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Cauchy Problem for the Heat Equation in a Bounded Domain Without Initial Value

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Abstract: We consider the determination of heat flux within a body from the Cauchy data. The aim of this paper is to seek an approach to solve the one-dimensional heat equation in a bounded domain without initial value. This problem is severely ill-posed and there are few theoretic results. A quasi-reversibility regularization method is used to obtain a regularized solution and convergence estimates are given. For numerical implementation, we apply a method of lines to solve the regularized problem. From numerical results, we can see that the proposed method is reasonable and feasible.

Keywords: Ill-posed problem, Quasi-reversibility method, Method of lines, Finite difference, Convergence estimate.

1 Introduction

In this paper, we consider an inverse heat conduction problem to determine the heat flux in a bounded domain without initial value. To our knowledge, this kind of inverse problem is very important for applications in science, engineering and bioengineering which has attracted great attention of many researchers in recent years. In this case, our goal is to determine the interior and surface heat flux on an inaccessible from Cauchy data on the accessible boundary. As we know, this kind of Cauchy problem is severely ill-posed in Hadamard's sense [Eldén (1987); Eldén, Berntsson, and Regińska (2000); Qian and Fu (2007); Hào, Reinhardt, and Schneider (2001); Weber (1981); Liu and Zhang (2013)], that is, small perturbations in Cauchy data can result in dramatically large errors in the solution. Hence, regularization techniques should be considered to stabilize the computations [Engl, Hanke, and Neubauer (1996); Groetsch (1984)]. In the past years, many regular-

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ization methods have been developed for the heat equation in an unbounded domain [Carasso (1982); Eldén, Berntsson, and Regińska (2000); Seidman and Elden (1990); Fu and Qiu (2003); Tautenhahn (1997); Xiong and Fu (2008); Berntsson (1999); Eldén (1995)]. These methods include Tikhonov method [Carasso (1982)], wavelet and spectral method [Eldén, Berntsson, and Regińska (2000); Fu and Qiu (2003); Xiong and Fu (2008)], conjugate gradient method [Lee, Yang, Chang, and Wu (2009)], optimal schemes [Tautenhahn (1997); Seidman and Elden (1990); Chang and Liu (2012)], boundary particle method and singular meshless method [Fu, Chen, and Zhang (2012); Chen and Fu (2009); Gu, Chen, and Fu (2013)], etc.

In this paper, we propose a quasi-reversibility regularization method to solve the Cauchy problem of the heat equation. The method of quasi-reversibility was first proposed by Lattès and Lions to deal with some ill-posed problems [Lattès and Lions (1969)]. The main idea of this method is by perturbing the equation in the ill-posed to obtain a well-posed problem. The similar regularization method was used in Eldén's papers [Eldén (1987, 1988)] where the author used the Fourier transform to get the exact solution for the sideways heat equation problem in a quarter plane. Qian et al. [Qian, Fu, and Xiong (2007)] rectified the defect of Eldén's papers and got the convergence in the whole solution domain for the heat flux distribution by the Fourier transform.

In many situations we do not know the initial condition because the heat process has already started before we estimate this problem. As we know, there are very few works to deal with the Cauchy problem without initial value [Dorroh and Ru (1999); Wang, Cheng, Nakagawa, and Yamamoto (2010)]. Based on the existing theory, Wang et al. [Wang, Cheng, Nakagawa, and Yamamoto (2010)] proved the uniqueness in determining both a boundary value and an initial value. Cannon and Douglas [Cannon and Douglas (1967)] established Hölder continuous dependence on the Cauchy data for solutions of the heat equation with an a priori bound. Dorroh and Ru [Dorroh and Ru (1999)] proved that the regularized solution for the exact Cauchy data converges the exact solution without initial value and did not provide a convergence estimate for the regularized solution corresponding to the noisy Cauchy data. In this paper, we apply a fourth-order modified method to obtain a regularized solution in a bounded domain without initial value. Convergence estimates are given based on the Fourier series. For numerical implementation, we apply a method of lines to obtain a stable approximate solution.

The outline of the paper is as follows. In Section 2, the formulation of the heat conduction problem and a quasi-reversibility regularization method are given. Section 3 gives the convergence estimates for the regularized solution. The method of lines is applied to obtain an approximate solution in Section 4. Several numerical examples are presented in Section 5 to illustrate the efficiency of the proposed method.

Finally, in Section 6 we give some concluding remarks.

2 Formulation of the heat conduction problem and a quasi-reversibility regularization method

We consider the heat conduction problem as follows

$$\begin{cases} u_t = u_{xx},\ 0 < x < 1,\ 0 < t < 2\pi, \\ u(0,t) = f(t),\ 0 \leqslant t \leqslant 2\pi, \\ u_x(0,t) = 0,\ 0 \leqslant t \leqslant 2\pi. \end{cases} \tag{1}$$

Suppose that $f(t) \in L^2[0,2\pi]$, so $f(t)$ can be written in its Fourier series. For the detail of this inverse problem, we refer to [Cannon (1984)Chap.2]. Uniqueness of the solution of problem (1) follows from the analyticity of the solution of the heat equation in the spatial variable x. Properties of uniqueness and continuous dependence are discussed in [Cannon (1984)Chap.11].

We can get the following formal solution of the problem (1), refer to [Dorroh and Ru (1999)],

$$u(x,t) = \sum_{n=-\infty}^{+\infty} C_n e^{int} \cosh(\sqrt{in}x) \tag{2}$$

and the heat flux is given by

$$u_x(x,t) = \sum_{n=-\infty}^{+\infty} C_n(\sqrt{in}) e^{int} \sinh(\sqrt{in}x) \tag{3}$$

where

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt,$$

and

$$\sqrt{in} = \sqrt{\frac{|\,n\,|}{2}}(1+\sigma i), \quad \sigma = sign(n), \quad n \in \mathbb{Z}.$$

Note that the real part of $\cosh(\sqrt{in}x)$ and $\sinh(\sqrt{in}x)$ are positive, the small error in the Dirichlet data $f(t)$ will be amplified by the factor $\Re(\cosh(\sqrt{in}x))$ or $\Re(\sinh(\sqrt{in}x))$ for $0 < x \leq 1$, so the problem (1) is severely ill-posed. We should employ a regularization method to deal with this problem. In this paper, we apply a quasi-reversibility method to construct an approximate solution for problem (1). That is to find the solution of the following problem

$$\begin{cases} v_t^\delta = v_{xx}^\delta - \alpha^2 v_{xxtt}^\delta,\ 0 < x < 1,\ 0 < t < 2\pi, \\ v^\delta(0,t) = f^\delta(t),\ 0 \le t \le 2\pi, \\ v_x^\delta(0,t) = 0,\ 0 \le t \le 2\pi. \end{cases} \tag{4}$$

Suppose $f^\delta(t) \in L^2[0,2\pi]$ is measured data and satisfies

$$\| f - f^\delta \| \le \delta \tag{5}$$

where $\| \cdot \|$ denotes the L^2-norm and the constant $\delta > 0$ represents a noise level.

Refer to [Dorroh and Ru (1999)], we know that the following formal solution of problem (4)

$$v^\delta(x,t) = \sum_{n=-\infty}^{+\infty} C_n^\delta e^{int} \cosh(\tau\sqrt{in}x) \tag{6}$$

and the heat flux is given by

$$v_x^\delta(x,t) = \sum_{n=-\infty}^{+\infty} C_n^\delta (\tau\sqrt{in}) e^{int} \sinh(\tau\sqrt{in}x) \tag{7}$$

where

$$C_n^\delta = \frac{1}{2\pi}\int_0^{2\pi} f^\delta(t) e^{-int} dt, \quad \tau = 1/\sqrt{1+n^2\alpha^2}.$$

It is well known that for an ill-posed problem an a priori assumption on the exact solution is necessary. To get a more sharp convergence rates for the regularized solution, the following a priori bound on the exact solution is needed

$$\| u(1,\cdot) \| \le E, \tag{8}$$

where E is a finite positive constant.

