresponding conservation laws can also be obtained simply by writing the corresponding weak-forms of both macroscopic and microscopic momentum balance laws, as well as their forms weighted by the "test function" of the first and second derivatives of the displacements, following Eq. (17). The concept of "force of defects" can also be determined through conservation laws [Gurtin (2002)]. Another extension of the Eshelby tensor is to micropolar materials, in which the strain energy function is dependent on the deformation curvature. Similar conservation laws can also be derived by involving the curvature terms [Lubarda and Markenscoff (2003)].

3 The use of the Eshelby Stress Tensor in computational finite deformation solid mechanics

Since the concept of "the force on defect" was introduced by Eshelby in 1951, the Eshelby stress tensor (or the energy-momenturm tensor) has been extended for continuum mechanics of solids by Eshelby (1975), and independently in Atluri (1982) in which the expression of the Eshelby tensor has been given in term of the strain energy in the deformed configuration. The duality of the Eshelby Stress and the Cauchy Stress was also discovered by Eshelby (1975) and Chadwick (1975), and was extended to finite strain. With the use of the states of inverse deformation [Shield (1967)] and of the dual reciprocal states in finite elasticity [Ogden (1975)], the Eshelby stress tensor has been widely explored. The concept of "Eshelbian Mechanics" was introduced by Maugin (1995) based on the configuration invariance of the energy conservation law of Noether's Theorem, as the Lagrangian-Hamiltonian-Noetherian formulation. In contrast, Newtonian mechanics is based on the conservation laws of linear and angular momenta, and leads to the energy conservation law in the rate form in the undeformed configuration. *As shown in Eqs.* (17) *and* (19)*, the balance law for* **T** *inherently involves the "weighted"* *weak-form of the momentum balance law for* **P**.

The Eshelby stress tensor and its alternate forms have been widely used in developing numerical methods, especially for problems with singularities or inhomogeneities. On the other hand, by its definition in Eq. (7), the Eshelby stress tensor is a quadratic function of the deformation gradient tensor even for linear elastic or seme-linear (involving a linear relation between **P** and **F**) materials undergoing infinitesimal deformations. It becomes very difficult to develop numerical methods explicitly based on using the Eshelby stress tensor as a direct variable in solving problems, since **T** is a nonlinear function of **F** even for small-strain linear elastic behavior. A few exceptions include the exact use of the Eshelby tensor for inhomogeneous inclusions (Eshelby 1957,1959), and in the boundary integral forms for micromechanics [Mura (1991)]. The Eshelby stress tensor has so far been widely used only in *a post processing computation* to evaluate the forces on defects (or the configurational forces), especially for mesh-based numerical methods, once the stress and deformation are already computed.

We now review the difference between the equilibrium equations of the Cauchy stress, and the balance laws for Eshelby stress. The strong form of the momentum balance equations for the Cauchy stress tensor $\boldsymbol{\sigma}$ are in the deformed configuration as in Eq. (6)a. The Cauchy stress tenor $\boldsymbol{\sigma}$ in the deformed configuration is analogous to the Eshelby stress tensor in the undeformed configuration, and the Cauchy stress tensor is also a quadratic function of the inverse of the deformation gradient tensor, i.e. $F_{Ij}^{-1} = \frac{\partial X_I}{\partial x_j}$ even for semi-linear elastic solids. It implies that both the equilibrium equations based on the Cauchy stress tensor and the balance law of the Eshelby stress tensor are not suitable for linearization for solving the corresponding strong forms even for linear elastic materials undergoing infinitesimal deformations. The well-known equilibrium equation based on the first Piola-Kirchhoff stress tensor **P** in Eq. (6)b can be obtained through the coordinate transformation as in Eq. (10). It can be easily linearilized in the undeformed configuration. Its weak forms over the solution domain have been widely used for developing numerical methods including the finite element methods. Without losing generality, the weak form of Eq. (6)b can be written with a test function w_j as,

$$\int_V P_{Ij,I} w_j \, dV = 0 \tag{65)a}$$

and for a continuous function w_j,

$$\int_{\partial V} N_I P_{Ij} w_j \, dS - \int_V P_{Ij} w_{j,I} \, dV = 0 \tag{65)b}$$

In order to make the strong form of the Cauchy stress tensor satisfied through the weak forms in Eq. (65)a, the trial functions of the first Piola-Kirchhoff stress tensor

P need to satisfy its strong form in Eq. (6)b. It also requires that the deformation gradient tensor satisfy the geometric identity [Shield (1967), Ogden (1975)],

$$(JF_{Ij}^{-1})_{,I} = 0 \tag{66}$$

as already discussed in Eqs. (6)(d-f).

If Eq. (66) is not satisfied in a numerical computation, various attendant numerical issues need special treatment, such as the well-known finite-element mesh compatibility issues of strong displacement continuity and strong traction reciprocity. For example, many "locking" issues in mesh-based numerical methods are related to the weak-form conditions,

$$\begin{aligned} &P_{Ij,I} \neq 0 \\ &P_{Ij,I} w_j = 0 \end{aligned} \tag{67}$$

Because of which, the extra configurational forces are introduced. The crack problem is another example of a "strong singularity" at the crack-tip, as the value of $(JF_{Ij}^{-1})_{,I}$ is infinite within V_ε.

