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Comparison of Four Multiscale Methods for Elliptic Problems

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Abstract: Four representative multiscale methods, namely asymptotic homogenization method (AHM), heterogeneous multiscale method (HMM), variational multiscale (VMS) method and multiscale finite element method (MsFEM), for elliptic problems with multiscale coefficients are surveyed. According to the features they possess, these methods are divided into two categories. AHM and HMM belong to the up–down framework. The feature of the framework is that the macroscopic solution is solved first with the help of effective information computed in local domains, and then the multiscale solution is resolved in local domains using the macroscopic solution when necessary. VMS method and MsFEM fall in the uncoupling framework. The feature of the framework is that the multiscale solution in the global domain is resolved directly making use of special functions defined in the macroscopic mesh. The cost of these multiscale methods is compared and their application to a functionally graded material is illustrated. Moreover, the manner to resolve multiscale solutions in AHM and HMM is compared, and it is found that AHM and HMM provide similar multiscale solutions.

Keywords: Multiscale method, homogenization, heterogeneous material, up–down framework, uncoupling framework.

1 Introduction

In recent years many works have concentrated on simulation of complex systems composed of heterogeneous materials or media [Yang, Yu, Ryu, Cho, Kyoung, Han, and Cho (2013); Hou and Efendiev (2009); Geers, Kouznetsova, and Brekelmans (2010); Hughes and Sangalli (2007)]. For example, the thermo–mechanical behavior of composite materials and flows in porous media are frequently investigated. These problems contain at least two spatial scales: one is the macroscale

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of structures and the other is the microscale of heterogeneities. Though classical numerical methods such as finite element method (FEM) could be used directly to simulate these multiscale problems, a main issue is that sufficiently large discrete systems should be solved which demand a tremendous amount of computer memory and CPU time. On the other hand, it is not necessary to obtain full details of the multiscale solution within the entire domain in some cases [E, Engquist, Li, Ren, and Vanden-Eijnden (2007); Zohdi and Wriggers (2005)]. Therefore some multiscale methods have been developed to serve different purposes and to solve the multiscale problems efficiently in the last few decades. In this paper, four representative multiscale methods are compared and their differences and similarities are shown through analytical and numerical results, in order to help the reader have a good choice of multiscale methods for solving multiscale problems for special purpose.

The first multiscale method is the asymptotic homogenization method (AHM) which was proposed in 1970s to solve two–scale problems for periodic structures [Bensoussan, Lions, and Papanicolaou (1978); Oleinik, Shamaev, and Yosifian (1992)]. AHM constructs a formal asymptotic expansion for solution of multiscale problem and provides the manner to compute zeroth–order, first–order and higher–order expanded terms. With the exception of the zeroth–order term, other terms are the combinations of microscopic functions and macroscopic functions to connect different scales. The different scales are connected further by solving auxiliary problems at the microscale to get effective coefficients of the homogenized problem defined at the macroscale [Cioranescu and Donato (1999)]. For structures made up of random heterogeneous materials, Papanicolaou et al. [Papanicolaou and Varadhan (1981)] extended AHM to random case with stationarity and ergodicity assumptions on multiscale coefficients. Considering the first–order homogenization provides microscopic fields with very low accuracy, Cui et al. [Han, Cui, and Yu (2010); Yu, Cui, and Han (2009)] extended AHM to second–order and proposed a statistical second–order two–scale method. AHM has been widely used to predict thermal, electrical and mechanical properties of heterogeneous materials [Ma and Cui (2013); Arabnejad and Pasini (2013); Chatzigeorgiou, Efendiev, Charalambakis, and Lagoudas (2012)].

Heterogeneous multiscale method (HMM) gives another kind of general framework for designing multiscale methods, and a finite element formulation is proposed for elliptic homogenization problem in [E and Engquist (2003)]. In general, HMM consists of two components: selecting a macroscopic solver and estimating the missing data in the solver from microscopic information. Scale separation such as periodicity and stationary randomness of the multiscale coefficients is the main special feature which HMM assumes. Diffusion problems, advection–diffusion

problems, Richards' equation, etc. have been considered by HMM [Ma and Zabaras (2011); Abdulle and Schwab (2005); Henning and Ohlberger (2010); Chen and Ren (2008)]. E et al. [E, Engquist, Li, Ren, and Vanden-Eijnden (2007)] reviewed the fundamental principles, theoretical analysis and some applications of HMM and also some obstacles that need to be solved.