To obtain convergence estimates, we should choose a suitable regularization parameter α. It is difficult to choose parameter α by an a-priori method. In this paper, we choose parameter α similar to μ in [Eldén (1987)] by

$$\alpha = \frac{1}{4(\ln(E/\delta))^2} \tag{9}$$

where E and δ are given in (8) and (5), respectively.

3 Convergence estimates

In this section, we give some error estimates for the heat flux in the interior of domain $0 < x < 1$ and on boundary $x = 1$, respectively. The a priori bound assumptions and the choices of regularization parameters are different for this two different cases.

Theorem 3.1 *Let $u(x,t)$ be the solution of problem (1) given by (2). Let $v^\delta(x,t)$ be the solution of problem (4) given by (6). The regularization parameter α is given by (9). Let the measurement temperature history at $x = 0$, $f^\delta(t)$, satisfies (5), and let the a priori assumption (8) hold. Then for fixed $x \in (0,1)$, we have*

$$\| u_x(x,\cdot) - v_x^\delta(x,\cdot) \| \le c(x) \frac{E}{16(\ln(E/\delta))^4} + \sqrt{2} E^x \delta^{1-x} \ln \frac{E}{\delta} \tag{10}$$

where $c(x) = \frac{\sqrt{2}}{c_1} \max\left\{ \left(\frac{5\sqrt{2}}{(1-x)e} \right)^5, \left(\frac{6\sqrt{2}}{(1-x)e} \right)^6 \right\}$, $c_1 = \frac{1}{2}\sqrt{1 - \sqrt{2}e^{-3\pi/4}}$.

PROOF. Let $v(x,t)$ be the solution of problem (4) with noise-free data, i.e., $\delta = 0$. By using the triangle inequality, we know

$$\| u_x(x,\cdot) - v_x^\delta(x,\cdot) \| \le \| u_x(x,\cdot) - v_x(x,\cdot) \| + \| v_x(x,\cdot) - v_x^\delta(x,\cdot) \| . \tag{11}$$

We start by estimating the second term on the right-hand side of (11). From (2) and (6), we have

$$f(t) = v(0,t) = \sum_{n=-\infty}^{+\infty} C_n e^{int}, \quad f^\delta(t) = u^\delta(0,t) = \sum_{n=-\infty}^{+\infty} C_n^\delta e^{int} \tag{12}$$

where

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt, \quad C_n^\delta = \frac{1}{2\pi} \int_0^{2\pi} f^\delta(t) e^{-int} dt.$$

In terms of the condition (5), we have

$$\begin{aligned} \| f - f^\delta \|^2 &= \| \sum_{n=-\infty}^{+\infty} C_n e^{int} - \sum_{n=-\infty}^{+\infty} C_n^\delta e^{int} \|^2 = \| \sum_{n=-\infty}^{+\infty} (C_n - C_n^\delta) e^{int} \|^2 \\ &= 2\pi \sum_{n=-\infty}^{+\infty} | (C_n - C_n^\delta) |^2 \le \delta^2. \end{aligned}$$

Consequently,

$$\begin{aligned}\| v_x(x,\cdot)-v_x^{\delta}(x,\cdot)\|^2 &= \|\sum_{n=-\infty}^{+\infty}(C_n-C_n^{\delta})(\tau\sqrt{in})e^{int}\sinh(\tau\sqrt{in}x)\|^2\\ &= 2\pi\sum_{n=-\infty}^{+\infty}|C_n-C_n^{\delta}|^2\,|\tau\sqrt{in}|^2\,|\sinh(\tau\sqrt{in}x)|^2\\ &\leq \sup_{n\in\mathbb{Z}}|\tau\sqrt{in}|^2\,|\sinh(\tau\sqrt{in}x)|^2\,\delta^2,\end{aligned}$$

i.e.,

$$\| v_x(x,\cdot)-v_x^{\delta}(x,\cdot)\| \leq \sup_{n\in\mathbb{Z}}\widetilde{B}(n)\delta$$

where

$$\widetilde{B}(n) = |\tau\sqrt{in}|\,|\sinh(\tau\sqrt{in}x)|\,.$$

Since $\tau=1/\sqrt{1+n^2\alpha^2}$ and $\tau\sqrt{\frac{|n|}{2}}\leq\frac{1}{2\sqrt{\alpha}}$, it is easy to see that

$$\begin{aligned}\widetilde{B}(n) &\leq \sqrt{2}\tau\sqrt{\frac{|n|}{2}}\left(\frac{|e^{\tau\sqrt{in}x}|+|e^{-\tau\sqrt{in}x}|}{2}\right)\\ &\leq \sqrt{2}\tau\sqrt{\frac{|n|}{2}}\left(\frac{|e^{\tau\sqrt{\frac{|n|}{2}}x}|+|e^{-\tau\sqrt{\frac{|n|}{2}}x}|}{2}\right)\\ &\leq \sqrt{2}\tau\sqrt{\frac{|n|}{2}}e^{\tau\sqrt{\frac{|n|}{2}}x}\leq\frac{\sqrt{2}}{2\sqrt{\alpha}}e^{\frac{x}{2\sqrt{\alpha}}},\end{aligned}\tag{13}$$

and combing with (9), we have

$$\| v_x(x,\cdot)-v_x^{\delta}(x,\cdot)\|\leq\sqrt{2}E^x\delta^{1-x}\ln\frac{E}{\delta}.\tag{14}$$

Now we estimate the first term on the right-hand side of (11). From (3) and (7), we have

$$\begin{aligned}\| u_x(x,\cdot)-v_x(x,\cdot)\|^2 &= \|\sum_{n=-\infty}^{+\infty}C_n(\sqrt{in})e^{int}[\sinh(\sqrt{in}x)-\tau\sinh(\tau\sqrt{in}x)]\|^2\\ &= 2\pi\sum_{n=-\infty}^{+\infty}|C_n|^2|\sqrt{in}|^2|\sinh(\sqrt{in}x)-\tau\sinh(\sqrt{in}x)|^2.\end{aligned}$$

From (8), the a priori assumption is equivalent to

$$\begin{aligned}\| u(1,\cdot) \|^2 &= \| \sum_{n=-\infty}^{+\infty} C_n e^{int} \cosh(\sqrt{in}) \|^2 = 2\pi \sum_{n=-\infty}^{+\infty} |C_n|^2 |\cosh(\sqrt{in})|^2 \\ &\leq E^2.\end{aligned}$$

Consequently,

$$\| u_x(x,\cdot) - v_x(x,\cdot) \| \leq \sup_{n\in\mathbb{Z}} \widetilde{A}(n) E \tag{15}$$

where

$$\widetilde{A}(n) = |\sqrt{in}| \, | \frac{\sinh(\sqrt{in}x) - \tau \sinh(\tau\sqrt{in}x)}{\cosh(\sqrt{in})} | .$$

We now estimate $\widetilde{A}(n)$ and rewrite it as follows

$$\widetilde{A}(n) = \frac{A(n)}{C(n)}, \tag{16}$$

where

$$A(n) = |\sqrt{in}| \, | \sinh(\sqrt{in}x) - \tau \sinh(\tau\sqrt{in}x) |, C(n) = |\cosh(\sqrt{in})|.$$

We should estimate $A(n)$ and $C(n)$, respectively. To estimate $A(n)$, we rewrite

$$\begin{aligned}A(n) &= \sqrt{|n|} \, | \sinh(\sqrt{in}x) - \tau \sinh(\tau\sqrt{in}x) | \\ &\leq \sqrt{|n|} \big(| \sinh(\sqrt{in}x) - \sinh(\tau\sqrt{in}x) | + (1-\tau) |\sinh(\tau\sqrt{in}x)| \big) \\ &\leq \sqrt{|n|}(A_1 + A_2),\end{aligned} \tag{17}$$

where

$$A_1 = |\sinh(\sqrt{in}x) - \sinh(\tau\sqrt{in}x)|, \quad A_2 = (1-\tau)|\sinh(\tau\sqrt{in}x)|.$$

