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Composite Simpson's Rule for Computing Supersingular Integral on Circle

Jin Li[1,2], Hongxing Ru[1] and Dehao Yu[3,4]

Abstract: The computation with Simpson's rule for the supersingular integrals on circle is discussed. When the singular point coincides with some priori known point, the convergence rate of the Simpson rule is higher than the globally one which is considered as the superconvergence phenomenon. Then the error functional of density function is derived and the superconvergence phenomenon of composite Simpson rule occurs at certain local coordinate of each subinterval. Based on the error functional, a modify quadrature is presented. At last, numerical examples are provided to validate the theoretical analysis and show the efficiency of the algorithms.

Keywords: Supersingular integral, Composite Simpson rule, Error Expansion, Superconvergence

1 Introduction

In this paper we consider the following supersingular integral on the circle

$$I^2(f,s) := \oint_c^{c+2\pi} \frac{\cos\frac{x-s}{2} f(x)}{\sin^3\frac{x-s}{2}} dx, \tag{1}$$

Following the definition of Hadamard finite-part integral, we have

$$I^2(f,s) = \lim_{\varepsilon\to 0}\left\{\int_c^{s-\varepsilon} \frac{f(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} dx + \int_{s+\varepsilon}^{c+2\pi} \frac{f(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} dx - \frac{f(s)}{\sin^2\frac{\varepsilon}{2}}\right\}. \tag{2}$$

[1] School of Mathematics,Shandong University,Jinan,250100,PR China
[2] School of Sciences, Shandong Jianzhu University, Jinan 250101, China.
[3] School of Mathematical Sciences, Xiamen University, Xiamen 361005, China.
[4] LSEC, ICMSEC, Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing 100190 , China.

The error analysis for Riemann integrals with the composite Simpson rule has been well done. The convergence rate for the usual Riemann integrals is $O(h^4)$. However, it is not true for Hadamard finite-part integrals due to the hypersingularity of the integrand. The Newton-Cotes methods for computation the hypersingular integral on interval was studied by [Linz (1985)]. Then in 1993, [Yu (1993)] gave a new quadrature formulae to compute the case of singular point coinciding with the mesh point which presented the error estimate is $O(h|\ln h|)$. In recent years, a modified trapezoidal rule was presented by [Wu and Yu (1999)] and the convergence rate is proved. Numerical methods have been extensively investigated for hypersingular integral on the interval[Abdou (2003); Akel and HusseinH (2011); Chen and Hong (1999); Klerk (2002); Monegato (1994); Zhou Li and Yu (2010); Choi, Kim, and Yun (2004); Hasegawa (2004); Hui and Shia (1999); Ioakimidis (1985); Kim and Jinn (2002); Li Wu and Yu (2009); Li and Yu (2011a,b); Li Zhang and Yu (2013); Zhou Li and Yu (2010)].

The Simpson rule for the computation of supersingular integral on interval was firstly discussed in [Du (2001)] and the $O(h)$ convergence rate is proved, then in the year of 2005, the trapezoidal rule for supersingular integral was presented in [Wu and Sun (2005)] where this rule was shown to be divergent in general, but exhibit the superconvergence phenomenon at certain special point. Then [Zhang, Wu and Yu (2009)], the superconvergence phenomenon of the composite Simpsonï¿½ï¿½s rule for the supersingular was studied and the superconvergence estimate was given. In recent paper [Li Zhang and Yu (2010)], the general Newton-Cotes rules for evaluating the supersingular integrals were investigated and the error expansion estimate of the Newton-Cotes rules were obtained.

The hypersingular integral in circle have not been studied widely, maybe reference [Yang (2013); Zhang, Wu and Yu (2010, 2009)] cover the whole area. In this paper, based on the error expansion of the density function, the error functional of the supersingular integral is obtained. We are concerning with the pointwise superconvergence phenomenon, i.e., when the singular point s coincides with some a priori known points, the convergence rate of the composite Simpson rule is higher than what is globally possible. We show that a convergence rate can reach $O(h^2)$ with the local coordinate equal to zero which depends upon the regularity of the density function.

The rest of this paper is organized as follows. In Sect.2, after introducing some basic formulas of the general (composite) Simpson rule and notations, we present our main result. In Sect.3 the corresponding theoretical analysis is given. Finally, several numerical examples are given to validate our analysis.

2 Main result

Let $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = c + 2\pi$ be a uniform partition of the interval $[c, c + 2\pi]$ with mesh size $h = 2\pi/n$ and $f_Q(x)$ be defined as the Simpson interpolation for $f(x)$ and $x_i = c + (i-1)h, x_{i-1/2} = x_i - h/2$, with basis function defined as below

$$\phi_{2i+1}(x) = \begin{cases} -\dfrac{4(x - x_{i+1})(x - x_i)}{h^2}, & x \in [x_i, x_{i+1}], \quad 0 \le i \le m-1, \\ 0, & otherwise. \end{cases}$$

$$\phi_{2i}(x) = \begin{cases} \dfrac{2(1-\delta_{i0})(x - x_{i-1})(x - x_{i-1/2})}{h^2}, & x \in [x_{i-1}, x_i], \\ \dfrac{2(1-\delta_{im})(x - x_{i+1})(x - x_{i+1/2})}{h^2}, & x \in [x_i, x_{i+1}], \quad 0 \le i \le m, \\ 0, & otherwise. \end{cases}$$

where $\delta_{ij} = 0$ with $i \neq j$ and $\delta_{ij} = 1$ with $i = j$.
Then we have

$$f_Q(x) = \frac{2(x - x_i)(x - x_{i-1/2})}{h^2} f(x_{i-1}) + \frac{2(x - x_{i-1})(x - x_{i-1/2})}{h^2} f(x_i) \\ -\frac{4(x - x_i)(x - x_{i-1})}{h^2} f(x_{i-1/2}), \;\; x \in [x_{i-1}, x_i], \;\; 0 \le i \le 2n, \tag{3}$$

and

$$f_Q(x) = \sum_{i=0}^{2n} \phi_i(x) f(x_{i/2}) \tag{4}$$

We also define a linear transformation

$$x = \hat{x}_i(\tau) := (\tau + 1)\frac{x_i - x_{i-1}}{2} + x_{i-1}, i = 0, 1, \cdots, n-1, \;\; \tau \in [-1, 1], \tag{5}$$

from the reference element $[-1, 1]$ to the subinterval $[x_{i-1}, x_i]$.
The new composite Simpson rule is given by $f_Q(x)$ to replacing $f(x)$ in Eq. 1

$$I_n^2(f_Q, s) := \int_c^{c+2\pi} \frac{f_Q(x) \cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} dx = \sum_{i=0}^{2n} \omega_i^2(s) f(x_{i/2}), \tag{6}$$