Variational multiscale (VMS) method was proposed by Hughes et al. [Hughes (1995); Hughes, Feijoo, Mazzei, and Quincy (1998)] to deal with multiscale problems in science and engineering. VMS method considers an overlapping sum decomposition of multiscale solution. One component of the decomposition is a sufficiently smooth function at the macroscale, and the other one is a rapidly oscillating function. The microscopic oscillating solution is localized to macroscopic elements and is semi–solved first, and then the macroscopic solution is solved when the microscopic solution is substituted into the macroscopic problem. VMS method has been used to simulate turbulent channel flows, laminar and turbulent flows, etc. [Hughes, Oberai, and Mazzei (2001); Hughes, Mazzei, Oberai, and Wray (2001); Gravemeier (2006)]. Stochastic variational multiscale method has also been proposed for multiscale problems with random heterogeneities [Asokan and Zabaras (2006); Zabaras and Ganapathysubramanian (2009)].

Hou et al. [Hou and Wu (1997); Hou, Wu, and Cai (1999)] proposed the multiscale finite element method (MsFEM) for solving multiscale problems. The main idea of MsFEM is to construct special finite element base functions, which capture the microscopic information, in the macroscopic mesh. Unlike classical smooth polynomial base functions, these base functions are oscillatory functions which reflect the effect of microscale on the macroscale. MsFEM has been applied for many multiscale problems such as groundwater flow [Ye, Xue, and Xie (2004); He and Ren (2006)], two–phase flow [Efendiev, Ginting, Hou, and Ewing (2006); Efendiev and Hou (2007)] and (un)saturated water flows [He and Ren (2009); Zhang, Fu, and Wu (2009)] in heterogeneous porous media. An extended multiscale finite element method was developed by Zhang et al. [Zhang, Wu, and Fu (2010a,b)] for solving mechanical problems of heterogeneous materials. Additional terms of base functions for the interpolation of the vector fields were introduced in their works.

Apart from multiscale methods, some homogenization methods have been developed to predict effective properties of heterogeneous materials or media, see [Ma, Temizer, and Wriggers (2011); Sab and Nedjar (2005); Wang and Pan (2008)] and references therein. Recently, Dong et al. developed a novel method termed Computational Grains for predicting effective properties of heterogeneous materials with arbitrary–shaped inclusions and voids [Dong and Atluri (2012a,b); Dong, Gamal, and Atluri (2013)]. The method is efficient because no meshing is required in each grain which contains a inclusion or a void and then the computational cost can

be reduced. With the effective properties at hand, one can solve the macroscopic solutions of the multiscale problems conveniently.

The remainder of this paper is organized as follows. An overview of the four multiscale methods is presented in section 2. In section 3 we analyze the differences and similarities of these methods and compare their computational cost. AHM and HMM in resolving multiscale solution in local domains is compared further in this section. A numerical example to illustrate the application of these methods to random heterogeneous materials is given in section 4. And conclusions are drawn in the last section.

2 Multiscale methods

We consider the classical second–order elliptic problem with homogeneous Dirichlet boundary condition

$$\begin{cases} -\nabla \cdot (a^{\varepsilon}(x)\nabla u^{\varepsilon}(x)) = f(x) & \text{in } \Omega \subset \mathbb{R}^d, d = 2,3, \\ u^{\varepsilon}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

This problem is widely applied in practical problems such as thermal or electrical conductivity, mechanical properties of composite materials and flows in porous media. The coefficients in this problem possess multiscale characteristic, which is described by a parameter ε. The source term f is a macroscopic function however. We assume that $a^{\varepsilon} = (a^{\varepsilon}_{ij})$ is symmetric and satisfies

$$c\xi \cdot \xi \leq \xi \cdot a^{\varepsilon}\xi \leq C\xi \cdot \xi, \quad \forall \xi \in \mathbb{R}^d, \tag{2}$$

with $0 < c \leq C$, such that the problem has a unique solution u^{ε} for any ε [Braess (2001)]. To get an accurate numerical solution, a full fine scale solver such as fine FEM with mesh $\mathscr{T}_h$ cannot be avoided. The mesh size h cannot be larger than the size of heterogeneities in order to offer some details of the multiscale solution. The fact leads to a big challenge of solving an extremely large discrete system, which is the reason why the following multiscale methods were proposed to obtain an approximation of the multiscale solution u^{ε}.

2.1 Asymptotic homogenization method

The macroscopic behavior of heterogeneous material may be approximated by that of a homogeneous fictitious one, when the characteristic length of the heterogeneities is sufficiently small compared to the characteristic length of the structure. Asymptotic homogenization method (AHM) shows that under some assumptions on the multiscale coefficients a^{ε}, there exists a homogenized problem defined at

the macroscale such that solution of the original multiscale problem converges to solution of the homogenized problem as the small parameter ε goes to zero. As a consequence, AHM first estimates the macroscopic behavior of the heterogeneous material by solving a macroscopic problem which could significantly reduce the amount of computation.