For estimating $A(n)$, we should estimate A_1 and A_2, respectively. We have

$$\begin{aligned}A_1 &= |\sinh(\sqrt{in}x) - \sinh(\tau\sqrt{in}x)| = \frac{1}{2}|e^{\sqrt{in}x} - e^{\sqrt{in}x} - e^{\tau\sqrt{in}x} + e^{-\tau\sqrt{in}x}| \\ &= \frac{1}{2}|(e^{\sqrt{in}x} - e^{\tau\sqrt{in}x})(1 + e^{-(1+\tau)\sqrt{in}x})| \\ &= \frac{1}{2}|e^{\sqrt{in}x}||(1 - e^{-(1-\tau)x\sqrt{in}})(1 + e^{-(1+\tau)x\sqrt{in}})| \\ &\leq \frac{1}{2}e^{\sqrt{\frac{|n|}{2}}x}(A_{11} \cdot A_{12})\end{aligned} \tag{18}$$

where

$$A_{11} = |1 - e^{-(1-\tau)x\sqrt{in}}|, \quad A_{12} = |1 + e^{-(1+\tau)x\sqrt{in}}|.$$

Using the inequality $1 - e^{-y} \le y (y \ge 0)$ and $0 < \tau \le 1$, we get

$$\begin{aligned}
A_{11} &= |1 - e^{-(1-\tau)x\sqrt{\frac{|n|}{2}}(1+\sigma i)}| \\
&= |1 - e^{-(1-\tau)x\sqrt{\frac{|n|}{2}}\sigma i} + e^{-(1-\tau)x\sqrt{\frac{|n|}{2}}\sigma i} - e^{-(1-\tau)x\sqrt{\frac{|n|}{2}}(1+\sigma i)}| \\
&\le |1 - e^{-(1-\tau)x\sqrt{\frac{|n|}{2}}\sigma i}| + |e^{-(1-\tau)x\sqrt{\frac{|n|}{2}}\sigma i}(1 - e^{-(1-\tau)x\sqrt{\frac{|n|}{2}}})| \\
&\le |1 - \cos\big((1-\tau)x\sqrt{\frac{|n|}{2}}\big) - \sigma i \sin\big((1-\tau)x\sqrt{\frac{|n|}{2}}\big)| + (1-\tau)x\sqrt{\frac{|n|}{2}} \\
&\le 2|\sin\big(\frac{1}{2}(1-\tau)x\sqrt{\frac{|n|}{2}}\big)| + (1-\tau)x\sqrt{\frac{|n|}{2}} \\
&\le 2(1-\tau)x\sqrt{\frac{|n|}{2}} \le 2(1-\tau)\sqrt{\frac{|n|}{2}}, \qquad (19)
\end{aligned}$$

and

$$A_{12} \le 1 + |e^{-(1+\tau)x\sqrt{in}}| = 1 + e^{-(1+\tau)x\sqrt{\frac{|n|}{2}}} \le 2. \qquad (20)$$

Inserting the inequalities (19) and (20) into equation (18), we have

$$A_1 \le \frac{1}{2}e^{\sqrt{\frac{|n|}{2}}x}\big(4(1-\tau)\sqrt{\frac{|n|}{2}}\big) = \sqrt{2}(1-\tau)\sqrt{|n|}e^{\sqrt{\frac{|n|}{2}}x}. \qquad (21)$$

We apply the method with the same as A_1 to estimate A_2. Since $0 < \tau \le 1$, we get

$$\begin{aligned}
A_2 &= \frac{1}{2}(1-\tau)|e^{\tau\sqrt{in}x} - e^{-\tau\sqrt{in}x}| \le \frac{1}{2}(1-\tau)(|e^{\tau\sqrt{in}x}| + |e^{-\tau\sqrt{in}x}|) \\
&\le (1-\tau)e^{\tau\sqrt{\frac{|n|}{2}}x} \le (1-\tau)e^{\sqrt{\frac{|n|}{2}}x}. \qquad (22)
\end{aligned}$$

Coming with (21), (22) and (17), we then have

$$\begin{aligned}
A(n) &= \sqrt{|n|}(\sqrt{2}\big(1-\tau)\sqrt{|n|}e^{\sqrt{\frac{|n|}{2}}x} + (1-\tau)e^{\sqrt{\frac{|n|}{2}}x}\big) \\
&= \sqrt{|n|}(\sqrt{2|n|}+1)(1-\tau)e^{\sqrt{\frac{|n|}{2}}x}. \qquad (23)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
C(n) &= \frac{1}{2}|e^{\sqrt{in}}+e^{-\sqrt{in}}| = \frac{1}{2}|e^{\sqrt{in}}(1+e^{-2\sqrt{in}})| \\
&= \frac{1}{2}e^{\sqrt{\frac{|n|}{2}}}|1+e^{-\sqrt{2|n|}(1+\sigma i)}| \\
&= \frac{1}{2}e^{\sqrt{\frac{|n|}{2}}}\sqrt{1+e^{-2\sqrt{2|n|}}+2e^{-\sqrt{2|n|}}\cos(\sqrt{2|n|})} \\
&\geq \frac{1}{2}e^{\sqrt{\frac{|n|}{2}}}\sqrt{1+2e^{-\sqrt{2|n|}}\cos(\sqrt{2|n|})}.
\end{aligned}$$

Since $2e^{-\sqrt{2|n|}}\cos(\sqrt{2|n|})$ has a minimum value of $-\sqrt{2}e^{-3\pi/4}$, which is a constant less than 1. Thus we have

$$C(n) \geq c_1 e^{\sqrt{\frac{|n|}{2}}}, \tag{24}$$

where $c_1 = \frac{1}{2}\sqrt{1-\sqrt{2}e^{-3\pi/4}}$.

Since $\sqrt{1+n^2\alpha^2} \leq 1+\frac{1}{2}n^2\alpha^2$ and $0 < 1-\tau \leq \frac{1}{2}n^2\alpha^2$. Therefore, inserting the inequalities (23) and (24) into equation (16), we finally get

$$\begin{aligned}
\widetilde{A}(n) &\leq \frac{\sqrt{|n|}(\sqrt{2|n|}+1)(1-\tau)e^{\sqrt{\frac{|n|}{2}}x}}{c_1 e^{\sqrt{\frac{|n|}{2}}}} \\
&\leq \frac{1}{c_1}e^{-\sqrt{\frac{|n|}{2}}(1-x)}(\sqrt{2}|n|+\sqrt{\frac{|n|}{2}})\frac{1}{2}n^2\alpha^2 \\
&= \frac{1}{c_1}e^{-\sqrt{\frac{|n|}{2}}(1-x)}(\frac{\sqrt{2}}{2}|n|^3+\frac{\sqrt{2}}{4}|n|^{\frac{5}{2}})\alpha^2.
\end{aligned} \tag{25}$$

Set $t := \sqrt{|n|}$, from the equation (25), we obtain

$$\widetilde{A}(n) \leq \frac{1}{c_1}e^{-\frac{(1-x)t}{\sqrt{2}}}(\frac{\sqrt{2}}{2}t^6+\frac{\sqrt{2}}{4}t^5)\alpha^2, \quad t \geq 0.$$

The function $h^i(t) = e^{-\frac{(1-x)t}{\sqrt{2}}}t^i$ $(i = 5,6)$ attains its maximum

$$h^5_{\max} = h\left(\frac{5\sqrt{2}}{1-x}\right) = \left(\frac{5\sqrt{2}}{(1-x)e}\right)^5, \quad h^6_{\max} = h\left(\frac{6\sqrt{2}}{1-x}\right) = \left(\frac{6\sqrt{2}}{(1-x)e}\right)^6$$

for $t = \frac{5\sqrt{2}}{1-x}$ and $t = \frac{6\sqrt{2}}{1-x}$, respectively. Since α is given by (9), we get

$$\begin{aligned}\widetilde{A}(n) &\leq \frac{1}{c_1}\sqrt{2}\max\{h_{\max}^5, h_{\max}^6\}\alpha^2 \\ &\leq c(x)\frac{1}{16(\log(E/\delta))^4}, \end{aligned} \tag{26}$$

where

$$c(x) = \frac{\sqrt{2}}{c_1}\max\left\{\left(\frac{5\sqrt{2}}{(1-x)e}\right)^5, \left(\frac{6\sqrt{2}}{(1-x)e}\right)^6\right\}.$$

Combing (26) and (15), we get

$$\| u_x(x,\cdot) - v_x(x,\cdot) \| \leq c(x)\frac{E}{16(\log(E/\delta))^4}. \tag{27}$$

The theorem 3.1 now follows by combing (14) and (27). □

From Theorem 3.1, we know that (6) is a stable approximation of the exact solution $u(x,t)$. However, the accuracy of the regularized solution becomes progressively lower as $x \to 1$. To obtain the continuous dependence of the solution at $x = 1$, we need to introduce a stronger a priori assumption

$$\left\| \left.\frac{\partial^p u(x,\cdot)}{\partial x^p}\right|_{x=1} \right\| \leq E, \tag{28}$$

where $p > 1$ is an integer.