Hence, the Eshelby stress tensor as well as its alternate forms have been widely used in computing the configurational forces in the scalar or vector forms. The attendant path-independent integrals have been computed using the contour integral method, the domain integral method [Nikishkov and Atluri(1987)], or the interaction integral method, as well as the T^* integral for dynamic nonlinear problems [Atluri (1982), Nishoka and Atluri (1983)]. Its vector form in Eq. (12) provides the directional strength of the singularities for crack initialization and propagation [Gurtin & Podio-Guidugli (1996), Kienler & Herrmann (2002)]. It has also been used to evaluate and correct the incompatibility of the mesh-based trial functions, in order to avoid configuration forces, such as in the selective integration scheme, in the assumed strain field methods, and in the use of high-order elements with proper terms etc. One of the recent applications has been to develop the locking-free mesh-based methods for Hamiltonian systems by *choosing only configurational-force-free terms* under high speed rotation [Garcia-Vallejo, Mikkola and Escalona (2007), Sugiyama, Gerstmaya, Shabana (2006), Zhao and Ren (2012)].

It is impossible to make the deformation gradient tensor satisfy Eq. (66) through mesh-based trial functions. The trial functions through the use of the meshless interpolations, such as the moving least squares approximations and the radial basis aroximations, have been widely studied. However, the high-order continuity of such meshless trial functions, through the global solution domain, does not imply that Eq. (66) is satisfied any better. It becomes even more computationally costly because a high-order numerical quadrature scheme is required if the global

Galerkin approach is adopted. In contrast, the local meshless trial functions as in the Meshless Local Petrov Galerkin (MLPG) methods of Atluri et al (1998,2004), can satisfy Eq. (66) better, especially with the use of low order polynomial basis. Various test functions can be also chosen for computational efficiency, through the Meshless Local Petrov Galerkin (MLPG) approach [Atluri, et al (1998, 2004)]. It should be pointed out that the mixed MLPG method [Atluri, Han and Rajendran (2004)] becomes even more promising since the strain or stress can be interpolated independently along with the displacements as "the generalized degrees of freedom". Hence, Eq. (66) can be satisfied in a better way. By mapping the deformation gradient or stress variables back to the nodal displacements, the "locking-free MLPG method" has been developed by [Atluri, Han and Rajendran (2004)]. Since the balance laws for the Eshelby stress tensor **T** are essentially "weighted" forms of the momentum balance laws for the first Piola-Kirchhoff stress **P**, we may also make use of the Eshelby stress tensor to remove the restriction in Eq. (66), and the MLPG method can be extended to allow discontinuity in deformation. The inhomogeneity can also be included as "the distributed force on the defect" in Eq. (19). We call the resulting computational approach to solve for the displacements and the stress in a finitely deformed solid as the MLPG-Eshelby Method for computational solid mechanics. *This represents a radical departure from the current state of computational solid mechanics.* While the MLPG-Eshelby method for general computational finite deformation solid mechanics will be fully described in our forthcoming papers, a simple one-dimensional example is provided in the next section.

4 A simple example of the application of the Eshelby Stress Tensor in Computational Solid Mechanics

4.1 Formulation

For illustration purposes, we use the identity for the Eshelby stress tensor, derived independently in Atluri (1982) [Eqs. (18)&(19) in Atluri (1982)]. While computational methods for finite deformations of hyperelastic solids will be considered in our forthcoming papers, we consider now only a homogeneous linear elastic bar in a one-dimensional domain Ω, with a boundary $\partial\Omega$, subjected to the continuous body force $b(X)$ and the surface traction $\bar{P}(X_n)$, and undergoing infinitesimal deformations. The bar is descretized into n segments, each of an arbitrary length, as shown in Fig. 2.

One may write the Eq. (18) in [Atluri (1982)] for one-dimensional elasto-static

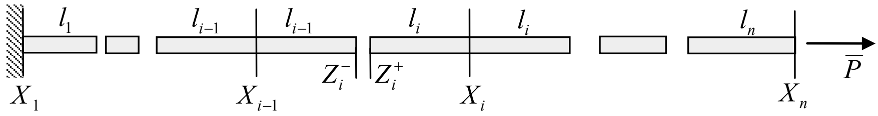

Figure 2: a bar divided into non-uniform segments

problems, as,

$$\int_V [\frac{dW}{dX} - \frac{d(P\cdot F)}{dX} - \rho_0 b(X)\cdot F]dV = 0 \qquad \text{for } \forall V \subset [X_1, X_n] \tag{68)a}$$

and the traction boundary conditions as

$$P(X_n) = \bar{P} \tag{68)b}$$

It is clearly seen that the combination of the first two terms in the left-hand side in Eq. (68)a leads to the definition of the Eshelby stress tensor in the present one-dimensional problem [i.e. $T = W - P\cdot F$]. The corresponding strong form balance law for the Eshelby stress tensor can be written as,

$$\frac{dT}{dX} = \frac{dW}{dX} - \frac{d(P\cdot F)}{dX} = \rho_0 b(X)\cdot F \quad \text{for } \forall X \in [X_1, X_n] \text{ and } X \notin \{Z_i\} \tag{69}$$