It is assumed that the multiscale coefficients a^ε take the form $a^\varepsilon(x) = a(x/\varepsilon)$ for simplicity. In this case a^ε may be a periodic function or a stationary random function defined in the entire domain, which can be used to describe a lot of heterogeneous materials. AHM provides a formal asymptotic expansion of the multiscale solution

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon^2 u^2(x, x/\varepsilon) + \cdots . \tag{3}$$

Due to

$$\nabla v^\varepsilon(x) = \nabla_x v(x,y) + \frac{1}{\varepsilon}\nabla_y v(x,y), \tag{4}$$

where $y = x/\varepsilon$, when we insert Eq. 3 into the multiscale problem (Eq. 1) and equate coefficients of the same powers of ε, it is found that u^0 is the solution of a homogenized problem

$$\begin{cases} -\nabla\cdot(\bar{a}\nabla u^0(x)) = f(x) & \text{in } \Omega, \\ u^0(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\bar{a} = (\bar{a}_{ij})$ is a constant tensor called effective coefficient tensor. In some cases $\bar{a}$ may be a function of x, but it is still independent of the microscopic variable y [Han, Cui, and Yu (2010); Yu, Cui, and Han (2009)].

In fact, $\bar{a}$ is a bridge connecting the microscale and the macroscale. It can be computed by solving the following auxiliary problems

$$\begin{cases} -\nabla\cdot(a(y)\nabla\omega_\lambda(y)) = 0 & \text{in } Y \subset \mathbb{R}^d, \\ \omega_\lambda(y) = \lambda\cdot y & \text{on } \partial Y. \end{cases} \tag{6}$$

Here Y is an arbitrary domain at the microscale, and λ in the Dirichlet boundary condition is a constant vector. It should be noted that other boundary conditions such as periodic boundary condition or Neumann boundary condition can be employed instead of Dirichlet boundary condition, see [Kanit, Forest, Galliet, Mounoury, and Jeulin (2003); Yue and E (2007)] for more details. A repeated unit cell (RUC) is used to compute effective coefficients for periodic structures. In contrast, for random heterogeneous materials, a representative volume element (RVE) is always used. A schematic diagram of RUC and RVE is presented in Fig. 1.

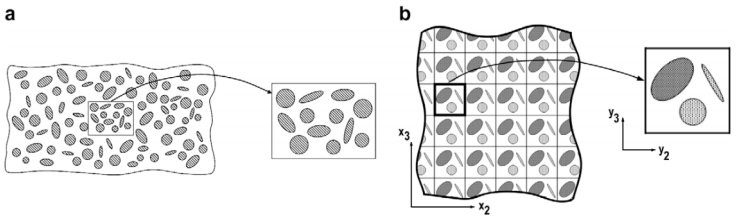


Figure 1: Description of RVE (a) and RUC (b) [Pindera, Khatam, Drago, and Bansal (2009)]

A non–rigorous definition of RVE is that it is a subdomain of random heterogeneous materials which is large enough so that the resulting effective coefficients have little dependence on the boundary condition applied in the auxiliary problems. It is expected that the characteristic length of RVE is much larger than that of heterogeneities but much smaller than that of macroscopic structures. In fact, theoretically the effective coefficients are defined in abstract probability spaces and cannot be explicitly obtained. According to the ergodic theory, the effective coefficients could be computed by solving auxiliary limit problems in the infinite physical space. This is the foundation of computing the effective coefficients in finite RVEs applied in many engineering problems. Note that the RVEs only provide approximations of the effective coefficients for random heterogeneous materials, and the accuracy of the approximations depends strongly on the size of RVEs and the boundary condition employed, see [Wu, Nie, and Yang (2014)] for detailed discussions.

The effective coefficients $\bar{a}$ can be computed by the following formula

$$\langle a(y)\nabla\omega_\lambda(y)\rangle = \bar{a}\cdot\langle\nabla\omega_\lambda(y)\rangle, \tag{7}$$

when the auxiliary problems (Eq. 6) are solved. Here $\langle\cdot\rangle$ means the volume average of the variables in Y, i.e., $\langle\cdot\rangle = \int_Y \cdot \mathrm{d}Y/|Y|$.

In some cases the macroscopic behavior is enough and it is not necessary to resolve all the details of the multiscale problem. AHM solves some microscopic auxiliary problems and a macroscopic homogenized problem numerically. Compared with the fine FEM, the amount of computation is extremely reduced. However, sometimes the multiscale solution should be resolved in some local domains or in the global domain. To this end, AHM provides a strategy to resolve the multiscale solution taking advantage of the formal asymptotic expansion, that is,

$$u^\varepsilon(x) = u^0(x) + \varepsilon\sum_{i=1}^{d}\chi_{e_i}(y)\frac{\partial u^0(x)}{\partial x_i} + \cdots, \tag{8}$$

where $\chi_{e_i} = \omega_{e_i} - e_i \cdot y$, and $e_i (i = 1, \ldots, d)$ are the canonical bases in $\mathbb{R}^d$. The first–order AHM approximates the multiscale solution u^ε by the first–order solution $u_1^\varepsilon = u^0 + \varepsilon u^1$ which includes additional microscopic information introduced by the microscopic functions χ_{e_i}. To improve the accuracy of the multiscale solution given by AHM, higher–order AHM has also been considered, see Cui et al. [Yang, Cui, Nie, and Ma (2012); Yang and Cui (2013)], Xiao et al. [Xiao and Karihaloo (2010)] and references therein. In fact, for periodic structures, the multiscale solution in the global domain is explicitly obtained when both the auxiliary problems and the homogenized problem are solved. However for random heterogeneous materials, additional work should be done to solve the auxiliary problems in some local domains which we are interested in or in the global domain to obtain the multiscale solution.

