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Flexural Wave Dispersion in Bi-material Compound Solid and Hollow Circular Cylinders

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Abstract: Flexural wave dispersion in a bi-material solid and hollow circular cylinders is investigated with the use of the three-dimensional linear theory of elastodynamics. It is assumed that on the interface surface of the cylinders the complete contact conditions satisfy. The analytical solution of the corresponding field equations is presented and, using these solutions, the dispersion equations for the cases under consideration are obtained. The dispersion equations are solved numerically and based on these solutions, dispersion curves are constructed for the concrete selected pairs of materials such as Tungsten (inner cylinder material) + Aluminum (outer cylinder material) and Steel (inner cylinder material) + Aluminum (outer cylinder material). The numerical results are obtained for the first and second lowest modes. According to these numerical results the influence of the problem parameters, such as the thicknesses of the external and inner cylinders and the materials of the inner cylinder material, on the character of the dispersion curves is analyzed.

Keywords: Dispersion, flexural waves, compound cylinder, complete contact.

1 Introduction

The detailed review of the investigations related to the flexural wave dispersion in a circular cylinders was given in a recently published paper by Akbarov (2013b). It follows from this review that up to now there is not any investigation related to the flexural wave dispersion in compound cylinders. However, up to now there are sufficient number investigations in which the dispersion of the axisymmetric waves in compound cylinders was studied. We consider a brief review of these investigations and begin this review with a work by Akbarov and Guz (2004) in

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which axisymmetric wave dispersion in a pre-stressed bi-material compound solid cylinder is studied. The investigations are made by utilizing the first version of the small initial deformation theory of the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB) (the definition of the various versions of the small initial deformation theories of the TLTEWISB are given in monograph by Guz (2004), as well as in papers by Akbarov (2012, 2013b). It is assumed that the elastic relations of the cylinders' materials are given through the Murnaghan potential described in a monograph by Murnaghan (1951). In a paper by Akbarov and Guliev (2009) the same problem was studied for the case where the constituents of the cylinder is fabricated from highly elastic materials and the corresponding investigations were carried out by utilizing the large (or finite) initial deformation version of the TLTEWISB. The materials of the constituents were assumed to be compressible and their elastic relations were given by the harmonic-type potential. Within the same assumptions, the influence of the finite initial strains on the axisymmetric wave dispersion in a circular cylinder, embedded in a compressible elastic medium, was studied in a work by Akbarov and Guliev (2010). Moreover, in a paper by Akbarov et al (2010) the problem considered in works by Akbarov and Guliev (2009, 2010) was developed for the case where the materials of the components of the system were incompressible and their stress-strain relations were given through the Treloar potential. Numerical results, regarding the influence of the initial strains in the cylinder and in embedded body on the wave dispersion, were presented and discussed. Note that in the foregoing papers related to the compound cylinders, it was assumed that complete contact conditions were satisfied on the interface surface between the constituents. However, in papers by Akbarov and Ipek (2010, 2012) this condition was refuted and, within the scope of the assumptions accepted in the paper by Akbarov and Guliev (2009), the influence of the imperfectness of the mentioned interface conditions on the dispersion of the axisymmetric longitudinal waves in the bi-material compound cylinder was studied.

In a paper by Akbarov (2012) within the scope of the assumptions accepted in the paper by Akbarov and Guz (2004) the influence of the third order elastic constants on the axisymmetric wave propagation in the bi-layered hollow cylinder was investigated. Moreover, the axisymmetric wave propagation in the double-walled CNT was considered in a paper by Akbarov (2013a).

A paper by Akbarov et al (2011a) within the scope of the second version of the small initial deformation theory of the TDLTEWISB it is investigated the dispersion relations of axisymmetric wave propagation in an initially twisted bi-material compound cylinder. It was assumed that the constituents of the compound cylinder were isotropic and homogeneous and, in particular, it was established that as a result of the existence of the initial twisting, at least in one constituent of the considered compound cylinder, the axisymmetric longitudinal and torsional waves could not be propagated separately, i.e. new axisymmetric waves, which differ from the axisymmetric torsional and longitudinal waves, must appear.

In papers by Ozturk and Akbarov (2008, 2009a, 2009b), within the scope of the second version of the small initial deformation theory of the TDLTEWISB, the axisymmetric torsional wave propagation in the initially uni-axially stretched bi-material compound cylinder was investigated. The elastic relations for the components of the compound cylinder were obtained from the Murnaghan potential. It should be noted that in all the foregoing investigations related to the torsional wave propagation in the pre-stressed bi-layered compound cylinder it was assumed that complete contact conditions were satisfied with respect to the contact surface between the inner and outer cylinders. In a paper by Kepceler (2010) the problems considered in the papers by Ozturk and Akbarov (2008, 2009a, 2009b) were examined for the case where the specified contact conditions were imperfect and the numerical results on the effects of this imperfection on the influence of the initial stresses on the wave propagation velocity are presented and discussed. In a paper by Cilli and Ozturk (2010), the torsional wave propagation in a pre-stretched multilayered solid cylinder was studied within the scope of the assumptions used in papers by Ozturk and Akbarov (2008, 2009a, 2009b).