By straightly calculation, we have

$$w_{2i}^2(s) = -\frac{4}{h^2}\left(\cot(\frac{x_i - s}{2}) - 4h\cot(\frac{x_{i-1} - s}{2}) + 8\log\left|\frac{\sin(\frac{x_i - s}{2})}{\sin(\frac{x_{i-1}-s}{2})}\right|\right);$$

and

$$w_{2i+1}^2(s) = \frac{1}{2h^2}\left(-2h\cot(\frac{x_{i+1} - s}{2}) - 6h\cot(\frac{x_i - s}{2}) + 8\log\left|\frac{\sin(\frac{x_{i+1} - s}{2})}{\sin(\frac{x_i-s}{2})}\right|\right)$$
$$+\frac{1}{2h^2}\left(-6h\cot(\frac{x_i - s}{2}) - 6h\cot(\frac{x_{i-1} - s}{2}) + 8\log\left|\frac{\sin(\frac{x_i - s}{2})}{\sin(\frac{x_{i-1}-s}{2})}\right|\right).$$

Now we present our main results below. The proof will be given in next section.

Theorem 1 *Assume $f(x) \in C^4[c, c+2\pi]$. For the Simpson rule $I_n(f,s)$ defined in Eq. 6, there exists a positive constant C, independent of h and s such that*

$$I^2(f,s) - I_n^2(f_Q,s) = 8hf^{(3)}(s)\log(2\cos\frac{\tau\pi}{2}) + \mathscr{R}_f(s), \tag{7}$$

where $s = x_{m-1} + (1+\tau)h/2, m = 1, 2, \cdots, n$ and

$$|\mathscr{R}_f(s)| \le C(|\ln h| + \gamma^{-2}(\tau))h^2 \tag{8}$$

and $\gamma(\tau)$ is defined as

$$\gamma(\tau) = \frac{1-|\tau|}{2} \quad \tau \in (-1,1). \tag{9}$$

In the following, C will denote a generic constant which is independent of h and s and may have different values in different places. In addition, we assume that $s \in (x_{m-1}, x_m)$, for some m and let $s = x_{m-1} + (\tau+1)h/2$ with $\tau \in (-1,1)$ denoting its local coordinate.

Now we define $\mathscr{I}_{n,i}(s)$ as below

$$\mathscr{I}_{n,i}(s) = \begin{cases} \displaystyle\int_{x_{i-1}}^{x_i} \frac{(x-x_i)(x-x_{i-1/2})(x-x_{i-1})\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx, & i \ne m, \\ \displaystyle\neq\!\!\!\!\!\!\int_{x_{m-1}}^{x_m} \frac{(x-x_m)(x-x_{m-1/2})(x-x_{m-1})\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx, & i = m. \end{cases} \tag{10}$$

Then we have

Lemma 1 *Assume $s = x_{m-1} + (\tau+1)h/2$ with $\tau \in (-1,1)$. Let $\mathscr{I}_{n,i}(s)$ be defined by (10). Then there holds that*

$$\begin{aligned}\mathscr{I}_{n,i}(s) &= h^2\sum_{k=1}^{\infty}\{\sin[k(x_m - s)] - \sin[k(x_{m-1}-s)]\}\\ &+12h\sum_{k=1}^{\infty}\frac{1}{k}\left[\cos k(x_m - s) + \cos k(x_{m-1}-s)\right]\\ &+24\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\sin k(x_m - s) - \sin k(x_{m-1}-s)\right].\end{aligned} \tag{11}$$

Proof For $i = m$, we set $F_m(x) = (x - x_m)(x - x_{m-\frac{1}{2}})(x - x_{m-1})$

$$\begin{aligned}&\mathscr{I}_{n,m}(s)\\ &= \lim_{\varepsilon\to 0}\left\{\left(\int_{x_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_m}\right)\frac{F_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx - \frac{F_m(\varepsilon)}{\sin^2\frac{\varepsilon}{2}}\right\}\\ &= -\lim_{\varepsilon\to 0}\left\{\left(\int_{x_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_m}\right)\frac{3(x-x_{m-1})(x-x_m) - \frac{h^2}{2}}{\sin^2\frac{x-s}{2}}dx - 4(2s - x_m - x_{m-1})\cot\frac{\varepsilon}{2}\right\}\\ &= h^2\cot\frac{x_m - s}{2} - h^2\cot\frac{x_{m-1}-s}{2} + 6\int_{x_{m-1}}^{x_m}(x - x_{m-\frac{1}{2}})\cot\frac{x-s}{2}dx.\end{aligned} \tag{12}$$

Similarly, for $i \neq m$, using integral by parts on the corresponding Riemann integral, we have

$$\begin{aligned}&\mathscr{I}_{n,i}(s) = h^2\cot\frac{x_i - s}{2} - h^2\cot\frac{x_{i-1}-s}{2}\\ &+6\lim_{\varepsilon\to 0}\left\{\left(\int_{x_{i-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_i}\right)(x - x_{i-\frac{1}{2}})\cot\frac{x-s}{2}dx\right\}\end{aligned} \tag{13}$$