Let $u(X)$ be the trial function for the displacements, over the solution domain, and let it be piece-wise continuous *except at several points* $X = Z_i$. Eq. (69) is simply the weighted weak form of the equilibrium equation for the Piola-Kirchhoff stress P for the one-dimensional problems, as

$$-\frac{dP}{dX}\cdot F = \rho_0 b(X)\cdot F \quad \text{for } \forall X \in [X_1, X_n] \text{ and } X \notin \{Z_i\} \tag{70}$$

in which the constitutive relation for P in Eq. (3) is applied. One may choose the trial function $u(X)$ to be piece-wise continuous over each local sub-domain $\Omega_i \equiv (X_i - l_i, X_i + l_i)$ (except X_1 and X_n have only one side), but not necessarily continuous at $Z_i = X_i - l_i$ for $i = 2, n$. Hence, there are $n-1$ gaps between the local sub-domains, at the points Z_i.

One may choose δX [a "variation" in the values of the coordinates of a material particle of the solid, in its initial configuration] to be a test function, in order to write the weak-form of Eq. (70). We take δX to be piece-wise continuous, except at several points $X = Z_i$. The equilibrium equations for the Eshelby stress, as in Eq. (70), can be written in a weak form, over each local domain Ω_i, as

$$\begin{aligned}&\int_{X_i-l_i}^{X_i+l_i}[-\frac{dP}{dX}\cdot F-\rho_0 b(X)\cdot F]\delta X\,dX\\&=\int_{X_i-l_i}^{X_i+l_i}[-\frac{dP}{dX}-\rho_0 b(X)]\delta x\,dX\end{aligned} \quad \text{for } \forall X\in[X_1,X_n] \text{ and } X\notin\{Z_i\} \tag{71)a}$$

or by applying the divergence theorem,

$$\begin{aligned}&\int_{X_i-l_i}^{X_i+l_i}[-\frac{dP}{dX}\cdot F-\rho_0 b(X)\cdot F]\delta X\,dX\\&=\int_{X_i-l_i}^{X_i+l_i}P\delta F\,dX-P\delta x|_{X_i-l_i}^{X_i+l_i}-\int_{X_i-l_i}^{X_i+l_i}\rho_0 b(X)\delta x\,dX\end{aligned} \tag{71)b}$$

for $\forall X\in[X_1,X_n]$ and $X\notin\{Z_i\}$

in which by definition,

$$\delta x\equiv F\delta X \quad \text{and} \quad \delta F\equiv\frac{d\delta x}{dX} \tag{72}$$

From Eq. (72) it is clear that $\delta x(X)$ is not in the same class of functions as the trial function, $u(X)$. Thus Eq. (71) for the Eshelby stress tensor necessarily implies a Petrov-Galerkin approach.

The summation of Eq. (71)b over the solution domain should be zero only if there are no gaps of the trial function [i.e. $u(X)$ is continuous at every point]. However, the trial function is not continuous but has gaps at $X = Z_i$. The alternate test function $\delta x(X)$ also becomes non-continuous over the gaps $X = Z_i$ due to the discontinuity of the displacement gradients, $F(X)$. If the actual solution contains no "real gaps" at $X = Z_i$ [i.e. δX is continuous], the corresponding gaps are introduced simply because of the discretization error, as well as the alternate test function $\delta x(X)$. However the weak form in Eq. (71)a is still valid over the domain $Z_i - \varepsilon \le X \le Z_i + \varepsilon$. For illustration purposes, we may use this weak form over the gaps in the undeformed configuration to preserve the balance of energy, rather than applying the balance law of the Eshelby stress tensor, to derive the corresponding weak form over the gaps. By ignoring the body force over the gaps and taking a small segment $2\varepsilon = Z_i^+ - Z_i^-$ centered at the gaps $X = Z_i$, one may take a linear continuous test function over the gaps as,

$$\delta x(X)=[\delta x(Z_i^-)(Z_i^+-X)+\delta x(Z_i^+)(X-Z_i^-)]/(2\varepsilon) \quad \text{for } \forall X\in[Z_i^-,Z_i^+] \tag{73}$$

with an average of $[\delta x(Z_i^+)+\delta x(Z_i^-)]/2$.