Finally, in a paper by Akbarov et al (2011b) by utilizing the finite initial deformation version of the TLTEWISB within the scope of the piecewise homogeneous body model, torsional wave dispersion in a pre-strained three-layered (sandwich) hollow cylinder was studied. The mechanical relations of the materials of the cylinders are described through their harmonic potential.

Thus, in the present paper the investigations reviewed above are developed for the flexural wave in the compound cylinder and namely by utilizing the field equations of the linear theory of elastodynamics the flexural wave dispersion in the bi-material compound solid and hollow cylinders are investigated. To the best of the author"s knowledge, the investigations carried out in the present paper are the first attempts on the flexural wave propagation in the compound cylinders.

2 Formulation of the problem and governing field equations

We consider the solid (Fig. 1a) and hollow (Fig. 1b) compound cylinders and assume that the radius of the cross section of the interface cylindrical surface between the cylinders is R. The thickness of the outer and inner hollow cylinders we denote through $h^{(1)}$ and $h^{(2)}$ respectively. We determine the position of the points of the cylinders in the cylindrical system of coordinates $Or\theta z$ (Fig. 1). The values related to the inner and outer hollow cylinders will be denoted by the upper indices (2) and (1), respectively. Within this framework, let us investigate the flexural wave propagation along the Oz axis in the cylinders using the coordinates r, θ and z in the framework of the linear theory of elastodynamics. Thus, we write the basic relations of the linear theory of elastodynamics for the case under consideration.

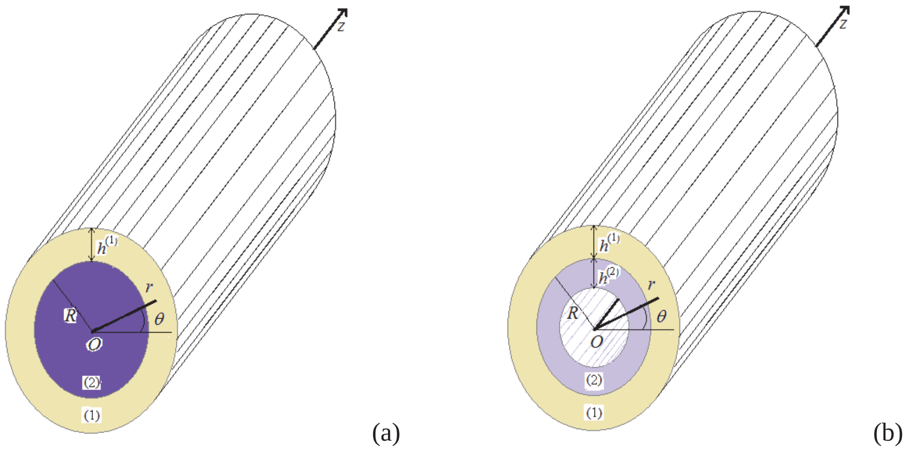


Figure 1: The geometry of the compound solid (a) and compound hollow (b) cylinders

The equations of motion:

$$\frac{\partial\sigma_{rr}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta r}^{(k)}}{\partial\theta}+\frac{\partial\sigma_{zr}^{(k)}}{\partial z}+\frac{1}{r}(\sigma_{rr}^{(k)}-\sigma_{\theta\theta}^{(k)})=\rho^{(k)}\frac{\partial^2u_r^{(k)}}{\partial t^2},$$

$$\frac{\partial\sigma_{r\theta}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}^{(k)}}{\partial\theta}+\frac{\partial\sigma_{z\theta}^{(k)}}{\partial z}+\frac{2}{r}\sigma_{r\theta}^{(k)}=\rho^{(k)}\frac{\partial^2u_\theta^{(k)}}{\partial t^2},$$

$$\frac{\partial\sigma_{rz}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}^{(k)}}{\partial\theta}+\frac{\partial\sigma_{zz}^{(k)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(k)}=\rho^{(k)}\frac{\partial^2u_z^{(k)}}{\partial t^2}. \quad (1)$$

The elastic relations:

$$\sigma_{rr}^{(k)}=(\lambda^{(k)}+2\mu^{(k)})\frac{\partial u_r^{(k)}}{\partial r}+\lambda^{(k)}\frac{1}{r}(\frac{\partial u_\theta^{(k)}}{\partial\theta}+u_r^{(k)})+\lambda^{(k)}\frac{\partial u_z^{(k)}}{\partial z},$$

$$\sigma_{\theta\theta}^{(k)}=\lambda^{(k)}\frac{\partial u_r^{(k)}}{\partial r}+(\lambda^{(k)}+2\mu^{(k)})\frac{1}{r}(\frac{\partial u_\theta^{(k)}}{\partial\theta}+u_r^{(k)})+\lambda^{(k)}\frac{\partial u_z^{(k)}}{\partial z},$$

$$\sigma_{zz}^{(k)} = \lambda^{(k)} \frac{\partial u_r^{(k)}}{\partial r} + \lambda^{(k)} \frac{1}{r} (\frac{\partial u_\theta^{(k)}}{\partial \theta} + u_r^{(k)}) + (\lambda^{(k)} + 2\mu^{(k)}) \frac{\partial u_z^{(k)}}{\partial z},$$

$$\sigma_{r\theta}^{(k)} = \mu^{(k)} \frac{\partial u_\theta^{(k)}}{\partial r} + \mu^{(k)} (\frac{1}{r} \frac{\partial u_r^{(k)}}{\partial \theta} - \frac{1}{r} u_\theta^{(k)}),$$

$$\sigma_{z\theta}^{(k)} = \mu^{(k)} \frac{\partial u_\theta^{(k)}}{\partial z} + \mu^{(k)} \frac{\partial u_z^{(k)}}{r \partial \theta}, \quad \sigma_{zr}^{(k)} = \mu^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + \mu^{(k)} \frac{\partial u_z^{(k)}}{\partial r}. \tag{2}$$

In (2) the conventional notation is used.