Theorem 3.2 *Let $u(x,t)$ be the solution of problem (1) which is given by (2) with exact data f. Let $v^\delta(x,t)$ be the solution of problem (4) which is given by (6) with measurement data f^δ. The measurement data f^δ satisfies (5) and let the a priori assumption (28) hold. The regularization parameter α is chosen as*

$$\alpha = \frac{1}{4(\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-p}))^2} \tag{29}$$

Then for $p > 1$, we get the error bound

$$\| u_x(1,\cdot) - v_x^\delta(1,\cdot) \| \leq (\varepsilon_1 + \varepsilon_2)E \tag{30}$$

where

$$\begin{aligned}\varepsilon_1 &:= \max\left\{\frac{2}{c_2}\alpha^{\frac{2}{5}(p-1)}, \frac{1}{c_2}(\frac{\sqrt{2}}{2}\alpha^{\frac{8}{5}} + \frac{\sqrt{2}}{4}\alpha^2)\right\}, \\ \varepsilon_2 &:= \sqrt{2}\Big((\ln\frac{E}{\delta})^{1-p} + (\ln(\ln\frac{E}{\delta})^{-p})(\ln\frac{E}{\delta})^{-p}\Big), c_2 = 1 - e^{-\sqrt{2}}.\end{aligned}$$

PROOF. From (2) and (28), we have

$$\left\| \frac{\partial^p u(x,\cdot)}{\partial x^p}\Big|_{x=1} \right\|^2 = \begin{cases} 2\pi\sum_{n=-\infty}^{+\infty}|C_n|^2|\sqrt{in}|^{2p}|\cosh(\sqrt{in})|^2, & p \textit{ is even} \\ 2\pi\sum_{n=-\infty}^{+\infty}|C_n|^2|\sqrt{in}|^{2p}|\sinh(\sqrt{in})|^2, & p \textit{ is odd} \end{cases} \le E^2$$

Since the procedure of the proof is completely similar whenever p is even or odd, thus we only discuss the case that p is even.

Taking a similar procedure of the proof of Theorem 3.1. From (5) and (28), we get

$$\begin{aligned} \| u_x(1,\cdot) - v_x^\delta(1,\cdot) \| &\le \| u_x(1,\cdot) - v_x(1,\cdot) \| + \| v_x(1,\cdot) - v_x^\delta(1,\cdot) \| \\ &\le \sqrt{2\pi \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} |C_n|^2|\sqrt{in}|^2|\sinh(\sqrt{in}) - \tau\sinh(\tau\sqrt{in})|^2} \\ &\quad + \sqrt{2\pi \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} |C_n - C_n^\delta|^2|\tau\sqrt{in}|^2|\sinh(\tau\sqrt{in})|^2} \\ &\le \sup_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \widehat{A}(n)E + \sup_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \widehat{B}(n)\delta, \end{aligned} \tag{31}$$

where

$$\widehat{A}(n) = \left| \frac{\sinh(\sqrt{in}) - \tau\sinh(\tau\sqrt{in})}{(\sqrt{in})^{p-1}\cosh(\sqrt{in})} \right|, \quad \widehat{B}(n) = \left| \tau\sqrt{in}\sinh(\tau\sqrt{in}) \right|.$$

We also start by estimating the second term on the right-hand side of (31). Let $x = 1$ in (13) and note that α is given by (29), we have

$$\begin{aligned} \widehat{B}(n)\delta &\le \frac{\sqrt{2}}{2\sqrt{\alpha}} e^{\frac{1}{2\sqrt{\alpha}}}\delta = \sqrt{2}(\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-p}))(\ln\frac{E}{\delta})^{-p}E \\ &= \sqrt{2}\Big((\ln\frac{E}{\delta})^{1-p} + (\ln(\ln\frac{E}{\delta})^{-p})(\ln\frac{E}{\delta})^{-p}\Big)E, \quad p > 1. \end{aligned} \tag{32}$$

To estimate the first term on the right-hand side of (31), we rewrite $\widehat{A}(n)$ as

$$\widehat{A}(n) = \frac{1}{|n|^{\frac{p-1}{2}}} \left| \frac{\sinh(\sqrt{in}) - \tau\sinh(\tau\sqrt{in})}{\cosh(\sqrt{in})} \right|. \tag{33}$$

To estimate (33), we distinguish two cases.

Case 1: when $|n| \ge \alpha^{-\frac{4}{5}} > 0$

Note that $0 < \tau \le 1$, we get

$$\begin{aligned}
\widehat{A}(n) &= \frac{1}{|n|^{\frac{p-1}{2}}} \frac{|\sinh(\sqrt{in})| + |\sinh(\tau\sqrt{in})|}{|\cosh(\sqrt{in})|} \\
&\le \frac{1}{|n|^{\frac{p-1}{2}}} \frac{e^{\sqrt{\frac{|n|}{2}}} + e^{\tau\sqrt{\frac{|n|}{2}}}}{e^{\sqrt{\frac{|n|}{2}}}\sqrt{1+e^{-2\sqrt{2|n|}} - 2e^{-\sqrt{2|n|}}}} \\
&\le \frac{1}{|n|^{\frac{p-1}{2}}} \frac{1}{c_2}(1+e^{-(1-\tau)\sqrt{\frac{|n|}{2}}}) \le \frac{1}{|n|^{\frac{p-1}{2}}} \frac{2}{c_2} \le \frac{2}{c_2}|n|^{-\frac{p-1}{2}} \qquad (34)
\end{aligned}$$

where $c_2 = 1 - e^{-\sqrt{2}} > 0$. Thus we get

$$\widehat{A}(n) \le \frac{2}{c_2}|n|^{-\frac{p-1}{2}} \le \frac{2}{c_2}\alpha^{\frac{2}{5}(p-1)}, \quad p > 1. \qquad (35)$$

Case 2: when $1 \le |n| < \alpha^{-\frac{4}{5}}$

Taking a similar procedure of the estimation of $\widetilde{A}(n)$. Let $x = 1$ in (25), we get

$$\widehat{A}(n) \le \frac{1}{|n|^{\frac{p}{2}}} \frac{1}{c_2}(\frac{\sqrt{2}}{2}|n|^3 + \frac{\sqrt{2}}{4}|n|^{\frac{5}{2}})\alpha^2 \le \frac{1}{c_2}(\frac{\sqrt{2}}{2}|n|^{\frac{6-p}{2}} + \frac{\sqrt{2}}{4}|n|^{\frac{5-p}{2}})\alpha^2. \qquad (36)$$

If $1 < p < 5$, from (36), we have

$$\begin{aligned}
\widehat{A}(n) &\le \frac{1}{c_2}(\frac{\sqrt{2}}{2}\alpha^{\frac{2}{5}(p-6)} + \frac{\sqrt{2}}{4}\alpha^{\frac{2}{5}(p-5)})\alpha^2 = \frac{1}{c_2}(\frac{\sqrt{2}}{2}\alpha^{\frac{2}{5}(p-1)} + \frac{\sqrt{2}}{4}\alpha^{\frac{2}{5}p}) \\
&\le \frac{\sqrt{2}}{c_2}\alpha^{\frac{2}{5}(p-1)}. \qquad (37)
\end{aligned}$$