Thus one may have the weak form for the gaps as

$$\begin{aligned}&\int_{Z_i^-}^{Z_i^+}[-\frac{dP}{dX}\cdot F-\rho_0 b(X)\cdot F]\delta X\,dX \\ &\quad=-[P(Z_i^+)-P(Z_i^-)][\delta x(Z_i^+)+\delta x(Z_i^-)]/2\end{aligned}\qquad \text{for } \forall X\in[Z_i^-,Z_i^+] \tag{74}$$

By adding Eqs. (74) and (71)b, the weak form of the equilibrium equations can be obtained for the numerical discretization problem as:

$$\begin{aligned}&\sum_{i=1}^{n}\int_{X_i-l_i}^{X_i+l_i}P\delta F\,dX+\sum_{i=2}^{n}[P(Z_i^+)+P(Z_i^-)][\delta x(Z_i^+)-\delta x(Z_i^-)]/2 \\ &\quad=\sum_{i=1}^{n}\int_{X_i-l_i}^{X_i+l_i}\rho_0 b(X)\delta x\,dX\end{aligned} \tag{75}$$

On the other hand, if defects exist at the gaps $X=Z_i$, the corresponding "forces on the defects" can be computed through Eq. (68). The test function δX has to be discontinuous at $X=Z_i$. The average movement of the defects can be defined as

$$\delta X(Z_i)=[\delta X(Z_i^+)+\delta X(Z_i^-)]/2 \tag{76}$$

With the use of Eq. (27) or the weak forms in Section 3 for V_ε, the Eshelby traction $\mathbf{t}^*$ can be computed in terms of the strain energy changes of $V\backslash V_\varepsilon$ which also be used to drive the defects or break elements [LSTC (2013)]. The applications of the Eshelby stress for discontinuous mechanics will be discussed in our following papers.

Eq. (75) may, at first glance, appear to be quite similar to the discretized equilibrium equations for the hybrid finite element methods [Atluri (1975)], which, however, are based on the direct "weak-form" of a momentum balance law for $\mathbf{P}$, rather than on the balance law for $\mathbf{P}$, weighted a priori with $\mathbf{F}$. However, the concepts behind Eq. (75), and the hybrid finite element methods, are quite different. For a given set of nodes in the undeformed initial configuration, $\{X_i\}$, and a trial function for the displacement $u(X)$, the corresponding stress $P(X)$ can be computed following the standard finite element procedures. The "weak-form" of the momentum balance law for $\mathbf{P}$ states that $u(X)$ is the solution, if the deformed domain $x=X+u(X)$ renders the "total potential energy" stationary. By taking any admissible test function $\delta u(X)$, the corresponding stress $\tilde{P}(X)$ computed from the neighboring solution $\tilde{u}(X)=u(X)+\delta u(X)$ must also satisfy the same "weak-form" of the momentum balance law for $\mathbf{P}$. It is also clear, that *in the usual Galerkin finite element methods, the test function $\delta u(X)$ [or the "variation" of $u(X)$] belongs to* *the same class of functions as* $u(X)$. In contrast, in present case of writing a "weak-form" of the balance law for the Eshelby stress tensor, namely for Eq. (70), one may keep the trial function $u(X)$ unchanged but change the coordinates in the initial configuration, namely X_i, to admissible neighboring points $Y_i = X_i + \delta X_i$, thus resulting in a new node set $\{Y_i\}$. Keeping the same trial function $u(X)$, the new trial function for the node Y_i can be computed through Eq. (72), as

$$u(Y_i) = u(X_i) + \delta X \left.\frac{du}{dX}\right|_{X=X_i} \equiv u(X_i) + \delta x(X_i) \tag{77}$$

or within a piece-wise continuously defined sub-domain,

$$u(Y) = u(X) + \frac{du(X)}{dX}\delta X = u(X) + F\delta X \equiv u(X) + \delta x(X) \tag{78}$$

It is clear that the test function $\delta x(X)$ used in Eq. (71) is dependent on both $\{X_i\}$ and $u(X)$, instead of being an admissible function of $\{X_i\}$ only, as in the usual finite element methods. It is the "change of the trial function" caused by "mesh changes". Within the general Galerkin approach, $\delta x(X)$ is replaced with the displacement variation and thus the weakform in Eq. (75) is reduced to the momentum balance law for **P**. The conservation law for the energy of the system can not be preserved in the usual finite element method. In other words, *the Galerkin approach enforces that the system's energy conservation law is preserved only in the undeformed configuration, instead of in any other configurations*. Noether's theorem [Noether (1918)] states that the energy conservation law must be configuration invariant. Hence, the test function in term of $\delta x(X)$ [as an alternate form of δX] should be chosen differently from the trial function of $u(X)$ which essentially leads to the Petrov-Galerkin approach.

With a continuous test function δX, the corresponding stress $P(Y)$ can also be computed in a same way. The "weak-form" of the balance law for **T** states that $u(X)$ is the solution if all "computed" solutions $u(Y)$ and $P(Y)$ also satisfy the same weak form. If not, the newly computed unbalanced nodal forces are the so-called configurational forces.