Now we write the boundary and contact conditions:

the boundary conditions on the outer surface of the outer hollow cylinder are

$$\left.\sigma_{rr}^{(1)}\right|_{r=R+h^{(1)}} = 0, \quad \left.\sigma_{r\theta}^{(1)}\right|_{r=R+h^{(1)}} = 0, \quad \left.\sigma_{rz}^{(1)}\right|_{r=R+h^{(1)}} = 0, \tag{3}$$

the contact conditions on the interface surface between the cylinders are

$$\left.\sigma_{rr}^{(1)}\right|_{r=R} = \left.\sigma_{rr}^{(2)}\right|_{r=R}, \quad \left.\sigma_{r\theta}^{(1)}\right|_{r=R} = \left.\sigma_{r\theta}^{(2)}\right|_{r=R}, \quad \left.\sigma_{rz}^{(1)}\right|_{r=R} = \left.\sigma_{rz}^{(2)}\right|_{r=R}$$

$$\left.u_r^{(1)}\right|_{r=R} = \left.u_r^{(2)}\right|_{r=R}, \quad \left.u_\theta^{(1)}\right|_{r=R} = \left.u_\theta^{(2)}\right|_{r=R}, \quad \left.u_z^{(1)}\right|_{r=R} = \left.u_z^{(2)}\right|_{r=R} \tag{4}$$

and boundary conditions on the inner surface of the inner hollow cylinder

$$\left.\sigma_{rr}^{(2)}\right|_{r=R-h^{(2)}} = 0, \quad \left.\sigma_{r\theta}^{(2)}\right|_{r=R-h^{(2)}} = 0, \quad \left.\sigma_{rz}^{(2)}\right|_{r=R-h^{(2)}} = 0. \tag{5}$$

Note that the boundary condition (3) and contact condition (4) occur both for the solid and hollow compound cylinders, but the boundary condition (5) occurs for the hollow compound cylinder only.

This completes formulation of the problem and consideration of the governing field equations.

3 Solution procedure and obtaining the dispersion equation

For solution of the eigenvalue problems (1) – (5) we use the representation proposed by Guz (2004):

$$u_r^{(k)} = \frac{1}{r} \frac{\partial}{\partial \theta} \Psi^{(k)} - \frac{\partial^2}{\partial r \partial z} \mathrm{X}^{(k)}, \quad u_\theta^{(k)} = -\frac{\partial}{\partial r} \Psi^{(k)} - \frac{1}{r} \frac{\partial^2}{\partial \theta \partial z} \mathrm{X}^{(k)},$$

$$u_z^{(k)} = (\lambda^{(k)} + \mu^{(k)})^{-1} \left((\lambda^{(k)} + 2\mu^{(k)}) \Delta_1 + \mu^{(k)} \frac{\partial^2}{\partial z^2} - \rho^{(k)} \frac{\partial^2}{\partial t^2} \right) \mathrm{X}^{(k)},$$

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{6}$$

Here the functions $\Psi^{(k)}$ and $X^{(k)}$ are the solutions of the equations

$$\left(\Delta_1 + \frac{\partial^2}{\partial z^2} - \frac{\rho^{(k)}}{\mu^{(k)}}\frac{\partial^2}{\partial t^2}\right)\Psi^{(k)} = 0, \quad \left[\left(\Delta_1 + \frac{\partial^2}{\partial z^2}\right)\left(\Delta_1 + \frac{\partial^2}{\partial z^2}\right) + \right.$$

$$\left. -\rho^{(k)}\frac{\lambda^{(k)}+3\mu^{(k)}}{\mu^{(k)}(\lambda^{(k)}+2\mu^{(k)})}\Delta_1 + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2} + \frac{(\rho^{(k)})^2}{\mu^{(k)}(\lambda^{(k)}+2\mu^{(k)})}\frac{\partial^4}{\partial t^4}\right] X^{(k)} = 0. \tag{7}$$

For the flexural waves we represent the functions Ψ and X as follows:

$$\Psi = \Psi_n(r)\sin n\theta \sin(kz - \omega t), \quad X = X_n(r)\cos n\theta\cos(kz-\omega t). \tag{8}$$