Now, by using the well-known identity(see, e.g.,[Andrews (2002); Yu (2002)]),

$$\frac{1}{2}\cot\frac{t}{2} = \sum_{k=1}^{\infty}\sin kt,$$

we can easily obtain (11) from (12) and (13). □

Lemma 2 *Under the same assumptions of Lemma 1, there holds that*

$$\sum_{i=1}^{n}\mathscr{I}_{n,i}(s) = 24h\ln 2\cos\frac{\tau\pi}{2}. \tag{14}$$

Proof By (11), we have

$$\begin{aligned}
\sum_{i=1}^{n}\mathscr{I}_{n,m}(s) &= h^2\sum_{k=1}^{\infty}\sum_{i=1}^{n}\{\sin[k(x_m-s)]-\sin[k(x_{m-1}-s)]\} \\
&+ 12h\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{1}{k}[\cos k(x_m-s)+\cos k(x_{m-1}-s)] \\
&+ 24\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{1}{k^2}[\sin k(x_m-s)-\sin k(x_{m-1}-s)] \\
&= 24h\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{1}{k}\cos k(x_1-s) \\
&= 24h\sum_{j=1}^{\infty}\frac{1}{j}\cos[nj(x_1-s)] \\
&= 24\pi\sum_{j=1}^{\infty}\frac{1}{j}\cos[j(1+\tau)\pi] \\
&= 24h\ln 2\sin\frac{(1+\tau)\pi}{2} \\
&= 24h\ln 2\cos\frac{\tau\pi}{2}
\end{aligned} \tag{15}$$

where we have used

$$\sum_{i=1}^{n}\sin k(x_i-s)=\begin{cases} n\sin k(x_1-s), & k=nj, \\ 0, & k\neq nj. \end{cases} \tag{16}$$

The proof of Lemma 2 is completed. □

Before presenting the main results, we firstly define $K_s(x)$

$$K_s(x)=\begin{cases} \dfrac{(x-s)^3\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} & x\neq s, \\ 8, & x=s. \end{cases} \tag{17}$$

Lemma 3 *Assume that $f(x)\in C^4[a,b]$ and $f_Q(x)$ be defined by Eq. 3, there holds*

$$\begin{aligned}
f(x)-f_Q(x)= & \frac{f^{(3)}(s)}{2}(x-x_i)(x-x_{i-\frac{1}{2}})(x-x_{i-1}) \\
& +\mathscr{R}_i^1(x)+\mathscr{R}_i^2(x)+\mathscr{R}_i^3(x)+\mathscr{R}_i^4(x),
\end{aligned} \tag{18}$$

where

$$\mathscr{R}_i^1(x) = \frac{F_i(x)}{12h^2}(x-x_{i-1})^3 f^{(4)}(\xi_{1i}), \tag{19}$$

$$\mathscr{R}_i^2(x) = \frac{F_i(x)}{12h^2}(x-x_i)^3 f^{(4)}(\xi_{2i}), \tag{20}$$

$$\mathscr{R}_i^3(x) = \frac{F_i(x)}{12h^2}(x-x_{i-1/2})^3 f^{(4)}(\xi_{3i}), \tag{21}$$

$$\mathscr{R}_i^4(x) = \frac{f^{(4)}(\alpha_i)}{6} F_i(x)(x-s) \tag{22}$$

where $\xi_{1i}, \xi_{2i}, \xi_{3i}, \alpha_i \in (x_{i-1}, x_i)$ *and*

$$|\mathscr{R}_i^j(x)| \leq Ch^4, j = 1,2,3. \tag{23}$$

By performing $f(x_i), f(x_{i-1/2}), f(x_{i-1})$ at the point x, the proof can is similarly as in reference[Li Zhang and Yu (2010)]. □

Setting

$$\mathscr{H}_m(x) \quad = f(x) - f_Q(x) - \frac{f^{(3)}(s)}{6}(x-x_m)(x-x_{m-\frac{1}{2}})(x-x_{m-1}). \tag{24}$$

Lemma 4 *Under the same assumptions of Theorem 1, for $\mathscr{H}_m(x)$ in Eq. 24, there holds that*

$$\left| \int_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} dx \right| \leq C\gamma^{-1}(\tau)h^2. \tag{25}$$

where $\gamma(\tau)$ is defined in Eq. 9.

Proof. By the definition of $\mathscr{H}_m(x)$, we have

$$|\mathscr{H}_m^{(l)}(x)| \leq Ch^{4-l}, l = 0,1,2. \tag{26}$$

As we known

$$\int_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}} dx = 8\int_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)}{(x-s)^3} dx + \int_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)[\kappa_s(x)-8]}{(x-s)^3} dx. \tag{27}$$

From the identity

$$\begin{aligned}\fint_a^b \frac{f(x)}{(x-s)^3}dx = & \frac{f(s)}{2}\left[\frac{1}{(a-s)^2}-\frac{1}{(b-s)^2}\right] \\ & -\frac{(b-a)f'(s)}{(b-s)(s-a)}+\frac{f''(s)}{2}\ln\frac{b-s}{s-a} \\ & +\int_a^b \frac{f(x)-f(s)-f'(s)(x-s)-f''(s)(x-s)^2/2}{(x-s)^3}dx,\end{aligned} \tag{28}$$

we have

$$\begin{aligned}\fint_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)}{(x-s)^3}dx = & \frac{\mathscr{H}_m(s)}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right] \\ & -\frac{h\mathscr{H}_m'(s)}{(x_m-s)(s-x_{m-1})} \\ & +\frac{\mathscr{H}_m''(s)}{2}\ln\frac{x_m-s}{s-x_{m-1}}+\int_{x_{m-1}}^{x_m}\frac{\mathscr{H}_m'''(\theta(x))}{6}dx\end{aligned} \tag{29}$$