2.2 Heterogeneous multiscale method

G–convergence [Zhikov, Kozlov, and Oleinik (1994)] in homogenization theory shows that for a sequence of multiscale coefficients $a^\varepsilon, \varepsilon > 0$ satisfying Eq. 2 for any ε, the solution u^ε of the multiscale problem (Eq. 1) converges to the solution of the homogenized problem (Eq. 5) as ε goes to zero. Based on this fact, the heterogeneous multiscale method (HMM) solves the homogenized problem by choosing a macroscopic solver such as the coarse FEM. It is assumed that $\bar{U}_H \subset H_0^1(\Omega)$ is the standard piecewise linear finite element space on a macroscopic mesh $\mathscr{T}_H$ with size H. Then the discrete variational formulation of the homogenized problem is: Find $u_{HMM} \in \bar{U}_H$ such that

$$a(u_{HMM}, v_H) = (f, v_H), \quad \forall v_H \in \bar{U}_H, \tag{9}$$

where

$$a(u_{HMM}, v_H) = \int_\Omega \nabla v_H \cdot \bar{a} \nabla u_{HMM} \, \mathrm{d}\Omega, \tag{10}$$

$$(f, v_H) = \int_\Omega f v_H \, \mathrm{d}\Omega. \tag{11}$$

Since generally $\bar{a}$ is unknown, the missing data above is the effective stiffness matrix. Therefore the next step of HMM is estimating the missing effective stiffness matrix. Let $\bar{\varphi}_i, \bar{\varphi}_j \in \bar{U}_H$ be two arbitrary macroscopic base functions. As illustrated in Fig. 2, in any element $K \in \mathscr{T}_H$, the effective element stiffness matrix could be estimated by numerical quadrature

$$a_K(\bar{\varphi}_i, \bar{\varphi}_j) \approx |K| \sum_{x_k \in K} \omega_k (\nabla \bar{\varphi}_i \cdot \bar{a} \nabla \bar{\varphi}_j)(x_k), \quad i, j = 1, \ldots, n_K, \tag{12}$$

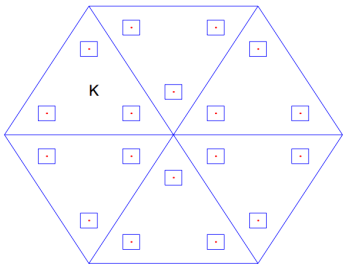


Figure 2: Illustration of HMM [E, Ming, and Zhang (2005)]

where x_k and ω_k are the macroscopic quadrature points and weights in element K, and n_K is the number of nodes in K. To couple the microscale and the macroscale, the following microscopic problems

$$\begin{cases} -\nabla \cdot (a^{\varepsilon}(x)\nabla \tilde{\varphi}_i(x)) = 0 & \text{in } I(x_k), \\ \tilde{\varphi}_i(x) = \bar{\varphi}_i(x) & \text{on } \partial I(x_k), \end{cases} \tag{13}$$

are solved around any quadrature point x_k and for any base function $\bar{\varphi}_i$. Here $I(x_k)$ is a small cube centered at x_k. It may be an RUC for periodic structures or an RVE for random heterogeneous materials in the x coordinates. Then the integrand at x_k in Eq. 12 is approximated by

$$(\nabla \bar{\varphi}_i \cdot \bar{a} \nabla \bar{\varphi}_j)(x_k) \approx \frac{1}{|I(x_k)|} \int_{I(x_k)} \nabla \tilde{\varphi}_i \cdot a^{\varepsilon} \nabla \tilde{\varphi}_j \, \mathrm{d}I(x_k). \tag{14}$$

When the effective stiffness matrix is computed, the variational problem (Eq. 9) is solved finally to get the HMM solution u_{HMM} which is an approximation of the homogenized solution u^0. In fact, HMM estimates the effective integrands in Eq. 12, but AHM directly estimates the effective coefficients $\bar{a}$ in Eq. 12 by solving auxiliary problems (Eq. 6) locally around every quadrature point x_k and then computing Eq. 7.