Substituting the expressions (8) into the equations (7) we obtain

$$\left(\Delta_{1n} + \zeta_1^2\right)\Psi_n = 0, \quad \left(\Delta_{1n} + \zeta_2^2\right)\left(\Delta_{1n} + \zeta_3^2\right)X_n = 0,$$

$$\Delta_{1n} = \frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{n^2}{r^2}, \tag{9}$$

where

$$\zeta_1^2 = k^2\left(\frac{\rho^{(k)}c^2}{\mu^{(k)}} - 1\right), \quad c = \frac{k}{\omega} \tag{10}$$

but the ζ_2^2 and ζ_3^2 are determined as solutions of the equation

$$\mu^{(k)}(\zeta^{(k)})^4 - k^2(\zeta^{(k)})^2\left[\rho^{(k)}c^2 - (\lambda^{(k)}+2\mu^{(k)}) + \frac{\mu^{(k)}}{\lambda^{(k)}+2\mu^{(k)}}\left(\rho^{(k)}c^2 - \mu^{(k)}\right) + \right.$$

$$\left. \frac{(\lambda^{(k)}+\mu^{(k)})^2}{\lambda^{(k)}+2\mu^{(k)}}\right] + k^4\left(\frac{\rho^{(k)}c^2}{\lambda^{(k)}+2\mu^{(k)}} - 1\right)\left(\rho^{(k)}c^2 - \mu^{(k)}\right) = 0. \tag{11}$$

In (10) and (11), c is the phase velocity of the flexural waves.
Thus, we find the solution of the equations in (9) as follows:
for the inner solid cylinder

$$\Psi_n^{(2)} = A_1^{(2)}E_n(\zeta_1^{(2)}kr), X_n^{(2)} = A_2^{(2)}E_n(\zeta_2^{(2)}kr) + A_3^{(2)}E_n(\zeta_3^{(2)}kr), \tag{12}$$

for the outer hollow cylinder

$$\Psi_n^{(1)} = A_1^{(1)}E_n(\zeta_1^{(1)}kr) + B_1^{(1)}D_n(\zeta_1^{(1)}kr),$$

$$\mathrm{X}_n^{(1)} = A_2^{(1)} E_n(\zeta_2^{(1)} kr) + A_3^{(1)} E_n(\zeta_3^{(1)} kr) + A_2^{(1)} D_n(\zeta_2^{(1)} kr) + A_3^{(1)} D_n(\zeta_3^{(1)} kr), \quad (13)$$

for the inner hollow cylinder

$$\Psi_n^{(2)} = A_1^{(2)} E_n(\zeta_1^{(2)} kr) + B_1^{(2)} D_n(\zeta_1^{(2)} kr),$$

$$\mathrm{X}_n^{(2)} = A_2^{(2)} E_n(\zeta_2^{(2)} kr) + A_3^{(2)} E_n(\zeta_3^{(2)} kr) + A_2^{(2)} D_n(\zeta_2^{(2)} kr) + A_3^{(2)} D_n(\zeta_3^{(2)} kr), \quad (14)$$

where

$$E_n(\zeta_j^{(m)} kr) = J_n(\zeta_j^{(m)} kr), \quad D_n(\zeta_j^{(m)} kr) = Y_n(\zeta_j^{(m)} kr) \text{ if } (\zeta_j^{(m)})^2 > 0,\ m = 1,2,$$

$$E_n(\zeta_j^{(m)} kr) = I_n(\zeta_j^{(m)} kr), \quad D_n(\zeta_j^{(m)} kr) = K_n(\zeta_j^{(m)} kr) \text{ if } (\zeta_j^{(m)})^2 < 0,\ j = 1,2,3 \quad (15)$$

In (15), $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kind of the $n-th$ order, $I_n(x)$ and $K_n(x)$ are Bessel functions of a purely imaginary argument of the $n-th$ order and Macdonald functions of the $n-th$ order, respectively.

Thus, using relations (6), (12) – (14) and (2) we obtain the dispersion equation

$$\det \left\| \beta_{ij}^{s} \right\| = 0, \quad i; j = 1,2,...,9, \quad (16)$$

for the compound solid cylinder from the boundary (3) and contact (4) conditions, as well as the dispersion equation

$$\det \left\| \beta_{ij}^{h} \right\| = 0, \quad i; j = 1,2,3,...,12, \quad (17)$$

for the hollow compound cylinder from the boundary (3), (5) and contact (4) conditions. We here do not give the explicit expressions of β_{ij}^{s} and β_{ij}^{h}, because they can easily be determined from the corresponding expressions given in a paper by Akbarov (2013b).

4 Numerical results and discussions

The numerical results are obtained for steel (St), tungsten (Tg) and aluminum (Al). Note that the material of the internal cylinder is selected as steel (St) (it will be denoted as Case 1) or tungsten (Tg) (it will be denoted as Case 2), but the material of the external hollow cylinder is selected as aluminum (Al). All mechanical characteristics of these materials and their notation, which will be used below, are given in Table 1. The values of the velocity of wave dilatation and bar velocity for these materials are given in Table 2. Note that the data given in the tables are selected

Table 1: The values of elastic constants of selected materials.