where $\theta(x)\in(x_{m-1},x_m)$. Since

$$\begin{aligned}&\left|\frac{\mathscr{H}_m(s)}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right| \\ &=\left|\frac{\mathscr{H}_m(s)-\mathscr{H}_m(x_{m-1})}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right| \\ &=\left|\frac{\mathscr{H}_m'(\xi_{m-1})(s-x_{m-1})}{2}\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right| \\ &\le C\gamma^{-1}(\tau)h^2,\end{aligned} \tag{30}$$

where $\xi_{m-1}\in(x_{m-1},x_m)$ and we have used $\mathscr{H}_m(x_{m-1})=0$.
Then we have

$$\left|\frac{h\mathscr{H}_m'(s)}{(x_m-s)(s-x_{m-1})}\right|\le C\gamma^{-1}(\tau)h^2, \tag{31}$$

$$\left|\frac{\mathscr{H}_m''(s)}{2}\ln\frac{x_m-s}{s-x_{m-1}}\right|\le C[|\ln\gamma(\tau)|+|\ln h|]h^2 \tag{32}$$

and

$$\left|\int_{x_{m-1}}^{x_m}\frac{\mathscr{H}_m'''(\theta(x))}{6}dx\right|\le Ch^2. \tag{33}$$

As for the second term,

$$\begin{aligned}
&\left|\fint_{x_{m-1}}^{x_m}\frac{\mathscr{H}_m(x)[\kappa_s(x)-8]}{(x-s)^3}dx\right| \\
&\leq \max_{x\in[x_{m-1},x_m]}|\mathscr{H}_m(x)|\int_{x_{m-1}}^{x_m}\frac{\kappa_s(x)-8}{(x-s)^3}\,dx \\
&= \max_{x\in[x_{m-1},x_m]}|\mathscr{H}_m(x)|\left\{\fint_{x_{m-1}}^{x_m}\frac{\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}\,dx-\fint_{x_{m-1}}^{x_m}\frac{8}{(x-s)^3}\,dx\right\} \\
&= \max_{x\in[x_{m-1},x_m]}|\mathscr{H}_m(x)|\left\{\frac{1}{\sin^2\frac{s-x_{m-1}}{2}}-\frac{1}{\sin^2\frac{s-x_m}{2}}+\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right\} \\
&= \max_{x\in[x_{m-1},x_m]}|\mathscr{H}_m(x)-\mathscr{H}_m(x_m)|\left\{\frac{1}{\sin^2\frac{s-x_{m-1}}{2}}-\frac{1}{\sin^2\frac{s-x_m}{2}}+\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right\} \\
&= \max_{x\in[x_{m-1},x_m]}|\mathscr{H}'_m(\xi_m)(s-x_m)|\left\{\frac{1}{\sin^2\frac{s-x_{m-1}}{2}}-\frac{1}{\sin^2\frac{s-x_m}{2}}+\left[\frac{1}{(x_{m-1}-s)^2}-\frac{1}{(x_m-s)^2}\right]\right\} \\
&\leq C\gamma^{-1}(\tau)h^2
\end{aligned} \tag{34}$$

Eq. 25 can be obtained by putting together from Eq. 29 to Eq. 34 which completes the proof. □

The proof of Theorem 1: According toEq. 24, we have

$$\fint_{x_{m-1}}^{x_m}\frac{[f(x)-f_Q(x)]\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx=\fint_{x_{m-1}}^{x_m}\frac{\mathscr{H}_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx$$
$$+\frac{f^{(3)}(s)}{6}\fint_{x_{m-1}}^{x_m}\frac{(x-x_m)(x-x_{m-\frac{1}{2}})(x-x_{m-1})\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx$$

then we have

$$\begin{aligned}
\fint_c^{c+2\pi}\frac{\cos\frac{x-s}{2}[f(x)-f_Q(x)]}{\sin^3\frac{x-s}{2}}dx &= \sum_{i=1}^n\fint_{x_{i-1}}^{x_i}\frac{\cos\frac{x-s}{2}[f(x)-f_Q(x)]}{\sin^3\frac{x-s}{2}}dx \\
&= 8hf^{(3)}(s)\log(2\cos\frac{\tau\pi}{2})+R_f(s),
\end{aligned}$$

where

$$R_f(s)=\mathscr{R}^1(s)+\mathscr{R}^2(s) \tag{35}$$

$$\mathscr{R}^1(s)=\fint_{x_{m-1}}^{x_m}\frac{\mathscr{H}_m(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx, \tag{36}$$

$$\begin{aligned}\mathscr{R}^2(s) = & \sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^1(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx+\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^2(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx \\ & +\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^3(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx+\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^4(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx.\end{aligned} \tag{37}$$

For the first part of Eq. 37,we have

$$\begin{aligned}&\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^1(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right| \\ &=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\cos\frac{x-s}{2}\frac{F_i(x)}{12h^2}(x-x_{i-1})^3 f^{(4)}(\xi_{1i})}{\sin^3\frac{x-s}{2}}dx\right| \\ &=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{K_s(x)\frac{F_i(x)}{12h^2}(x-x_{i-1})^3 f^{(4)}(\xi_{1i})}{(x-s)^3}dx\right| \\ &\leq C\max_{x\in(x_{i-1},x_i)}\{K_s(x)\}h^4\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{1}{|x-s|^3}dx \\ &\leq C\gamma^{-2}(\tau)\max_{x\in(c,c+2\pi)}\{K_s(x)\}h^2.\end{aligned} \tag{38}$$

For the second part of Eq. 37,we have

$$\begin{aligned}&\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^2(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right| \\ &=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\cos\frac{x-s}{2}\frac{F_i(x)}{12h^2}(x-x_i)^3 f^{(4)}(\xi_{2i})}{\sin^3\frac{x-s}{2}}dx\right| \\ &=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{K_s(x)\frac{F_i(x)}{12h^2}(x-x_i)^3 f^{(4)}(\xi_{2i})}{(x-s)^3}dx\right| \\ &\leq C\max_{x\in(x_{i-1},x_i)}\{K_s(x)\}h^4\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{1}{|x-s|^3}dx \\ &\leq C\gamma^{-2}(\tau)\max_{x\in(c,c+2\pi)}\{K_s(x)\}h^2.\end{aligned} \tag{39}$$