If $5 \le p < 6$, from (36), we have

$$\widehat{A}(n) \le \frac{1}{c_2}(\frac{\sqrt{2}}{2}|n|^{\frac{1}{2}} + \frac{\sqrt{2}}{4})\alpha^2 = \frac{1}{c_2}(\frac{\sqrt{2}}{2}\alpha^{\frac{8}{5}} + \frac{\sqrt{2}}{4}\alpha^2). \qquad (38)$$

If $p \ge 6$, note that $|n| \ge 1$, from (36), we have

$$\widehat{A}(n) \le \frac{1}{c_2}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4})\alpha^2 \le \frac{1}{c_2}(\frac{\sqrt{2}}{2}\alpha^{\frac{8}{5}} + \frac{\sqrt{2}}{4}\alpha^2). \qquad (39)$$

Summarizing (34)-(39), we complete the estimate of the first term on the right-hand side of (31), i.e.,

$$\widehat{A}(n)E \le \max\Big\{\frac{2}{c_2}\alpha^{\frac{2}{5}(p-1)}, \frac{1}{c_2}(\frac{\sqrt{2}}{2}\alpha^{\frac{8}{5}} + \frac{\sqrt{2}}{4}\alpha^2)\Big\}E \qquad (40)$$

The theorem 3.2 now follows from (31), (32) and (40). □

Remark 1 *Since the regularization parameter $\alpha \to 0$ as the measured error $\delta \to 0$, we can easily find that, for $p > 1$, $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0 (\delta \to 0)$. Thus*

$$\lim_{\delta \to 0} \| u_x(1,\cdot) - v_x^\delta(1,\cdot) \| = 0, \quad p > 1. \tag{41}$$

Remark 2 *Note that the regularization parameter in Theorem 3.1 differs from in Theorem 3.2. However, we can use only one regularization parameter in Theorem 3.1 and Theorem 3.2 by making no more efforts. In Theorem 3.1, if we let the regularization parameter α be given by (29) and a stronger priori bound E given by (28). Using the procedure of the proof of Theorem 3.1, we can easily get the similar error estimate as*

$$\begin{aligned} \| u_x(x,\cdot) - v_x^\delta(x,\cdot) \| \quad \leq \quad & \sqrt{2}(E)^x \delta^{1-x} \Big(\ln \frac{E}{\delta} \Big)^{-px} \ln \Big(\frac{E}{\delta} \Big(\ln \frac{E}{\delta} \Big)^{-p} \Big) \\ & + \widetilde{c}(x) \frac{E}{16(\ln(\frac{E}{\delta}(\ln \frac{E}{\delta})^{-p}))^4} \end{aligned} \tag{42}$$

where $p \geq 0$ and $\widetilde{c}(x)$ is the similar constant $c(x)$ in Theorem 3.1. If we take $p = 0$ in (42), we have $\widetilde{c}(x) = c(x)$ and can easily get the same error estimate as Theorem 3.1. Thus we conclude that (42) generalizes Theorem 3.1.

4 The method of lines

In order to obtain the approximate computed solution for the heat flux in a bounded domain without initial value, we use a method of lines [Eldén (1997)] to solve problem (4).

Rewrite (4) in a block operator equation in which the subscript x denotes the spatial derivative

$$\begin{bmatrix} I & 0 \\ 0 & I - \alpha^2 \frac{\partial^2}{\partial t^2} \end{bmatrix} \begin{bmatrix} v(x,t) \\ v_x(x,t) \end{bmatrix}_x = \begin{bmatrix} 0 & I \\ \frac{\partial}{\partial t} & 0 \end{bmatrix} \begin{bmatrix} v(x,t) \\ v_x(x,t) \end{bmatrix}, \tag{43}$$

and Cauchy conditions become

$$\begin{bmatrix} v(0,t) \\ v_x(0,t) \end{bmatrix} = \begin{bmatrix} f^\delta(t) \\ 0 \end{bmatrix}. \tag{44}$$

Partition the time interval $[0,2\pi]$ *as* $0=t_0<t_1<\cdots<t_n=2\pi$ *where* $t_j=\tau\cdot j$ $(j=0,1,\cdots,n)$ *and* $\tau=\frac{2\pi}{n}$ *is the step size. Denote* $v(x,t)$ *at the discrete times by the following vector*

$$V(x)=\left[v(x,t_0),v(x,t_1),\cdots,v(x,t_n)\right]^T. \tag{45}$$

According to (44), we have

$$V(0)=\left[f^\delta(t_0),f^\delta(t_1),\cdots,f^\delta(t_n)\right]^T, V_x(0)=\left[0,0,\cdots,\cdots,0\right]^T. \tag{46}$$

The time derivative $\frac{\partial}{\partial t}$ *can be approximated by the forward difference scheme, we have*

$$\begin{bmatrix} v_t(x,t_1) \\ \vdots \\ v_t(x,t_{n-1}) \end{bmatrix} \approx \frac{1}{\tau}\begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(n-1)\times n} \begin{bmatrix} v(x,t_1) \\ \vdots \\ v(x,t_n) \end{bmatrix}. \tag{47}$$

Since the initial data is unknown, we should apply some neighborhood points to approximate the initial data. Thus we have the following approximation

$$\begin{aligned} v_t(x,t_0) &\approx 2v_t(x,t_1)-v_t(x,t_2) \\ &\approx 2\frac{v(x,t_2)-v(x,t_0)}{2\tau}-\frac{v(x,t_3)-v(x,t_1)}{2\tau} \\ &= \frac{-v(x,t_0)+\frac{1}{2}v(x,t_1)+v(x,t_2)-\frac{1}{2}v(x,t_3)}{\tau}. \end{aligned} \tag{48}$$

By the similar method, we get

$$\begin{aligned} v_t(x,t_n) &\approx 2v_t(x,t_{n-1})-v_t(x,t_{n-2}) \\ &\approx 2\frac{v(x,t_n)-v(x,t_{n-2})}{2\tau}-\frac{v(x,t_{n-1})-v(x,t_{n-3})}{2\tau} \\ &= \frac{\frac{1}{2}v(x,t_{n-3})-v(x,t_{n-2})-\frac{1}{2}v(x,t_{n-1})+v(x,t_n)}{\tau}. \end{aligned} \tag{49}$$

From (48), (49) and (47), the unbounded operator $\frac{\partial}{\partial t}$ *can be expressed by*

$$V_t(x)\approx\Psi\cdot V(x), \tag{50}$$

where the coefficient matrix Ψ *is given by*

$$\Psi=\frac{1}{\tau}\begin{bmatrix} -1 & \frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & \ddots & \ddots & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{bmatrix}_{(n+1)\times(n+1)}. \tag{51}$$

The second order derivative $\frac{\partial^2}{\partial t^2}$ *can be approximated by the central difference scheme as*

$$V_{tt}(x) \approx \Phi \cdot V(x), \tag{52}$$

where Φ *is a three-diagonal matrix, with nonzero elements as follows*

$$(\Phi)_{i,i-1} = \frac{1}{\tau^2},\ (\Phi)_{i,i} = -\frac{2}{\tau^2},\ (\Phi)_{i,i+1} = \frac{1}{\tau^2},\ i = 2,\ \cdots,\ n-1, \tag{53}$$

and the first and the last row of the matrix Φ *can be obtained by the following deductions. The* 1*-st component of the vector* $V_{tt}(x)$ *is given by*

$$\begin{aligned} v_{tt}(x,t_0) &\approx \frac{v_t(x,t_1) - v_t(x,t_0)}{\tau} \\ &\approx \frac{2v_t(x,t_2) - v_t(x,t_3) - 2v_t(x,t_1) + v_t(x,t_2)}{\tau} \\ &\approx \frac{1}{\tau}\left(3\frac{v(x,t_3)-v(x,t_1)}{2\tau} - \frac{v(x,t_4)-v(x,t_2)}{2\tau} - 2\frac{v(x,t_2)-v(x,t_0)}{2\tau}\right) \\ &= \frac{v(x,t_0) - \frac{3}{2}v(x,t_1) - \frac{1}{2}v(x,t_2) + \frac{3}{2}v(x,t_3) - \frac{1}{2}v(x,t_4)}{\tau^2}, \end{aligned} \tag{54}$$

thus the first row of the matrix Φ *is*

$$(\Phi)_{1,1\cdots n+1} = \frac{1}{\tau^2}(1,\ -\frac{3}{2},\ -\frac{1}{2},\ \frac{3}{2},\ -\frac{1}{2},\ 0,\ \cdots,\ 0). \tag{55}$$