If the mapping operation between the mesh changes δX and the changes of the trial function $\delta x(X) = F\delta X$ is invertible, through using a continuous trial function, any admissible test function $\delta x(X)$ can be used in Eq. (75), instead of δX explicitly. However, such compatible invertible relations can be not defined for 2-D or 3-D problems for a global mesh-based or even a meshless interpolation. One may choose $\delta x(X)$ to be same as $\delta u(X)$, which has been done in most global meshless Galerkin methods by ignoring the actual inverted functionδXthrough F. It may cause some numerical issues, such as i) the continuity requirement not allowing any discontinuity or defects within the whole solution domain; ii) higher order quadrature schemes for high-order nonlinear integrands in the domain integrals requiring more material integration points which is computationally time costly for nonlinear materials; and iii) difficulty in enforcing the essential boundary conditions as δX is mapped to different class of funtions. On the other hand, a compatible and invertible relations between $\delta x(X)$ and δX can be easily constructed in a closed form within "a local spatial patch" [or a local sub domain]. Such techniques have been widely used in the error estimation and mesh adaptivity with higher order accuracy. Hence the weak-forms within a local sub domain become more convenient through the Meshless Local Petrov-Galerkin (MLPG) approach [Atluri (1998, 2004)].

4.2 Numerical implementation

There are many ways to choose the trial function $u(X)$, over the solution domain with various orders of continuity, in the initial coordinates. First the moving least squares (MLS) approximation is used to construct the trial function based on the fictitious nodal value $\hat{u}^{(i)}$ [Atluri (2004)], as

$$u^{MLS}(X) = \sum_{i=1}^{n} \Phi^{(i)}(X)\hat{u}^{(i)} \tag{79}$$

The continuity of the trial function u^{MLS} is dependent on the weight functions $w^{(i)}(X)$ in the MLS interpolation [Atluri (2004)]. In the present study, we choose the fourth-order spline function as the weight function, which leads to a continuous trial function.

Secondly, the mixed interpolation for both $F(X)$ and $u(X)$ can also be used to construct the trial functions based on the fictitious nodal value $\hat{u}^{(i)}$ and $F^{(i)}$ [Atluri, Han and Rajendran (2004)] over each segment, as

$$\begin{aligned} F^{MIX}(X) &= F^{(i)} \equiv \left.\frac{du^{MLS}}{dX}\right|_{X=X_i} \qquad \text{for } \forall X \in [X_{i-1}, X_i] \\ u^{MIX}(X) &= u(X^{(i)}) + F^{(i)} \cdot (X - X^{(i)}) \end{aligned} \tag{80}$$

It is clear that the trial function $u^{MIX}(X)$ is piece-wise linear, but discontinuous at points Z_i (i.e. with gaps).

The test function δX does not explicitly appear in Eq. (75), instead only δx appears. Hence, one may define δx over the solution domain, and the corresponding δX may be computed through Eq. (72) which is not involved in the numerical computation. It needs to be pointed out that Eq. (72) can not be defined globally (except in their global boundary/domain weighted integral forms) rather than within a local sub-domain. It is another reason which precludes the Eshelby stress tensor from being implemented through the global mesh-based methods [Such as the usual finite element methods, or the global Galerkin methods]. Eq. (75) becomes nonlinear in δx if the test function δX is assumed first, and δx then is computed through Eq. (72).

The test function δx can be chosen to be piece-wise linear for a simple mapping relations between δX and δx. First the element-based simple polynomial shape functions are chosen to interpolate the test function based on the nodal values $v^{(i)}$ over each segment, as,

$$\delta x^{FEM}(X) = N^{(i-1)}\delta x^{(i-1)} + N^{(i)}\delta x^{(i)} \qquad \text{for } \forall X \in [X_i - l_i, X_i + l_i] \tag{81}$$

in which the shape functions $N^{(i-1)} = 1-\xi$ and $N^{(i)} = \xi$ are linear. The test function δx^{FEM} is piece-wise linear and possesses C^0 continuity (i.e. no gaps), and a linear relation between δX and δx can be obtained within elements if the trial function is continuous, such as u^{MLS}.

On the other hand, the mixed MLS interpolation is also chosen to construct the discontinuous trial function u^{MIX}. A simple continuous test function δX can be chosen as discussed in Section 2, as:

$$\delta X^{MIX}(X) = \delta X^{(i)} + \delta\lambda^{(i)} \cdot (X - X^{(i)}) \qquad \text{for } \forall X \in [X_i - l_i, X_i + l_i] \tag{82}$$

where $\delta X^{(i)}$ and $\delta\lambda^{(i)}$ are two independent nodal variables.

Thus a linear relation between δX and δx can be computed within each local sub-domain, by definition, as

$$\begin{aligned} \delta x^{MIX}(X) &= F(X) \cdot \delta X(X) = F^{(i)}[\delta X^{(i)} + \delta\lambda^{(i)} \cdot (X - X^{(i)})] \\ &= F^{(i)}\delta X^{(i)} + F^{(i)}\delta\lambda^{(i)} \cdot (X - X^{(i)}) \\ &\equiv \delta x^{(i)} + \delta F^{(i)} \cdot (X - X^{(i)}) \qquad \text{for } \forall X \in [X_i - l_i, X_i + l_i] \end{aligned} \tag{83}$$

in which the nodal values $\delta x^{(i)}$ and $\delta F^{(i)}$ are independent variables and different from those used in Eq. (80).