HMM assumes that the multiscale problem (Eq. 1) satisfies separation of scales. That is, characteristic length of the microscale $O(l)$ is much smaller than characteristic length of the macroscale $O(1)$, and then there exist RUCs or RVEs of size δ such that $l \ll \delta \ll 1$. With scale separation assumption, HMM simply solves some microscopic problems (Eq. 13) and a macroscopic problem (Eq. 9) to capture the macroscopic behavior of the multiscale problem. Therefore its computational cost is much less than that of fine FEM.

To recover the details of the multiscale solution in some local domains, HMM adopts a strategy proposed by Oden et al. [Oden and Vemaganti (2000)]. After the

homogenized problem is solved, the following problem

$$\begin{cases} -\nabla\cdot(a^{\varepsilon}(x)\nabla\hat{u}_h^{\varepsilon}(x)) = f(x) & \text{in } \Omega_L \subset \Omega, \\ \hat{u}_h^{\varepsilon}(x) = u_{HMM}(x) & \text{on } \partial\Omega_L, \end{cases} \tag{15}$$

is solved with fine mesh $\mathscr{T}_h$ to obtain the multiscale solution $\hat{u}_h^{\varepsilon}$ in a local domain Ω_L. The HMM solution u_{HMM} acts as a known Dirichlet boundary condition on the boundary of Ω_L. We should note that though a more accurate approximation $\hat{u}_h^{\varepsilon}$ of original multiscale solution u^{ε} in Eq. 1 can be obtained, the characteristic length of Ω_L should be much smaller than the characteristic length of Ω considering the limit of capability of computing resources, but should be much larger than the characteristic length of heterogeneities because the boundary condition has great effect on the accuracy of $\hat{u}_h^{\varepsilon}$ when Ω_L is very small. In computation, the size of Ω_L may be very close to the RVE (RUC) size δ.

2.3 Variational multiscale method

The variational multiscale (VMS) method assumes that solution of the multiscale problem (Eq. 1) can be decomposed into two parts, namely $u^{\varepsilon} = \bar{u} + u'$, where $\bar{u}$ represents a smooth function and u' a rapidly oscillating function, as shown in Fig. 3. Therefore the variational formulation of Eq. 1 needs to be decomposed into a macroscopic variational formulation and a microscopic variational formulation. VMS method semi–solves the microscopic variational problem first to get an expression of microscopic solution u', and then substitutes it into the macroscopic variational problem to solve the macroscopic solution $\bar{u}$.

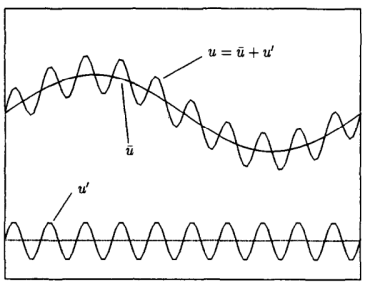


Figure 3: An overlapping sum decomposition of multiscale solution [Hughes, Feijoo, Mazzei, and Quincy (1998)]

Specifically, the variational formulation of Eq. 1 is:
Find $u^{\varepsilon} \in X \subset H_0^1(\Omega)$ such that

$$a(u^{\varepsilon}, v^{\varepsilon}) = (f, v^{\varepsilon}), \quad \forall v^{\varepsilon} \in X. \tag{16}$$

The left–hand side and the right–hand side of the equation are defined similar to Eq. 10 and Eq. 11, respectively. This equation can be rewritten as

$$a(\bar{u}+u',\bar{v}+v') = (f,\bar{v}+v'), \quad \forall \bar{v}+v' \in X, \tag{17}$$

or separated artificially as

$$a(\bar{u},\bar{v}) + a(u',\bar{v}) = (f,\bar{v}), \quad \forall \bar{v} \in \bar{X}, \tag{18}$$

$$a(u',v') = -(a(\bar{u},v') - (f,v')), \quad \forall v' \in X'. \tag{19}$$

Here the solution space X is decomposed as $X = \bar{X} \bigoplus X'$ where $\bar{X}$ is the solution space for the macroscale and X' is the solution space for the microscale. Eq. 18 and Eq. 19 are the macroscopic variational formulation and microscopic variational formulation respectively. Though u' is a nonlocal function, we can only estimate it locally at the microscale. Assuming that u' equals to zero on the boundaries of the macroscopic mesh $\mathscr{T}_H$ for solving $\bar{u}$, VMS method suggests to introduce residual–free bubbles which are defined in every macroscopic element $K \in \mathscr{T}_H$ in order to compute u' locally. Since $\bar{u}$ in the element K can be expressed as

$$\bar{u}|_K = \sum_{i=1}^{n_K} \bar{\varphi}_{i,K} \bar{u}_{i,K}, \tag{20}$$

where $\bar{\varphi}_{i,K} (i = 1,\ldots,n_K)$ are the macroscopic base functions defined in the element K and n_K is the number of nodes in K, Eq. 19 is rewritten locally as

$$a_K(u',v') = -\left(\sum_{i=1}^{n_K} \bar{u}_{i,K} \cdot a_K(\bar{\varphi}_{i,K},v') - (f,v') \right), \quad \forall v' \in X'_K. \tag{21}$$