For the third part of Eq. 37,we have

$$\begin{aligned}
&\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^3(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right| \\
&=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\cos\frac{x-s}{2}\frac{F_i(x)}{12h^2}(x-x_{i-1/2})^3f^{(4)}(\xi_{3i})}{\sin^3\frac{x-s}{2}}dx\right| \\
&=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{K_s(x)\frac{F_i(x)}{12h^2}(x-x_{i-1/2})^3f^{(4)}(\xi_{3i})}{(x-s)^3}dx\right| \\
&\leq C\max_{x\in(x_{i-1},x_i)}\{K_s(x)\}h^4\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{1}{|x-s|^3}dx \\
&\leq C\gamma^{-2}(\tau)\max_{x\in(c,c+2\pi)}\{K_s(x)\}h^2.
\end{aligned} \tag{40}$$

For the last part of Eq. 37,we have

$$\begin{aligned}
&\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\mathscr{R}_i^4(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx\right| \\
&=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{\cos\frac{x-s}{2}\frac{f^{(4)}(\alpha_i)}{6}(x-x_{i-1})(x-x_{i-1/2})(x-x_i)(x-s)}{2\sin^3\frac{x-s}{2}}dx\right| \\
&=\left|\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{K_s(x)\frac{f^{(4)}(\alpha_i)}{6}(x-x_{i-1})(x-x_{i-1/2})(x-x_i)}{(x-s)^2}dx\right| \\
&\leq C\max_{x\in(x_{i-1},x_i)}\{K_s(x)\}h^2\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\frac{1}{(x-s)^2}dx \\
&\leq C\max_{x\in(c,c+2\pi)}\{K_s(x)\}\gamma^{-1}(\tau)h^2.
\end{aligned} \tag{41}$$

From Eq. 38 to Eq. 41 and Lemma 4, we have

$$\begin{aligned}
|R_f(s)| &\leq |\mathscr{R}^1(s)|+|\mathscr{R}^2(s)| \\
&\leq C\max_{x\in(c,c+2\pi)}\{K_s(x)\}[|\ln h|+\gamma^{-2}(\tau)]h^2.
\end{aligned} \tag{42}$$

Then the proof is completed. □

From the above analysis, we obtain the following modify Simpson rule,

$$\tilde{I}_n^2(f;s)=I_n^2(f;s)-8hf^{(3)}(s)\log(2\cos\frac{\tau\pi}{2}), \tag{43}$$

Based on the theorem 1, we present the modify Simpson rule

$$\tilde{I}_n^2(f;s) = I_n^2(f;s) - 8hf^{(3)}(s)\log(2\cos\frac{\tau\pi}{2}), \tag{44}$$

and

$$\tilde{E}_n^2(f;s) = \not\int_c^{c+2\pi} \frac{f(x)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx - \tilde{I}_n^2(f;s) \tag{45}$$

then we have

Corollary 1 *Under the same assumption of theorem 1,we have*

$$\tilde{E}_n^2(f;s) \leq C \max_{x\in(c,c+2\pi)}\{K_s(x)\}[|\ln h| + \gamma^{-2}(\tau)]h^2 \tag{46}$$

where $\gamma(\tau)$is defined as Eq. 9.

For simplicity of our presentation, we confine to the special case with $f(x) = x^3, s = x_{m-1} + \frac{(\tau+1)h}{2}$. By the definition Eq. 1, we have

$$\begin{aligned}
&I^2(f,s) - I_n^2(f_Q,s) \\
&= \not\int_c^{c+2\pi} \frac{\cos\frac{x-s}{2}f(x)}{\sin^3\frac{x-s}{2}}dx - \not\int_c^{c+2\pi} \frac{\cos\frac{x-s}{2}f_Q(x)}{\sin^3\frac{x-s}{2}}dx \\
&= \not\int_c^{c+2\pi} \frac{\cos\frac{x-s}{2}[f(x) - f_Q(x)]}{\sin^3\frac{x-s}{2}}dx \\
&= \not\int_c^{c+2\pi} \frac{(x-x_{i-1})(x-x_{i-1/2})(x-x_i)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx \\
&= \left(\not\int_{x_{m-1}}^{x_m} + \sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\right)\frac{(x-x_{i-1})(x-x_{i-1/2})(x-x_i)\cos\frac{x-s}{2}}{\sin^3\frac{x-s}{2}}dx \\
&= \left(\not\int_{x_{m-1}}^{x_m} + \sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\right)\frac{3(x-x_{i-1})(x-x_i) - \frac{h^2}{2}}{\sin^2\frac{x-s}{2}}dx
\end{aligned}$$