Materials	Density	Young's moduli	Pois.'s ratio
Steel (St) Steel (St)	$\rho_{St}\times10^{-3}=7.795kg/m^3$	$E_{St}\times10^{-4}=19.6MPa$	$\nu_{St}=0.27$
Tungsten (Tg)	$\rho_{Tg}\times10^{-3}=19.3kg/m^3$	$E_{Tg}\times10^{-4}=34.3MPa$	$\nu_{Tg}=0.28$
Aluminum (Al)	$\rho_{Al}\times10^{-3}=2.77kg/m^3$	$E_{Al}\times10^{-4}=7.28MPa$	$\nu_{Al}=0.30$

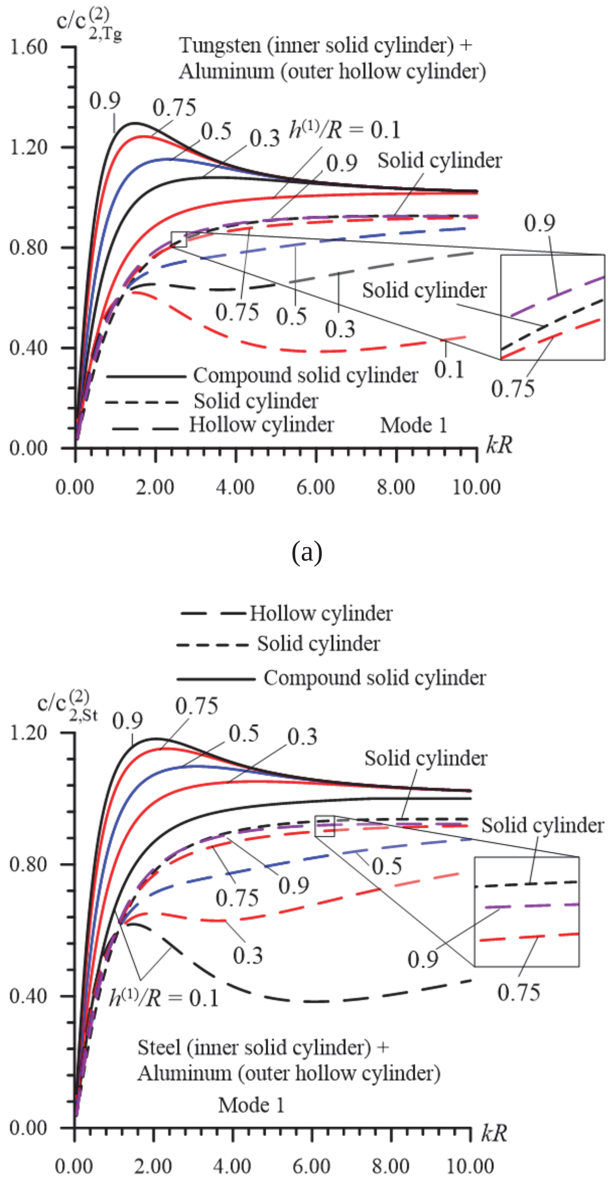
Table 2: The wave velocity of the selected materials.

Materials	Velocity of wave of dilatation	Bar velocity	The values of velocity ratio
Steel (St)	$c_{2,St}^{(1)}\times10^{-3}=3.152$m/s	$c_{b,St}^{(1)}\times10^{-3}=5.025$m/s	$c_{b,St}^{(1)}/c_{2,St}^{(1)}=1.594$
Tungsten (Tg)	$c_{2,Tg}^{(1)}\times10^{-3}=2.63$m/s	$c_{b,Tg}^{(1)}\times10^{-3}=4.219$m/s	$c_{b,Tg}^{(1)}/c_{2,Tg}^{(1)}=1.604$
Aluminum (Al)	$c_{2,Al}^{(2)}\times10^{-3}=3179$m/s	$c_{b,Al}^{(2)}\times10^{-3}=5.126$m/s	$c_{b,Al}^{(2)}/c_{2,Al}^{(2)}=1.612$

according to Guz (2004) and Guz and Makhort (2000). Assume that $n = 1$ in (8) – (15).

The dispersion equations (16) (for the solid compound cylinder) and (17) (for the hollow compound cylinder) are solved numerically by employing the algorithm and PC programs which were used in the previous papers by the author, such as Akbarov (2013b), Akbarov and Ipek (2010, 2012) and others. We recall that in the paper by Akbarov (2013b) the dispersion of the flexural wave dispersion in the finite pre-strained solid and hollow cylinders made of highly elastic material was studied and programs which were used under this studying were tested by the known classical results obtained for example in papers by Abramson (1957). Therefore, the PC programs used in the investigations carried out in the paper by Akbarov (2013b) after corresponding development and change are employed for the numerical solution of the equations (16) and (17). Consequently, the algorithm and PC programs used in the present numerical investigations has been already tested, although we will also consider below some fragments on the mentioned testing.

Thus, we analyze numerical results and begin this analyze with graphs given in Fig. 2 which show dispersion curves related to the first mode in Case 1 (Fig. 2a) and in Case 2 (Fig. 2b) for the solid compound cylinder (Fig. 1a). In Fig. 2 the corresponding dispersion curves for the inner solid cylinder made of Tungsten and for the hollow cylinder made of Aluminum are also illustrated. It follows from Fig. 2 that the flexural wave propagation velocity in the compound solid cylinder is greater than the flexural wave propagation velocity in each constituents of this

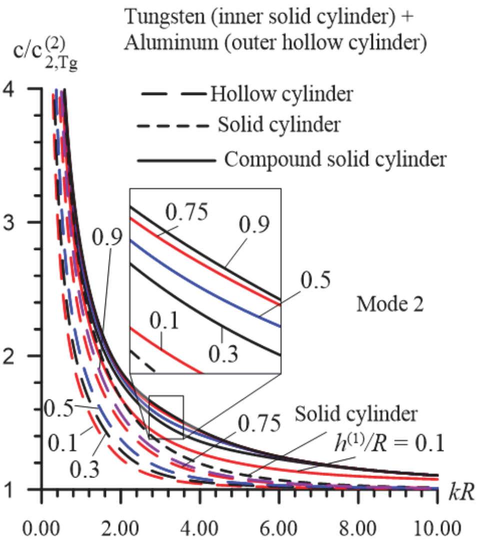


(a)