Let

$$u'|_K = \sum_{i=1}^{n_K} \varphi'_{i,K} \bar{u}_{i,K}, \tag{22}$$

then $\varphi'_{i,K}$ satisfies

$$a_K(\varphi'_{i,K},v') = -(a_K(\bar{\varphi}_{i,K},v') - (f,v')), \quad \forall v' \in X'_K, i = 1,\ldots,n_K. \tag{23}$$

Here $\varphi'_{i,K}$ are the residual–free bubbles which are zero on element boundary ∂K. Eq. 23 in every macroscopic element are solved first to obtain all of the residual–free bubbles, and then the macroscopic solution can be resolved by solving the macroscopic variational problem (Eq. 18) with mesh $\mathscr{T}_H$ taking advantage of Eq. 22. It should be mentioned that since the microscopic solution is constructed by the linear combination of residual–free bubbles with macroscopic solution at the nodes as the coefficients, the multiscale solution has been obtained by VMS method.

2.4 Multiscale finite element method

The main idea of the multiscale finite element method (MsFEM) is to construct special finite element base functions at the macroscale which can capture the microscopic information. These base functions are obtained by solving some problems in every macroscopic element. And then a small discrete system is solved at the macroscale to get the macroscopic solution.

To solve the multiscale problem (Eq. 1), the domain is partitioned first into macroscopic finite elements $\mathscr{T}_H$ (see Fig. 4). Let $\bar{U}_K = \text{span}\{\bar{\varphi}_{i,K}, i = 1,\ldots,n_K\}$ where $\bar{\varphi}_{i,K}$ are the linear base functions defined in the element $K \in \mathscr{T}_H$ and n_K is the number of nodes in K. It is a common knowledge that the standard piecewise linear finite element space $\bar{U}_H = \{\bar{u}|_K \in \bar{U}_K, K \in \mathscr{T}_H\} \subset H_0^1(\Omega)$ cannot resolve all the features of the multiscale solution. MsFEM solves the following problems

$$\begin{cases} -\nabla \cdot (a^\varepsilon(x)\nabla\tilde{\varphi}_{i,K}(x)) = 0 & \text{in } K \in \mathscr{T}_H, \\ \tilde{\varphi}_{i,K}(x) = \bar{\varphi}_{i,K}(x) & \text{on } \partial K, \end{cases} \tag{24}$$

to construct special multiscale base functions $\tilde{\varphi}_{i,K}$ for every K and $1 \le i \le n_K$. Compared with linear functions $\bar{\varphi}_{i,K}$, these $\tilde{\varphi}_{i,K}$ are oscillatory functions in every element but share the same trace with $\bar{\varphi}_{i,K}$. When Eq. 24 are numerically solved with mesh $\mathscr{T}_h$ in every macroscopic element (see microscopic mesh in Fig. 4), an approximate solution of the multiscale problem (Eq. 1) is found in the special multiscale solution space $\tilde{U}_H = \{\tilde{u}|_K \in \tilde{U}_K, K \in \mathscr{T}_H\} \subset H_0^1(\Omega)$ where $\tilde{U}_K = \text{span}\{\tilde{\varphi}_{i,K}, i = 1,\ldots,n_K\}$. Though only the approximate values of the multiscale solution at every macroscopic node are obtained, $\sum_{i=1}^{n_K} \tilde{\varphi}_{i,K}\bar{u}_{i,K}$ can give an approximation of multiscale solution due to the fact that the base functions $\tilde{\varphi}_{i,K}$ have captured the microscopic information.

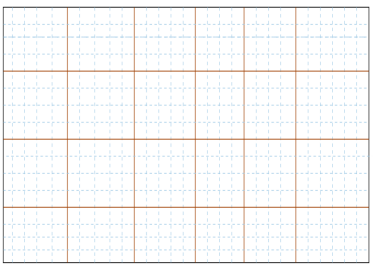


Figure 4: Macroscopic mesh and microscopic meshes in MsFEM [Hou and Efendiev (2009)]

The function $\tilde{u} \in \tilde{U}_H$ constructed by multiscale base functions $\tilde{\varphi}_{i,K}$ is continuous across the boundaries of the macroscopic elements. However, though the multi-

scale base functions are oscillatory within every element, they are linear along the inner boundaries of elements as same as $\bar{\varphi}_{i,K}$, and then the resulting function $\tilde{u}$ is linear along the inner boundaries of the elements. To obtain oscillatory multiscale solution in the entire domain, another oscillatory boundary condition has also been proposed, cf. [Hou and Wu (1997)]. In addition, an over–sampling technique was proposed to reduce the effect of boundary conditions for the multiscale base functions on the accuracy of the multiscale solution. More relative work refers to [Hou, Wu, and Zhang (2004); Efendiev, Chu, Ginting, and Hou (2008)].