$$
\begin{aligned}
&= h^2 \cot\frac{x_m - s}{2} - h^2 \cot\frac{x_{m-1} - s}{2} \\
&+6 \lim_{\varepsilon\to 0}\left\{\left(\int_{x_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_m}\right)(x - x_{m-1/2})\cot\frac{x-s}{2}\,dx\right\} \\
&+ \sum_{i=1, i\neq m}^{n}\left(h^2\cot\frac{x_i - s}{2} - h^2\cot\frac{x_{i-1}-s}{2} - 2\int_{x_{i-1}}^{x_i}(x - x_{i-1/2})\cot\frac{x-s}{2}\,dx\right) \\
&= h^2 \cot\frac{x_m - s}{2} - h^2 \cot\frac{x_{m-1} - s}{2} \\
&+12h\ln\left|\sin\frac{x_m - s}{2}\sin\frac{x_{m-1}-s}{2}\right| - 24\lim_{\varepsilon\to 0}\left\{\left(\int_{x_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_m}\right)\ln\left|\sin\frac{x-s}{2}\right|dx\right\} \\
&+ \sum_{i=1, i\neq m}^{n}\left(h^2\cot\frac{x_i - s}{2} - h^2\cot\frac{x_{i-1}-s}{2}\right) \\
&+ \sum_{i=1, i\neq m}^{n} 12h\ln\left|\sin\frac{x_i - s}{2}\sin\frac{x_{i-1}-s}{2}\right| \\
&- \sum_{i=1, i\neq m}^{n} 24\lim_{\varepsilon\to 0}\left\{\left(\int_{x_{i-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_i}\right)\ln\left|\sin\frac{x-s}{2}\right|dx\right\} \\
&= h^2\sum_{i=1}^{n}\sum_{k=1}^{\infty}\{\sin[k(x_m - s)] - \sin[k(x_{m-1}-s)]\} \\
&+12h\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{1}{k}[\cos k(x_m - s) + \cos k(x_{m-1}-s)] \\
&+24\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{1}{k^2}[\sin k(x_m - s) - \sin k(x_{m-1}-s)] \\
&= h^2\sum_{k=1}^{\infty}\sum_{i=1}^{n}\{\sin[k(x_m - s)] - \sin[k(x_{m-1}-s)]\} \\
&+12h\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{1}{k}[\cos k(x_m - s) + \cos k(x_{m-1}-s)] \\
&+24\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{1}{k^2}[\sin k(x_m - s) - \sin k(x_{m-1}-s)] \\
&= 24h\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{1}{k}\cos k(x_1 - s)
\end{aligned}
$$

$$
\begin{aligned}
&= 24h \sum_{j=1}^{\infty} \frac{1}{j} \cos[nj(x_1 - s)] \\
&= 24\pi \sum_{j=1}^{\infty} \frac{1}{j} \cos[j(1+\tau)\pi] \\
&= 24h \ln 2 \sin \frac{(1+\tau)\pi}{2} \\
&= 24h \ln 2 \cos \frac{\tau\pi}{2}.
\end{aligned} \tag{47}
$$

The above equation implies that, for the finite part integral Eq. 1, the composite Simpson rule can reach the convergencerate $O(h)$ in general. While for the error functional $\ln 2 \cos \frac{\tau\pi}{2} = 0$, the convergence rate will be higher than $O(h)$.

3 Numerical example

In this section, computational results are reported to confirm our theoretical analysis.

Example 1 *Consider the supersingular integral*

$$
\oint_{c}^{c+2\pi} \frac{\cos \frac{x-s}{2} f(x)}{\sin^3 \frac{x-s}{2}} dx = g(s), s \in (c, c+2\pi) \tag{48}
$$

with $f(x) = 1 + \sin(x) + \cos(x)$ *and the exact analysis is* $4\pi[\sin(s) - \cos(s)]$
The left of table 1 show that when the local coordinate of singular point $\tau = \pm\frac{2}{3}$*, the quadrature reach the convergence rate of* $O(h^2)$ *as for the non-supersingular point*

Table 1: Errors of the Simpson rule and mod-Simpson rule $s = a + x_{[n/4]} + (1+\tau)h/2$

	$I_n^2(f;s)$			$\tilde{I}_n^2(f;s)$		
n	$\tau = 0$	$\tau = 2/3$	$\tau = 1/2$	$\tau = 0$	$\tau = 1/2$	$\tau = 2/3$
32	-5.9448e-1	-2.2314e-2	-3.1512e-1	-1.0606e-1	-8.5810e-2	-2.2314e-2
64	-2.8515e-1	-6.2555e-3	-1.4747e-1	-2.6632e-2	-2.1749e-2	-6.2555e-3
128	-1.3939e-1	-1.6441e-3	-7.0968e-2	-6.6702e-3	-5.4694e-3	-1.6441e-3
256	-6.8878e-2	-4.2073e-4	-3.4764e-2	-1.6689e-3	-1.3710e-3	-4.2073e-4
512	-3.4233e-2	-1.0638e-4	-1.7198e-2	-4.1739e-4	-3.4320e-4	-1.0638e-4
1024	-1.7064e-2	-2.6743e-5	-8.5528e-3	-1.0437e-4	-8.5853e-5	-2.6743e-5
h^α	1.0245	1.9409	1.0407	1.9978	1.9930	1.9409

Table 2: Errors of the Simpson rule and mod-Simpson rule $s = b - (1+\tau)h/2$

	$I_n^2(f;s)$			$\tilde{I}_n^2(f;s)$		
n	$\tau=0$	$\tau=2/3$	$\tau=1/2$	$\tau=0$	$\tau=1/2$	$\tau=2/3$
32	5.9448e-1	2.2314e-2	3.1512e-1	1.0606e-1	8.5810e-2	2.2314e-2
64	2.8515e-1	6.2555e-3	1.4747e-1	2.6632e-2	2.1749e-2	6.2555e-3
128	1.3939e-1	1.6441e-3	7.0968e-2	6.6702e-3	5.4694e-3	1.6441e-3
256	6.8878e-2	4.2073e-4	3.4764e-2	1.6689e-3	1.3710e-3	4.2073e-4
512	3.4233e-2	1.0638e-4	1.7198e-2	4.1739e-4	3.4320e-4	1.0638e-4
1024	1.7064e-2	2.6744e-5	8.5528e-3	1.0437e-4	8.5854e-5	2.6744e-5
h^α	1.0245	1.9409	1.0407	1.9978	1.9930	1.9409