The $(n+1)$*-th component of the vector* $V_{tt}(x)$ *is given by the same method*

$$\begin{aligned} v_{tt}(x,t_n) &\approx \frac{v_t(x,t_n) - v_t(x,t_{n-1})}{\tau} \\ &\approx \frac{2v_t(x,t_{n-1}) - v_t(x,t_{n-2}) - 2v_t(x,t_{n-2}) + v_t(x,t_{n-3})}{\tau} \\ &\approx \frac{1}{\tau}\left(2\frac{v(x,t_n)-v(x,t_{n-2})}{2\tau} - 3\frac{v(x,t_{n-1})-v(x,t_{n-3})}{2\tau} + \frac{v(x,t_{n-2})-v(x,t_{n-4})}{2\tau}\right) \\ &= \frac{-\frac{1}{2}v(x,t_{n-4}) + \frac{3}{2}v(x,t_{n-3}) - \frac{1}{2}v(x,t_{n-2}) - \frac{3}{2}v(x,t_{n-1}) + v(x,t_n)}{\tau^2}, \end{aligned} \tag{56}$$

thus we can obtain the last row of the matrix Φ *as follows*

$$(\Phi)_{n+1,1\cdots n+1} = \frac{1}{\tau^2}(0,\ \cdots,\ 0,\ -\frac{1}{2},\ \frac{3}{2},\ -\frac{1}{2},\ -\frac{3}{2},\ 1). \tag{57}$$

Therefore, the coefficient matrix Φ is given by

$$\Phi = \frac{1}{\tau^2}\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & \cdots & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ & & & \ddots & \ddots & \ddots & & & & \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix}_{(n+1)\times(n+1)}. \tag{58}$$

In terms of (50) and (52), the problem (43) can be discreted to be a system of ordinary differential equations

$$\begin{bmatrix} I & 0 \\ 0 & I-\alpha^2\Phi \end{bmatrix}\begin{bmatrix} V \\ V_x \end{bmatrix}_x = \begin{bmatrix} 0 & I \\ \Psi & 0 \end{bmatrix}\begin{bmatrix} V \\ V_x \end{bmatrix}, \tag{59}$$

where I is the identity matrix. Further from (59), we can prove that Φ has non-positive eigenvalues, i.e., Φ is a semi-negative definite matrix, therefore the matrix $I-\alpha^2\Phi$ is invertible. We finally get

$$\begin{bmatrix} V \\ V_x \end{bmatrix}_x = \begin{bmatrix} 0 & I \\ (I-\alpha^2\Phi)^{-1}\Psi & 0 \end{bmatrix}\begin{bmatrix} V \\ V_x \end{bmatrix}. \tag{60}$$

The Cauchy conditions (44) become

$$\begin{bmatrix} V \\ V_x \end{bmatrix}(0) = \left[f^\delta(t_0), \cdots, f^\delta(t_n), 0, \cdots, 0\right]^T. \tag{61}$$

There are many feasible methods to solve the ODES (60). In our numerical implementation, we use the fourth order Kutta method for solving the system of equations (60). Therefore, we get

$$\begin{aligned}\begin{bmatrix} V \\ V_x \end{bmatrix}(x_{k+1}) \quad = \quad & \frac{1}{8}[I+\Delta x(C(\Phi,\Psi)+3C(\Phi,\Psi)(I+\frac{1}{3}C(\Phi,\Psi)) \\ & +3C(\Phi,\Psi)(I+\frac{2}{3}\Delta xC(\Phi,\Psi)) \\ & +C(\Phi,\Psi)(I+\Delta xC(\Phi,\Psi)))]\begin{bmatrix} V \\ V_x \end{bmatrix}(x_k), \end{aligned} \tag{62}$$

where $\Delta x = x_{k+1}-x_k$ is a step size for spatial variable and

$$C(\Phi,\Psi) = \begin{bmatrix} 0 & I \\ (I-\alpha^2\Phi)^{-1}\Psi & 0 \end{bmatrix}.$$

Combining (61) with (62), it is easy to obtain the heat flux in the solution domain.

5 Numerical experiments

In this section, we test numerical examples to demonstrate the feasible of our approach. In order to check the effect of numerical computations, we compute the root mean square error at fixed x by the following formula

$$e(u_x) = \left(\frac{1}{n+1} \sum_{j=0}^{n} (\tilde{u}_x(\cdot,t_j) - u_x(\cdot,t_j))^2 \right)^{\frac{1}{2}} \tag{63}$$

where $\tilde{u}_x$ *is the regularized solution,* u_x *is the exact solution, and* $\{t_j\}$ *is a set of discrete times in internal* $[0,2\pi]$.

The noise Cauchy data are generated by

$$f^{\delta}(t_j) = f(t_j)(1+\varepsilon \cdot rand(j))$$

where $f(t_j)$ *is the exact data,* $rand(j)$ *is a random number uniformly distributed in* $[-1,1]$ *and the magnitude* ε *indicates a relative noise level. Therefore, we take* $\delta = \varepsilon \parallel f(t) \parallel$ *in the proof of Theorem.*

5.1 Examples

In this section, we will present three examples to illustrate the effectiveness of the proposed method. All numerical results show that the proposed numerical approach is feasible and stable.

Example 1: *Let the exact solution for the problem (1) be*

$$u(x,t) = 1 - \frac{1}{2}(e^{x/\sqrt{2}}\cos(x/\sqrt{2}+t) + e^{-x/\sqrt{2}}\cos(x/\sqrt{2}-t)) \tag{64}$$

The Cauchy data can be calculated as $f(t) = 1-\cos t$ *and* $q(t) = 0$. *We consider to impose the stronger a priori bound on* $\left\| \left. \frac{\partial^p u(x,\cdot)}{\partial x^p} \right|_{x=1} \right\|$ *where* $p=2$. *We can calculate by Matlab that* $\parallel u(1,t) \parallel_{L^2} = 3.1543$, *so we might as well choose* $E = 3.6$.

We apply two methods to recover the surface heat flux in a bounded domain. One method is the method of lines (ML) given in Section 5 and the other method is Fourier series method (MS) given by (3), refer to [Dorroh and Ru (1999)]. Figure 1 shows the numerical comparison of the exact solution and its approximations with ML and MS where we take the regularization $\alpha \approx 0.0158$ *from (29). For the MS solution we choose* $n = 10$ *and for the computation of the ML solution the stepsize for x is* $1/100$, *for t is* $2\pi/380$. *The root mean square errors are* $e_{ML} = 0.0235$ *and* $e_{MS} = 0.013$ *for* $\varepsilon = 0.001$, *respectively. Since the exact solution* $u(x,t)$ *is periodic function with t, the MS solution converges the exact solution everywhere. Both methods work very well for such a periodic example.*

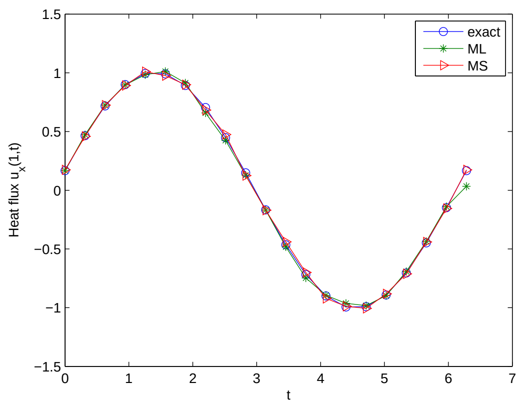

Figure 1: Approximate solutions with the method of lines (ML) and the method of Fourier Series (MS) at $x = 1$ for $\varepsilon = 0.001$.