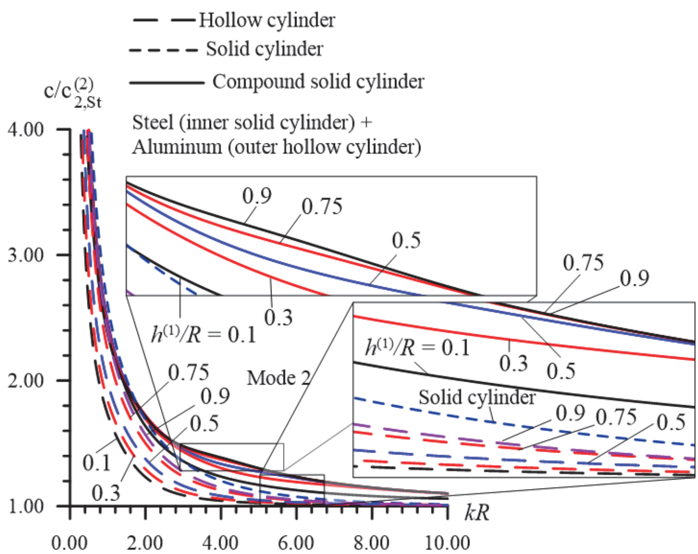
(b)

Figure 2: Dispersion curves for the compound solid cylinder related to the first mode in Case 1 (a) and in Case 2 (b)

cylinder.



(a)



(b)

Figure 3: Dispersion curves for the compound solid cylinder related to the second mode in Case 1(a) and in Case 2 (b)

The same type results are also observed for the second mode of the flexural wave propagation velocity in the compound solid cylinder the dispersion curves of which are given in Fig. 3 in Case 1 (Fig. 3a) and Case 2 (Fig. 3b). Note that in Fig. 3 the corresponding dispersion curves related to the inner solid and outer hollow cylinders are also illustrated. Consequently, it can be conclude that the compounding of the cylinders causes to increase of the flexural wave propagation velocity in that with respect to the flexural wave propagation velocity in its constituents. This can be explained with the fact that the dilatational wave propagation velocity $c_{2,Al}^{(2)}$ for Aluminum is greater than $c_{2,St}^{(2)}$ in Steel and $c_{2,Tg}^{(2)}$in Tungsten. Moreover, the foregoing increasing of the wave propagation velocity in the compound cylinder is also caused by the change of the geometry of the cross section of the cylinder after compounding procedure.

The comparison of the dispersion curves constructed in Case 1 with the corresponding ones constructed in Case 2 shows that the flexural wave propagation velocity in Case 1 is greater than that in Case 2. This statement can be explained with the fact that the dilatational wave propagation velocity $c_{2,Tg}^{(2)}$ in Tungsten is less than $c_{2,St}^{(2)}$ in Steel.

Now we analyze the limit values of the flexural wave propagation velocity in the first mode as $kR \to 0$ and as $kR \to \infty$. It follows from the graphs given in Fig. 2 that

$$c/c_{2.Tg} \to 0, \quad c/c_{2.St} \to 0 \text{ as } kR \to 0 \tag{18}$$

and

$$c/c_{2.Tg} \to c_{R.Al}/c_{2.Tg}, \quad c/c_{2.St} \to c_{R.Al}/c_{2.St} \text{ as } kR \to \infty, \tag{19}$$

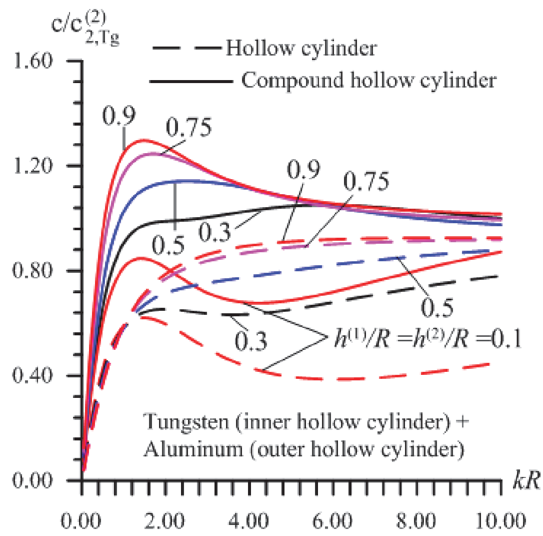
where $c_{R.Al}$ is Rayleigh wave velocity in Aluminum. Note that the limit relations (18) and (19) do not depend on the ratio $h^{(1)}/R$. Moreover note that, in general, if there exists the Stoneley wave for the selected pair of materials of the constituents of the compound cylinder, then we must write

$$\begin{aligned} &c/c_{2.Tg} \to \min\{c_{R.Al}/c_{2.Tg}; c_S/c_{2.Tg}\}, \\ &c/c_{2.St} \to \min\{c_{R.Al}/c_{2.St}; c_S/c_{2.St}\}; \text{ as } kR \to \infty, \end{aligned} \tag{20}$$