3 Comparison of the four multiscale methods

3.1 Two frameworks

Since the characteristic length of microscale is much smaller than the characteristic length of macroscale, it is impossible to find high–precision numerical solutions for multiscale problems by the fine FEM in practice. Therefore approximations of the multiscale solutions should be considered by designing effective multiscale methods. The four methods recalled in the above section can be divided broadly into two categories. AHM and HMM belong to the up–down framework, while VMS method and MsFEM belong to the uncoupling framework.

As can be seen in Fig. 5, based on the homogenization theory, the up–down framework focuses first on the macroscopic behavior of the multiscale problem. The homogenized problem is considered to be solved with macroscopic mesh. Since the effective coefficients are unknown a priori, one needs to solve some auxiliary problems in some local domains at the microscale (e.g., around quadrature points in macroscopic elements), and then to compute effective information for solution of homogenized problem. Since the microscopic auxiliary problems in different local domains are independent problems, these problems can be solved effectively by parallel computing. The number of auxiliary problems depends on the special feature of the multiscale problem. If the feature of the multiscale coefficients is uniform in the global domain, the effective coefficients are uniform and the auxiliary problems could be solved in any local domain. Otherwise, the effective coefficients depend on the macroscopic positions and one needs to solve the auxiliary problems in concerned local domains. The above procedure reflects microscopic information up to the macroscale. In some cases the macroscopic information should be transferred to the microscale, because more accurate solution reflecting the microscopic information is more useful in many applications. With the help of macroscopic solution in the global domain, one solves some problems in some local domains where the multiscale solution is required, and then the multiscale solution is computed. These local domains are independent of those ones for obtaining effective

information.

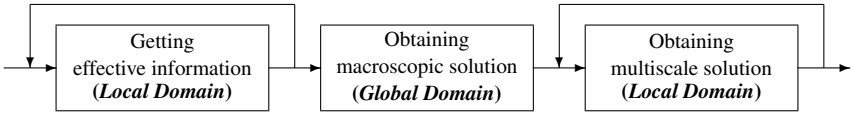


Figure 5: Flowchart of the up–down framework

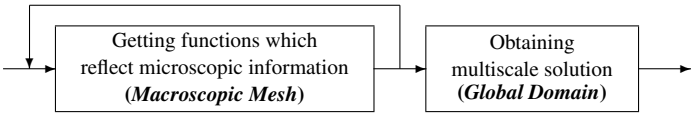


Figure 6: Flowchart of the uncoupling framework

Compared to the up–down framework, the uncoupling framework obtains the multiscale solution directly in the global domain, see Fig. 6. Since the full fine scale solution of the multiscale problem cannot be solved directly in the global domain, the uncoupling framework first partitions the global domain into some small subdomains (macroscopic mesh), and tries to resolve some details of the multiscale solution in every macroscopic element independently. Special substructural functions reflecting some microscopic information are constructed in every macroscopic element and computed by solving some problems in these elements. These substructural problems can be solved in parallel because the substructural functions are independent. With the help of the substructural functions, the multiscale solution is then resolved with macroscopic mesh by solving a small discrete system. In the uncoupling framework, two levels of meshes are used, and the scale of the substructural problems depends on the macroscopic mesh size. Since the substructural functions are connected with macroscopic nodes, the multiscale solution in every fine element is derived by the macroscopic nodal solutions and the substructural functions.

Generally speaking, the two frameworks solve the multiscale problem in different ways, and provide different amount of multiscale solution. The up–down framework solves the multiscale problem in two steps. The macroscopic solution is computed first and the second step is obtaining the multiscale solution in some local domains. The multiscale solution in the global domain can also be computed when necessary. If only the macroscopic solution is required, the second step is omitted. In contrast, the uncoupling framework computes the multiscale solution directly in the global domain. In addition, the computational cost of these two frameworks

is different. We compare the total degrees of freedom (DOFs) of discrete systems in these two frameworks with that of fine FEM for convenience. This is reasonable when the cost of numerical method for solving linear systems in the multiscale methods is proportional to the DOFs of the linear systems [Ming and Yue (2006)]. The DOFs of the up–down framework depend on the user requirement for the multiscale solution. The DOFs are fewer than that of fine FEM when only the macroscopic solution and the multiscale solution in some small local domains are required. The DOFs will be comparable to that of fine FEM when the multiscale solution in the global domain is required. In contrast, the DOFs of the uncoupling framework are comparable to that of fine FEM because the multiscale solution in the global domain is obtained directly. Furthermore, in the up–down framework, the separation of scales is assumed, because the size of local domains for obtaining effective information should be much smaller than the macroscopic mesh size for the purpose of computing effective information more efficiently. However, the scale separation assumption is not necessary in the uncoupling framework.