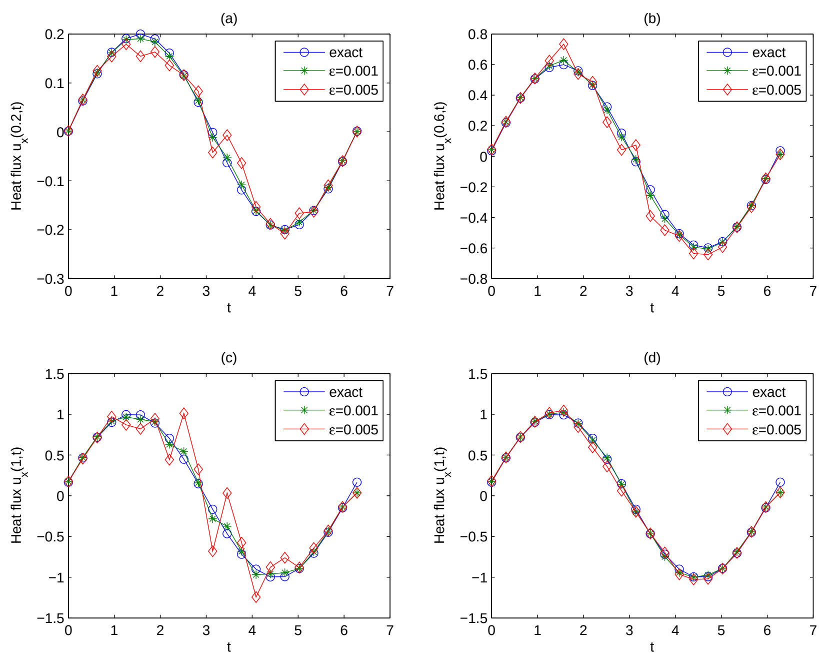

Figure 2: The ML solutions compared with the exact solution for different x. (a) $x = 0.2$ (b) $x = 0.6$(c) $x = 1$(d) $x = 1$

Table 1: The root mean square errors for the ML solutions for Example 1 with $\varepsilon = 0.001$ and $\varepsilon = 0.005$.

x	0.1	0.3	0.6	0.8	0.9	1	1
$e_{0.001}(u_x)$	0.0039	0.0058	0.0132	0.0229	0.0310	0.0419	0.0214
$e_{0.005}(u_x)$	0.0174	0.0264	0.0570	0.0969	0.1308	0.1760	0.0373

Numerical results at different locations x for two noise levels $\varepsilon = 0.001,\ 0.005$ *are computed by ML, see Figure 2. We choose the regularization parameters* $\alpha \approx 0.0037,\ 0.0058$ *from (9) for noise levels* $\varepsilon = 0.001,\ 0.005$ *in Figure 2(a)-(c), respectively and in 2(d) we take* $p = 2$ *and the regularization parameters* $\alpha \approx 0.0158, 0.0316$ *chosen by (29) for* $\varepsilon = 0.001,\ 0.005$*, respectively. In Table 1, we display the root means square errors in line with Figure 2.*

We can see that the accuracy of the regularized solution becomes lower for the same noise level from Figure 2(a) to Figure 2(c). The far the distance between x and Cauchy data is, the large the root mean square error of heat flux between approximation solution and exact solution is from the second column to the seventh column in Table 1.

From Figure 2(c) and the seventh column in Table 1, we know that the accuracy is worst and the root mean square error is largest on the boundary $x = 1$*. These results are consistent with the conclusion of Theorem 3.1, that is the accuracy of the regularized solution becomes progressively lower as* $x \to 1$*. As we know, it is difficult to recover the heat flux far away from Cauchy data without initial value. In order to obtain fairly accurate approximate solution, we use a stronger a priori bound (28) and the regularization parameter (29) to solve this Cauchy problem. Compared with Figure 2(c) and 2(d) or the last column in Table 1, it can seen that the numerical solution is more accurate for recovering the heat flux on the boundary* $x = 1$ *with (28) and (29). These results are consistent with the conclusion of Theorem 3.2.*

In Table 2, we display the root means square errors for different noise levels at the location $x = 0.4$*. For the second row in Table 2, a priori bound and the regularization parameter are given by (8) and (9), respectively. For the third row in Table 2, a stronger a priori bound and the regularization parameter are given by (28) and (29), respectively. From Table 2, we can see that the larger the noise levels are, the larger the root means square errors are between the approximate solution and exact solution. The root means square errors of the third row are less than errors*

Table 2: The root mean square errors for the different noise levels in Example 1 with location at $x = 0.4$.

ε	0.001	0.003	0.005	0.008	0.01	0.03	0.05
$e(u_x)$	0.0071	0.0185	0.0295	0.0451	0.0551	0.1476	0.2323
$e(u_x)$	0.0053	0.0106	0.0139	0.0169	0.0182	0.0221	0.0227

of the second row for the same noise level. Thus a stronger a priori bound (28) and the regularization parameter (29) can obtain better convergence and stability which is consistent with Remark 2 in Section 3.

Example 2: *Take the exact solution for the problem (1) as*

$$u(x,t) = 1 - e^{-t}\cos x \tag{65}$$

The Cauchy data can be calculated as $f(t) = 1 - e^{-t}$ *and* $q(t) = 0$. *We consider to impose the stronger a priori bound on* $\left\| \left. \frac{\partial^p u(x,\cdot)}{\partial x^p} \right|_{x=1} \right\|$ *where* $p = 2$. *We can calculate by Matlab that* $\| u(1,t) \|_{L^2} = 2.5068$, *so we might as well choose* $E = 2.6$.

Figure 3 shows the comparison of the exact solution and the ML solution and MS solution at $x = 1$ *for noise level* $\varepsilon = 0.001$. *In the computation of the MS solution, we take the regularization parameter* $\alpha \approx 0.0179$ *from (29) and* $n = 30$. *For the*

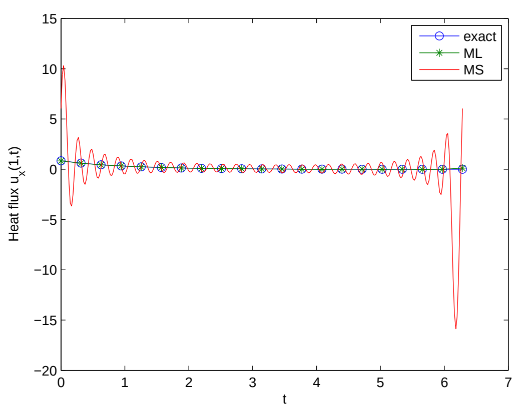

Figure 3: Approximate solutions with the method of lines (ML) and the method of Fourier Series (MS) at $x = 1$ for $\varepsilon = 0.001$.