Table 3: Errors of the Simpson rule and mod-Simpson rule $s = a + (1+\tau)h/2$

	$I_n^2(f;s)$			$\tilde{I}_n^2(f;s)$		
n	$\tau=0$	$\tau=2/3$	$\tau=1/2$	$\tau=0$	$\tau=1/2$	$\tau=2/3$
32	4.8788e-1	-3.1837e-2	2.2001e-1	-1.0726e-1	-8.9180e-2	-3.1837e-2
64	2.5844e-1	-7.4543e-3	1.2357e-1	-2.6785e-2	-2.2175e-2	-7.4543e-3
128	1.3271e-1	-1.7942e-3	6.4985e-2	-6.6895e-3	-5.5229e-3	-1.7942e-3
256	6.7208e-2	-4.3952e-4	3.3268e-2	-1.6713e-3	-1.3777e-3	-4.3952e-4
512	3.3815e-2	-1.0873e-4	1.6824e-2	-4.1769e-4	-3.4403e-4	-1.0873e-4
1024	1.6960e-2	-2.7037e-5	8.4593e-3	-1.0441e-4	-8.5958e-5	-2.7037e-5
h^α	0.9693	2.0403	0.9402	2.0009	2.0038	2.0403

the the convergence rate is $O(h)$ which agree with our theorematically analysis. From the right of the table 1 shows the modify Simpson rule have the convergence rate of $O(h^2)$ at both the superconvergence point and non-superconvergence point which coincide with our Corollary 1. For the case of $s = b-(1+\tau)h/2$, table 2 show that the convergence rate of $O(h^2)$ for the superconvergence point the same as the case of $s = a+(1+\tau)h/2$ because of no influence of the boundary condition which coincide with our theoretically analysis. For the case of $s = a+(1+\tau)h/2$, table 3 show that the convergence rate of $O(h^2)$ for the superconvergence point the same as the case of $s = x_{[n/4]}+(1+\tau)h/2$ because of no influence of the boundary condition which coincide with our theoretically analysis.

Example 2 *Consider the supersingular integral*

$$\oint_c^{c+2\pi} \frac{\cos\frac{x-s}{2} f(x)}{\sin^3\frac{x-s}{2}} dx = g(s), s \in (c, c+2\pi) \tag{49}$$

with $f(x) = 1+\sin(2x)+\cos(3x)$ and the exact analysis is $4\pi[9\sin(3s)-4\cos(2s)]$ The left of table 4 show that when the local coordinate of singular point $\tau = \pm\frac{2}{3}$, the quadrature reach the convergence rate of $O(h^3)$ which is higher than $O(h^2)$

Table 4: Errors of the Simpson rule and mod-Simpson rule $s = a + x_{[n/4]} + (1+\tau)h/2$

	$I_n^2(f;s)$			$\tilde{I}_n^2(f;s)$		
n	$\tau=0$	$\tau=2/3$	$\tau=1/2$	$\tau=0$	$\tau=1/2$	$\tau=2/3$
64	9.4168e+0	-1.6219e-1	4.6053e+0	-2.0131e-2	-5.7144e-2	-1.6219e-1
128	4.7496e+0	-2.0548e-2	2.3618e+0	-2.5977e-3	-7.2912e-3	-2.0548e-2
256	2.3800e+0	-2.5790e-3	1.1884e+0	-3.2916e-4	-9.1797e-4	-2.5790e-3
512	1.1906e+0	-3.2282e-4	5.9512e-1	-4.1403e-5	-1.1507e-4	-3.2282e-4
1024	5.9541e-1	-4.0373e-5	2.9768e-1	-5.1907e-6	-1.4401e-5	-4.0373e-5
2048	2.9771e-1	-5.0579e-6	1.4885e-1	-6.4922e-7	-1.8049e-6	-5.0579e-6
h^α	0.9966	2.9938	0.9903	2.9841	2.9901	2.9938

Table 5: Errors of the Simpson rule and mod-Simpson rule $s = b - (1+\tau)h/2$

	$I_n^2(f;s)$			$\tilde{I}_n^2(f;s)$		
n	$\tau=0$	$\tau=2/3$	$\tau=1/2$	$\tau=0$	$\tau=1/2$	$\tau=2/3$
64	-1.0890e+0	5.5584e-1	-1.1546e-1	2.4100e-4	1.5643e-1	5.5584e-1
128	-8.1702e-1	1.4048e-1	-3.0060e-1	1.3574e-4	3.9808e-2	1.4048e-1
256	-4.7658e-1	3.5067e-2	-2.1136e-1	2.7743e-5	9.9438e-3	3.5067e-2
512	-2.5526e-1	8.7448e-3	-1.2092e-1	4.1534e-6	2.4788e-3	8.7448e-3
1024	-1.3187e-1	2.1825e-3	-6.4258e-2	5.6251e-7	6.1841e-4	2.1825e-3
2048	-6.6992e-2	5.4510e-4	-3.3077e-2	7.5543e-8	1.5442e-4	5.4510e-4
h^α	0.8046	1.9988	0.9580	2.8965	1.9969	1.9988

Table 6: Errors of the Simpson rule and mod-Simpson rule $s = a + (1+\tau)h/2$

	$I_n^2(f;s)$			$\tilde{I}_n^2(f;s)$		
n	$\tau=0$	$\tau=2/3$	$\tau=1/2$	$\tau=0$	$\tau=1/2$	$\tau=2/3$
64	-3.2405e+0	-5.1775e-1	-2.0251e+0	5.0221e-3	-1.4294e-1	-5.1775e-1
128	-1.3573e+0	-1.3569e-1	-7.8354e-1	4.7400e-4	-3.8103e-2	-1.3569e-1
256	-6.1181e-1	-3.4467e-2	-3.3245e-1	4.9159e-5	-9.7298e-3	-3.4467e-2
512	-2.8908e-1	-8.6696e-3	-1.5121e-1	5.5001e-6	-2.4520e-3	-8.6696e-3
1024	-1.4032e-1	-2.1731e-3	-7.1832e-2	6.4742e-7	-6.1506e-4	-2.1731e-3
2048	-6.9106e-2	-5.4392e-4	-3.4971e-2	7.7841e-8	-1.5400e-4	-5.4392e-4
h^α	1.1102	1.9789	1.1711	3.1955	1.9717	1.9789

as for the non-supersingular point the the convergence rate is $O(h)$ which agree with our theorematically analysis. From the modify Simpson rule the table 4 shows the convergence rate of is $O(h^3)$ at the non-superconvergence point which coincide with our Corollary 1. For the case of $s = b-(1+\tau)h/2$, table 5 and 6 show that the convergence rate of $O(h^2)$ for the superconvergence point no influence of the boundary condition which coincide with our theoretically analysis while for the modify Simpson rule the convergence rate can reach $O(h^2)$ which agree our Corollary 1.