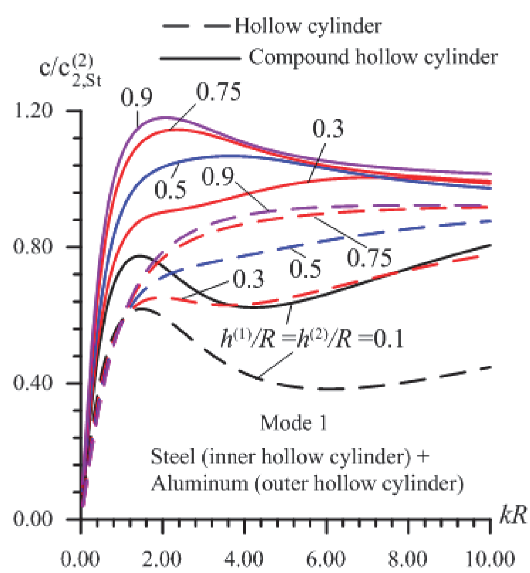
instead of the relation (19).

The observation of the dispersion curves related to the first mode (Fig. 2) shows that for the relatively great values of the ratio $h^{(1)}/R$, for instance, for $h^{(1)}/R \geq 0.5$, there appear the points (denote it as $kR = (kR)_*$) at which

$$\left.\frac{d\left(c\big/c_{2.Tg}^{(2)}\right)}{d(kR)}\right|_{kR=(kR)_*} = 0, \quad \left.\frac{d\left(c\big/c_{2.St}^{(2)}\right)}{d(kR)}\right|_{kR=(kR)_*} = 0, \tag{21}$$

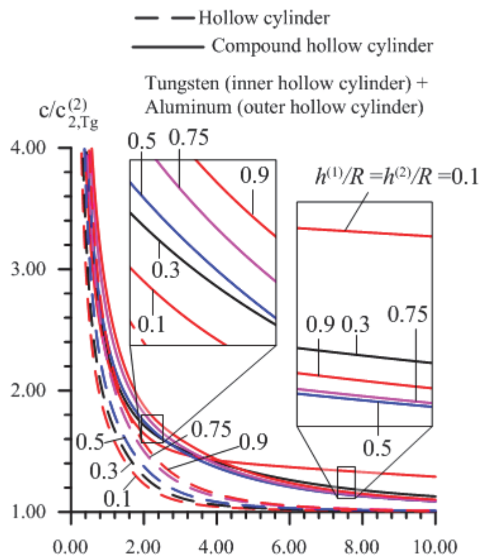


(a)

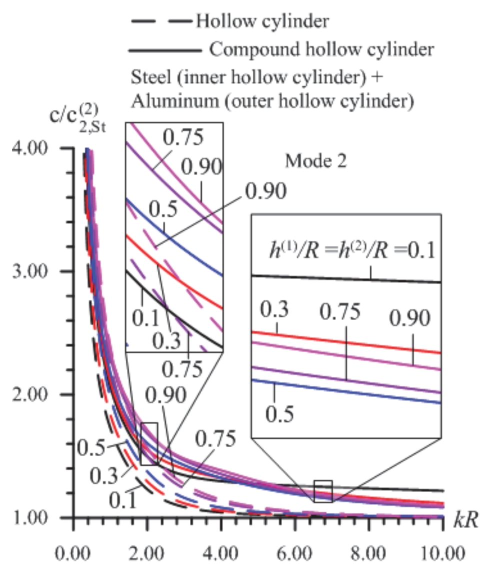


(b)

Figure 4: Dispersion curves for the compound hollow cylinder related to the first mode in Case 1 (a) and in Case 2 (b)



(a)



(b)

Figure 5: Dispersion curves for the compound hollow cylinder related to the second mode in Case 1 (a) and in Case 2 (b)

and the values of the $(kR)_*$ decrease with the $h^{(1)}/R$.

The equation (21) means that there exists such value of the dimensionless wavenumber kR, i.e. the value $kR = (kR)_*$, under which the group velocity of the flexural wave in the compound solid cylinder becomes equal to the corresponding phase velocity of that. It should be noted that such statement does not appear for the solid cylinder, but appears for the corresponding hollow cylinder under relatively small values of the $h^{(1)}/R$, for instance, in the cases where $h^{(1)}/R \leq 0.3$. However, in the cases where $h^{(1)}/R > 0.3$ the foregoing type points do not appear in the dispersion curves related to the solid compound cylinder.

Thus, it follows from the foregoing discussions that the dispersion curves of the flexural waves in the compound solid cylinder differ from those in the corresponding solid and hollow cylinders not only in the quantitative sense, but also in the qualitative sense.

Now we consider numerical results related to the compound hollow cylinder. The dispersion curves of the compound hollow cylinder related to the first mode are given in Fig. 4 for Case 1 (Fig. 4a) and for Case 2 (Fig. 4b). In Fig. 4 the corresponding dispersion curves related to the external hollow cylinder are also given. It follows from these graphs that as a result of the compounding of the cylinders the flexural wave propagation velocity increases and, as in the compound solid cylinder, this increasing can be explained with the inequalities $c_{2,Al}^{(2)} > c_{2,Tg}^{(2)}$and $c_{2,Al}^{(2)} > c_{2,St}^{(2)}$, as well as with the fact that the thickness of the compound hollow cylinder is greater than the thickness of the outer hollow cylinder.