The MLPG approach based on the Eshelby stress tensor [hereafter labeled as the MLPG-Eshelby Method] is presented here first by choosing u^{MLS} as the trial function, and δx^{FEM} as the test function. We call this the "Primal MLPG-Eshelby Method". The domain integrals in Eq. (75) are performed over the solution domain without any singular gaps.

The second MLPG-Eshelby method is formulated by choosing u^{MIX} as the trial function and δx^{MIX} as the test functions, and labeled as the "Mixed MLPG Eshelby method". The domain integrals in Eq. (75) can be simplified by evaluating the constant terms, as

$$\sum_{i=1}^{n} \delta x^{(i)} \int_{X_i-l_i}^{X_i+l_i} \rho_0 b(X)\,dX = \sum_{i=2}^{n} (P^{(i-1)} l_{i-1} \delta F^{(i-1)} + P^{(i)} l_i \delta F^{(i)})$$
$$+ \sum_{i=2}^{n} \frac{(P^{(i)} + P^{(i-1)})}{2} (\delta x^{(i)} - \delta x^{(i-1)} - l_i \delta F^{(i)} - l_{i-1} \delta F^{(i-1)})$$
$$= \sum_{i=2}^{n} \frac{(P^{(i)} + P^{(i-1)})}{2} (\delta x^{(i)} - \delta x^{(i-1)}) + \sum_{i=2}^{n} \frac{(P^{(i)} - P^{(i-1)})}{2} (l_i \delta F^{(i)} - l_{i-1} \delta F^{(i-1)}) \tag{84}$$

It is interesting that no interpolation is involved in Eq. (84) other than evaluating the nodal values, including the displacement gradients and stresses. Essentially Eq. (84) becomes a "Particle" method which is computationally efficient. In addition, the second order term in $\delta F^{(i)}$ can be omitted and the system can be written as,

$$\sum_{i=2}^{n} \frac{(P^{(i)} + P^{(i-1)})}{2} (\delta x^{(i)} - \delta x^{(i-1)}) = \sum_{i=1}^{n} \delta x^{(i)} \int_{X_i-l_i}^{X_i+l_i} \rho_0 b(X)\,dX \tag{85}$$

4.3 Numerical results

The bar is fixed at the left hand end and subjected to four loading conditions

i) uniform tension with zero body force (constant stress), as
$b(X) = 0$ and $P(X_n) = 1$

ii) constant body force as gravity load (linear stress), as
$b(X) = 1$ and $P(X_n) = 0$

iii) linear body force as centrifugal force (second order nonlinear stress) , as
$b(X) = X$ and $P(X_n) = 0$

iv) second order body force (third order nonlinear stress) , as
$b(X) = X^2$ and $P(X_n) = 0$

The bar is discretized regularly into 10 sub-domains, or irregularly with maximum 30% random variation from the regular sub-domains. All nodal coordinates are listed in Table 1.

The normalized relative displacement errors are shown in Figs. 3-6 for the regular sub-domains, and in Figs. 7-10 for the irregular sub-domains. The numerical

Table 1: Node coordinates of a bar

Node#	1	2	3	4	5	6	7	8	9	10	11
Regular sub-domains	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Irregular sub-domains	0.0	0.115	0.211	0.307	0.417	0.505	0.582	0.672	0.781	0.902	1.0

results show that the "Primal MLPG Eshelby Method" and the "Mixed MLPG Eshelby Method" pass the patch test and are quite stable.

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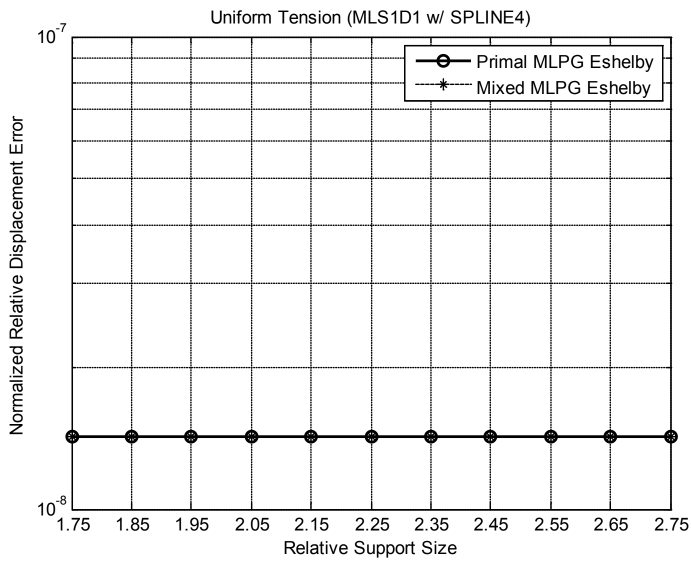