ML solution, the stepsize for x is $1/100$*, for t is* $2\pi/300$*. The root mean square errors are* $e_{ML} = 0.0171$ *and* $e_{MS} = 2.2278$ *for* $\varepsilon = 0.001$*, respectively. Since the exact solution* $u(x,t)$ *is not periodic to t, the computed surface heat flux for MS is drastically oscillatory on the boundary, especially at the neighbourhood two endpoints. Therefore, MS fails to recover the surface heat flux in a bounded domain. From Figure 3, it can be seen that ML is much more effective.*

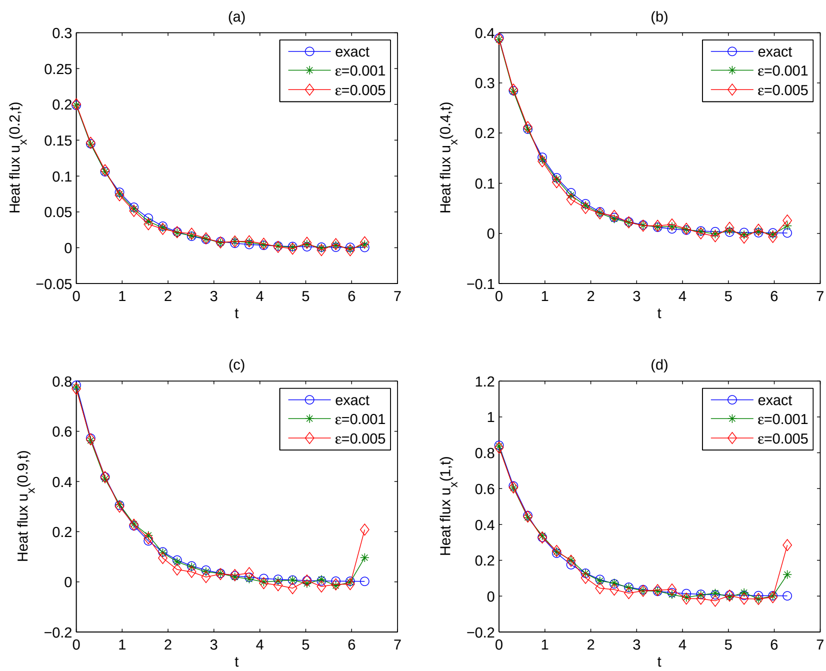

Figure 4: The ML solutions compared with the exact solution for different x. (a) $x = 0.2$ (b) $x = 0.4$(c) $x = 0.9$(d) $x = 1$

From the analysis of Remark 2 in Section 3, we let the regularization parameter α *be given by (29) and a stronger priori bound E given by (28), then we can easily get the same error estimate for both of interior and boundary heat flux. For reconstructing the interior and surface heat flux, we take* $p = 2$ *and* $E = 2.6$ *and the regularization parameters* $\alpha \approx 0.0179, 0.0373$ *given by (3.19) for* $\varepsilon = 0.001$*,* 0.005*, respectively. Numerical results by ML for noise levels* $\varepsilon = 0.001$*,* 0.005 *are presented in Figure 4 for different fixed x. We can see that numerical approximations* *are satisfactory for both of interior and surface heat flux. Meanwhile, numerical results are stable to the increase of noise levels.*

Example 3: *In this example, we consider a more complicated problem. The exact solution for problem (1) is unknown and the surface heat flux is a piecewise smooth function as follows*

$$u_x(1,t) = \begin{cases} -e^{\sin t}, & 0 \le t \le \pi, \\ -5, & \pi < t \le 2\pi. \end{cases} \tag{66}$$

The Neumann boundary data $q(t) = 0$ *and the Dirichlet data at* $x = 0$ *is obtained by solving a direct problem*

$$\begin{cases} u_{xx} = u_t, \\ u_x(0,t) = 0, \\ u_x(1,t) = \begin{cases} -e^{\sin t}, & 0 \le t \le \pi, \\ -5, & \pi < t \le 2\pi, \end{cases} \\ u(x,0) = \frac{3}{2}x^2 - x^3. \end{cases} \tag{67}$$

We apply the finite difference method of Crank-Nicolson scheme to solve this direct problem to get $f(t)$*, then use ML to solve the inverse problem.*

We consider to impose the stronger a priori bound on $\left\| \left. \frac{\partial^p u(x,\cdot)}{\partial x^p} \right|_{x=1} \right\|$ *where* $p = 2$*. Since we do not know the exact solution of the problem (1), we might as well choose* $E = 2$*. In the computation, we apply the proposed numerical method to solve the Cauchy problem where the stepsize for* x *is* $1/100$*, for* t *is* $2\pi/200$.

Figure 5 shows the comparison of the exact solution and the ML solution and MS solution at $x = 1$ *for noise level* $\varepsilon = 0.001$*. In the computation of the MS solution, we take the regularization parameter* $\alpha \approx 0.0199$ *from (29) and* $n = 10$*. The root mean square errors are* $e_{ML} = 0.3520$ *and* $e_{MS} = 17.9424$ *for* $\varepsilon = 0.001$*, respectively. From Figure 5, it can be seen that MS fails and ML is much more effective to recover the heat flux on the boundary for problem (1) without exact solution.*

Numerical results for various levels δ *of relative noises are computed by ML in Figure 6. From (29), we choose the regularization parameters* $\alpha \approx 0.0199, 0.0329, 0.0430$ *for* $\varepsilon = 0.001, 0.003, 0.005$*, respectively. The root mean square errors are* $e_{0.001} = 0.3520$*,* $e_{0.003} = 0.6354$ *and* $e_{0.005} = 0.8868$ *for* $\varepsilon = 0.001, 0.003, 0.005$*, respectively. We can see that the numerical results at* $x = 1$ *are convergent to the exact boundary value if choosing the regularization parameter* α *from* (29) *which is consistent with Theorem 3.2.*

From the numerical results, we can see that the proposed ML is much more effective than MS.

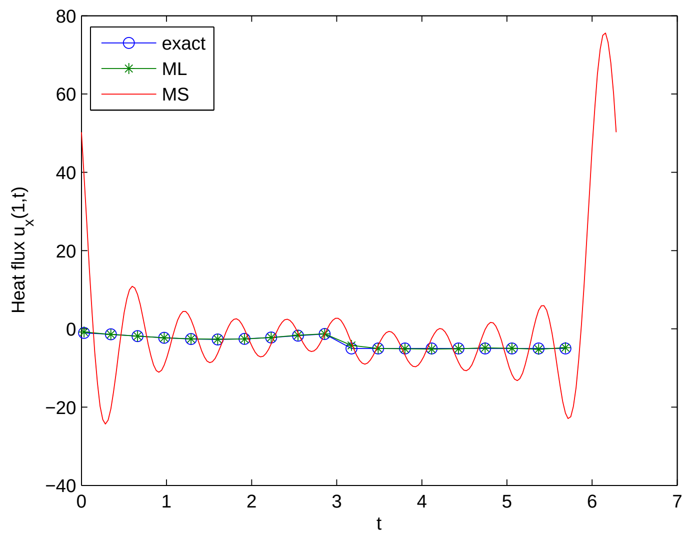

Figure 5: Approximate solutions with the method of lines (ML) and the method of Fourier Series (MS) at $x = 1$ for $\varepsilon = 0.001$.

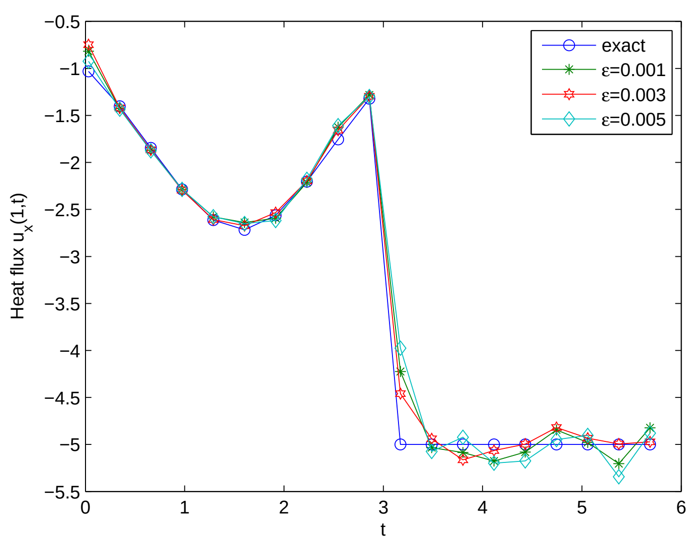

Figure 6: Exact and computed solution at $x = 1$.

6 Conclusion

In this paper, we study an inverse heat conduction problem in a bounded domain without initial value. This problem is severely ill-posed, we apply a quasi-reversibility regularization method to reconstruct heat flux. Under a certain choice of the regularization parameter, we can obtain some logarithmic convergence estimates with respect to the noise level in the Cauchy data. With a stronger assumption on the regularity of the solution, the convergence estimate is obtained for the whole domain, including boundary. The numerical results are consistent with our theoretic results and also show that the proposed method is reasonable, feasible and stable.

Acknowledgments

The research of J.C. Liu was supported by the Fundamental Research Funds for the Central Universities (2014QNA57).

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