The relation (18) occurs also for limit values of the flexural wave propagation velocity in the compound hollow cylinder in the first mode as $kR \to 0$. However, the relation related to the limit values of the flexural wave propagation velocity in the compound hollow cylinder in the first mode as $kR \to \infty$ is more complicate than that for the compound solid cylinder, i.e. than that given in Eq. (19). Namely, we obtain the following limit relations instead of the Eq. (19) for the compound hollow cylinder from the asymptotic analyses of the dispersion equation (17) and from the physical consideration related to the problem under investigation.

$$c/c_{2.Tg} \to \min\left\{c_{R.Al}/c_{2.Tg}; c_{R.Tg}/c_{2.Tg}\right\},$$

$$c/c_{2.St} \to \min\left\{c_{R.Al}/c_{2.St}; c_{R.St}/c_{2.St};\right\} \text{ as } kR \to \infty, \tag{22}$$

and, if there exists the Stoneley waves for the pair of materials of the constituents of the compound hollow cylinder we must write the following one instead of the Eq. (22).

$$c/c_{2.Tg} \to \min\{c_{R.A1}/c_{2.Tg}; c_{R.Tg}/c_{2.Tg}; c_S/c_{2.Tg}\}$$

$$c/c_{2.St} \to \min\{c_{R.A1}/c_{2.St}; c_{R.St}/c_{2.St}; c_S/c_{2.St}\} \quad as \quad kR \to \infty \tag{23}$$

In the relations (22) and (23) the velocities of the Rayleigh and Stoneley waves velocities are indicated through the lower index R and S respectively.

According to the graphs given in Fig. 4 we can conclude that the relation (21) occurs also for the compound hollow cylinder. However, for the compound hollow cylinder this relation takes place not only for the relatively great values of the $h^{(1)}/R(= h^{(2)}/R)$ (for instance, in the cases where $h^{(1)}/R \geq 0.5$), but also for the relatively small values of the $h^{(1)}/R(= h^{(2)}/R)$ (for instance, in the case where $h^{(1)}/R \geq 0.1$). Consequently, there exists also such values of the $h^{(1)}/R(= h^{(2)}/R)$(for instance, in the case where $h^{(1)}/R = 0.3$) under which the relation (21) does not satisfy.

Finally, we note the following statement. According to the mechanical consideration, the results obtained for the compound hollow cylinder must approach to the corresponding ones obtained for the compound solid cylinder with $h^{(1)}/R(= h^{(2)}/R)$. The comparison of the results calculated for the compound hollow cylinder in the cases where $h^{(1)}/R(= h^{(2)}/R)$= 0.75 and 0.90 (Fig. 4) with the results calculated for the compound solid cylinder in the cases where $h^{(1)}/R$= 0.75 and 0.90 (Fig. 2) proves the foregoing consideration and again validate the algorithm and PC programs used in the present investigations.

5 Conclusions

Thus, in the present paper it has been made the attempt to study the flexural wave dispersion in a bi-material solid and hollow circular cylinders with the use of the three-dimensional linear theory of elastodynamics. It is assumed that on the interface surface of the cylinders the complete contact conditions satisfy. The analytical solution of the corresponding field equations is presented and, using these solutions, the dispersion equations for the cases under consideration are obtained. The dispersion equations are solved numerically and based on these solutions, dispersion curves are constructed for the concrete selected pairs of materials such as Tungsten (inner cylinder material) + Aluminum (outer cylinder material) and Steel (inner cylinder material) + Aluminum (outer cylinder material).

The numerical results are obtained for the first and second lowest modes. According to these numerical results the influence of the problem parameters, such as the thicknesses of the external and inner cylinders and the materials of the inner cylinder material, on the character of the dispersion curves is analyzed. According to these analyses it is established that the dispersion curves of the flexural waves in the compound solid cylinder differ from those in the corresponding solid and hollow cylinders not only in the quantitative sense, but also in the qualitative sense.

Moreover, according to these analyses, it can be made the following concrete conclusions:

- the flexural wave propagation velocity in the compound cylinder is greater than that in the constituents of this cylinder;
- the low wavenumber limit values as $kR \to 0$ of the first mode of the flexural wave propagation velocity in the compound cylinders, as the same limit values of that in the constituents of the cylinder, are determined with the expression (18);
- the high wavenumber limit values as $kR \to \infty$ of the first mode of the flexural wave propagation velocity in the compound solid cylinder are determined by the expressions (19) and (20), but in the compound hollow cylinder with the expressions (22) and (23);
- an increase in the thickness of the outer hollow cylinder of the solid compound cylinder causes to appear a such value of the dimensionless wavenumber kR under which the relation (21) takes place, i.e. under which the group velocity of the flexural wave propagation velocity becomes equal to its phase velocity;
- the foregoing type behavior of the dispersion curves takes also place for the flexural wave propagation in the compound hollow cylinder. However, in the latter case this behavior occurs not only for the relatively great values of the thickness of the inner and outer cylinders, but also for the relatively small values of that and there exist such values of this thickness under which the wave propagation velocity of the flexural wave increase monotonically with the dimensionless wavenumber.

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