Figure 3: Normalized relative displacement errors of a bar under uniform tension (regular sub-domains)

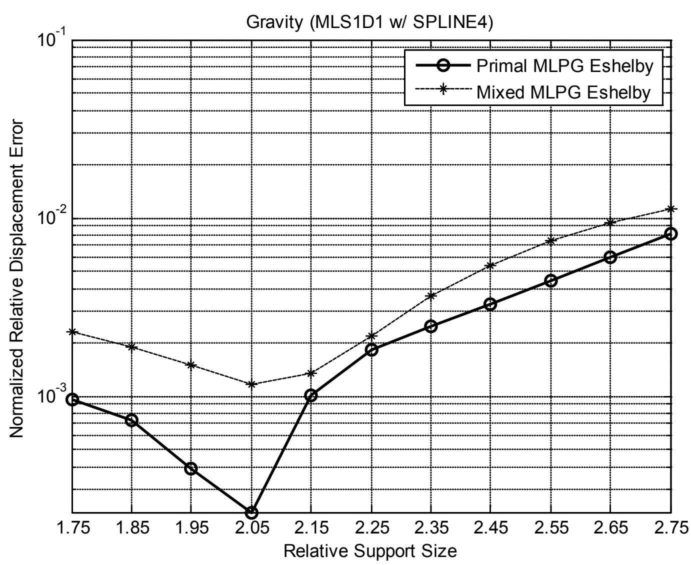

Figure 4: Normalized relative displacement errors of a bar under gravity load (regular sub-domains)

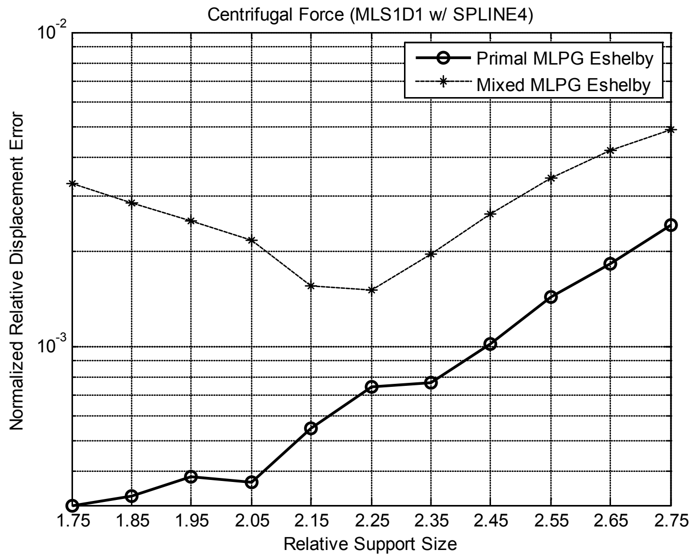

Figure 5: Normalized relative displacement errors of a bar under centrifugal force (regular sub-domains)

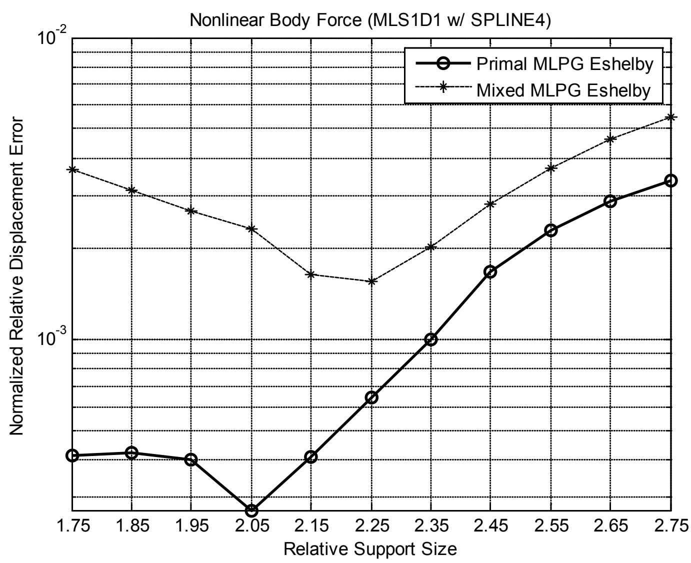

Figure 6: Normalized relative displacement errors of a bar under second order body force (regular sub-domains)