4 Conclusion

In this paper, we study the composite Simpson's rule for numerical evaluation supersingular integrals defined on circle. Based on the error expansion in each subinterval, the superconvergence phenomenon is obtained. The results in this paper show a possible way to improve the accuracy of the collocation method for supersingular integral equations by choosing the superconvergence points to be the collocation points.

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References

Andrews,L. C. (2002): *Special Functions of Mathematics for Engineers*. McGraw-Hill, Inc.

Abdou,M.A.(2003): Fredholm-Volterra integral equation with singular kernel. *Appl. Math. Comput.*, vol. 137, pp. 231-243.

Akel,M. HusseinH.(2011): Numerical treatment of solving singular integral equations by using Sinc approximations. *Appl. Math. Comput.*, vol. 218 pp.3565-3573.

Chen, J. T.; Hong, H.K. (1999): Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series, *Applied Mechanics Reviews, ASME*, vol.52, no.1, pp.17-33.

Choi, U.J.; Kim, S.W.; Yun, B.I.(2004): Improvement of the asymptotic behaviour of the Euler-Maclaurin formula for Cauchy principal value and Hadamard finite-part integrals. *Int. J. Numer. Methods.*, vol.61, pp. 496-513.

Du, Q.K. (2001): Evaluations of certain hypersingular integrals on interval. *Int. J. Numer. Methods.*, vol. 512, pp. 1195-1210.

Hasegawa, T. (2004): Uniform approximations to finite Hilbert transform and its derivative, *J. Comput. Appl. Math.*, vol. 163, pp. 127-138.

Hui, C.Y.; Shia, D.(1999): Evaluations of hypersingular integrals using Gaussian quadrature. *Int. J. Numer. Methods.*, vol. 44, pp. 205-214.

Ioakimidis, N.I.(1985): On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals and their derivatives. *Math. Comp.*, vol 44, pp. 191-198.

Kim, P.; Jin, U.C.(2003): Two trigonometric quadrature formulae for evaluating hypersingular integrals. Inter, *Inter. J. Numer. Methods Eng.*, vol. 56, pp. 469-486.

Klerk,J.(2002): Solving strongly singular integral equations by Lp approximation methods. *Appl. Math. Comput.*, vol. 127, pp. 311-326.

Li.J; Wu,J.M; Yu, D. H.(2009): Generalized extrapolation for computation of hypersingular integrals in boundary element methods. *CMES:Computer Modeling in Engineering & Sciences,* vol.42,no.2, pp.151-175.

Li, J.; Yu, D.H.(2011): The Superconvergence of Certain Two-Dimensional Cauchy Principal Value Integrals. *CMES:Computer Modeling in Engineering & Sciences*, vol.71 , no.4, pp. 331-346.

Li, J.; Yu, D.H.(2011): The Superconvergence of Certain Two-Dimensional Hilbert singular Integrals. *CMES:Computer Modeling in Engineering & Sciences*, vol. 84, no.4, pp. 233-252.

Li.J; Zhang X.P.; Yu, D. H.(2010): Superconvergence and ultraconvergence of Newton-Cotes rules for supersingular integrals. *J. Comput. Appl. Math.,* vol.233,no.11, pp.2841-2854.

Li.J; Zhang X.P.; Yu, D. H.(2013): Extrapolation methods to compute hypersingular integral in boundary element methods. *Science China Mathematics,* vol.56,no.8, pp.1647-1660.

Linz, P.(1985): On the approximate computation of certain strongly singular integrals. *Computing*, vol. 35, pp. 345-353.

Monegato, G.(1994): Numerical evaluation of hypersingular integrals. *J. Comput. Appl. Math.* vol.50. pp. 9-31

Yu, D. H.(2002): *Natural Boundary Integrals Method and its Applications*. K-luwer Academic Publishers.

Yu, D. H.(1993): *Mathematical Theory of Natural Boundary Element Method*. Science Press, Beijing.

Wu, J.M.; Yu,D.H.(1999): The approximate computation of hypersingular integrals on interval. *Chinese J. Numer. Math. Appl.*, vol.21 pp. 25-33.

Wu, J.M.; Sun, W.W. (2005): The superconvergence of the composite trapezoidal rule for Hadamard finite part integrals. *Numer. Math.*, vol. 102, pp. 343–363.

Yang,C.X (2013): A unified approach with spectral convergence for the evaluation of hypersingular and supersingular integrals with a periodic kernel. *J. Comput. Appl. Math.* , vol.239, pp.322-332.

Zhang, X.P., Wu, J.M., Yu, D.H.(2009): Superconvergence of the composite Simpsons rule for a certain finite-part integral and its applications,., *J. Comput. Appl. Math,* vol.223 pp. 598-613.

Zhang, X.P., Wu, J.M., Yu, D.H.(2010): The Superconvergence of composite trapezoidal rule for Hadamard finite-part integral on a circle and its application., *Inter J Comput Math* vol.87,pp.855-876

Zhang, X.P., Wu, J.M., Yu, D.H.(2009): The superconvergence of composite Newton-Cotes rules for Hadamard finite-part integral on a circle., *Computing* vol.85, pp.219-244.

Zhou,Y.T. Li,J. Yu,D.H.,(2010): Numerical solution of hypersingular equation using recursive wavelet on invariant set. *Appl. Math. Comput.* , vol.217, pp.861-868.