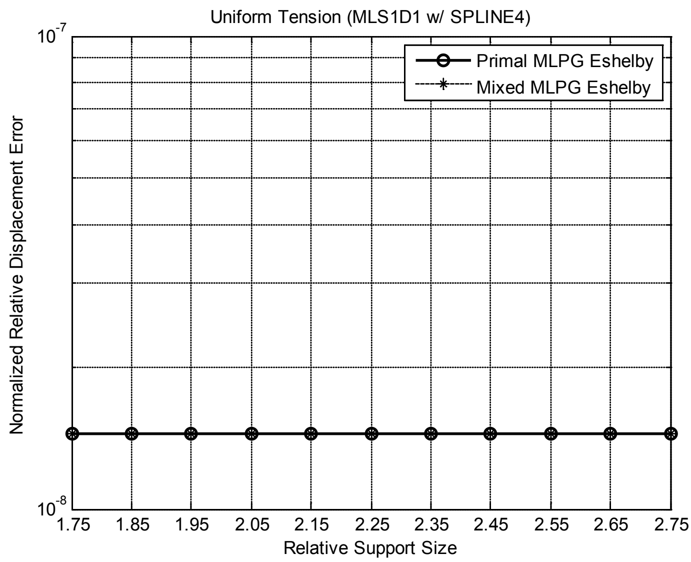

Figure 7: Normalized relative displacement errors of a bar under uniform tension (irregular sub-domains)

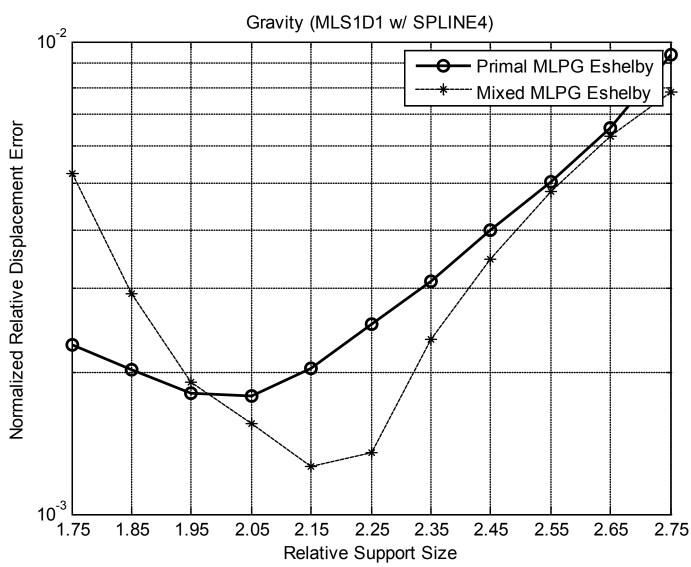

Figure 8: Normalized relative displacement errors of a bar under gravity load (irregular sub-domains)

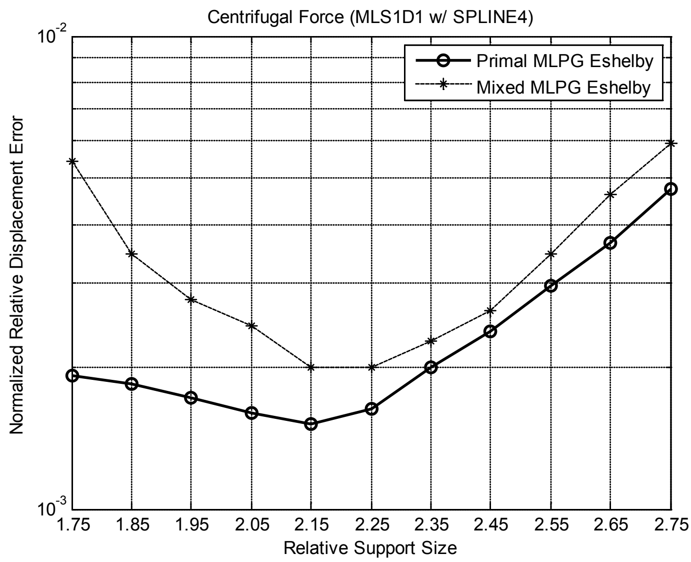

Figure 9: Normalized relative displacement errors of a bar under centrifugal force (irregular sub-domains)

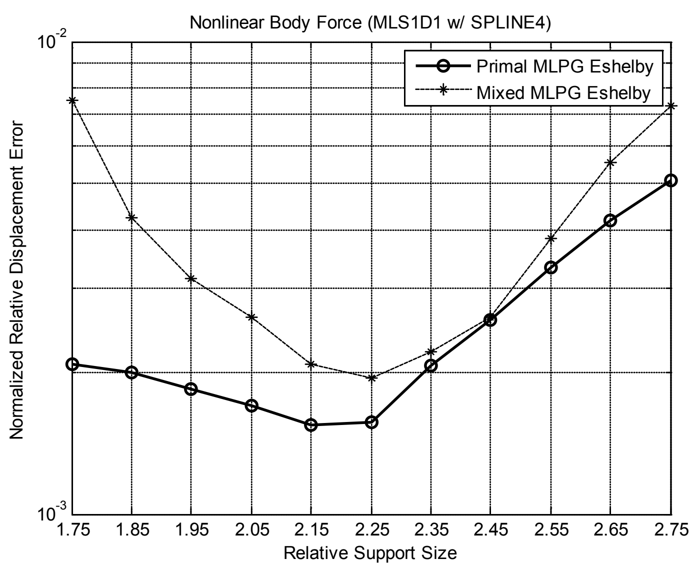

Figure 10: Normalized relative displacement errors of a bar under second order body force (irregular sub-domains)

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