Though the cost of multiscale methods in the two frameworks are comparable to that of fine FEM when the detailed multiscale solution is required, multiscale methods are still more effective than fine FEM. This is because the multiscale methods partition the multiscale problem into some independent small problems, so both the amount of computer memory and CPU time will be dramatically reduced, and it is very suitable to implement parallel computing. In contrast, fine FEM is always powerless for solving multiscale problems because of the limit of computer memory and CPU time. However we should note that the price by using multiscale methods is the accuracy of the multiscale solution.

3.2 Comparison of AHM and HMM in resolving multiscale solution

One main difference of AHM and HMM in the up–down framework is the manner they resolve multiscale solution in the local domains, as displayed in Fig. 7 and Fig. 8. AHM solves some auxiliary problems in a local RVE to produce auxiliary functions (e.g. χ in Eq. 8) which reflect microscopic information. The multiscale solution is then computed by the combination of the macroscopic solution and the microscopic auxiliary functions based on the asymptotic expansion of the multiscale solution. The number of the auxiliary problems depends on the spatial dimension. In contrast, to resolve multiscale solution, HMM solves only one problem in a local domain which takes the macroscopic solution on the boundary of the local domain as Dirichlet boundary condition. The source term f of the multiscale problem is considered in this local problem because the macroscopic solution acts as a boundary condition, while in AHM the effect of source term is represented in the macroscopic solution in the local RVE and the auxiliary problems do not

consider it. For resolving multiscale solution in a local domain, HMM is more economic than AHM because only one problem should be solved. However, according to the asymptotic expansion form, AHM can provide higher–order multiscale solution when more work has been done.

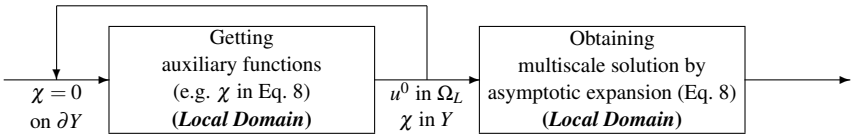


Figure 7: Flowchart of AHM in resolving multiscale solution

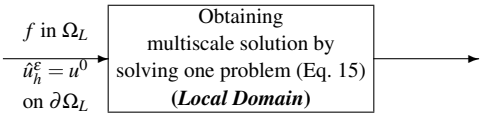


Figure 8: Flowchart of HMM in resolving multiscale solution

Though the manner to resolve multiscale solution in AHM and HMM is different, these two methods provide similar multiscale solutions in numerical computation. For simplicity, we illustrate the similarity for one–dimensional problem first.

As mentioned in section 2, to get an approximate solution of homogenized problem, HMM computes effective stiffness matrix while AHM computes effective coefficients. In fact the same strategy to obtain effective coefficients as in AHM was also adopted in HMM, see [E, Engquist, Li, Ren, and Vanden-Eijnden (2007); Yue and E (2007)]. For the convenience of displaying the similarity of multiscale solutions in AHM and HMM, it is assumed that the same macroscopic solution has been obtained by these two methods.

Let u^0 be the numerical solution of the homogenized problem. To resolve multiscale solution, HMM needs to solve

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(a^\varepsilon(x)\frac{\mathrm{d}\hat{u}^\varepsilon(x)}{\mathrm{d}x}\right) = f(x) & \text{in } \Omega_L = (x^1, x^2), \\ \hat{u}^\varepsilon(x) = u^0(x) & \text{on } \partial\Omega_L, \end{cases} \tag{25}$$

where Ω_L is a local domain in Ω (see Fig. 9).

In AHM, we consider the first–order multiscale solution

$$u_1^\varepsilon(x) = u^0(x) + \varepsilon\chi(y)\frac{\mathrm{d}u^0(x)}{\mathrm{d}x}, \tag{26}$$

where χ is the first–order auxiliary function and is the solution of the following problem

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}y}\left(a(y)\frac{\mathrm{d}\chi(y)}{\mathrm{d}y}\right) = \frac{\mathrm{d}a(y)}{\mathrm{d}y} & \text{in } Y, \\ \chi(y) = 0 & \text{on } \partial Y, \end{cases} \tag{27}$$

where $Y = (0,1)$ is the unit cell.

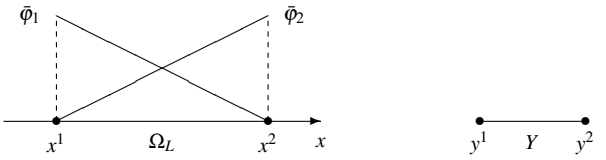


Figure 9: A diagram of Ω_L and Y in $\mathbb{R}$

To correlate the multiscale solutions of AHM and HMM, it is assumed that Ω_L is the ε–cell with size ε and Y is the corresponding normalized cell of Ω_L, cf. [Han, Cui, and Yu (2010)]. We rewritten $\chi(y)$ in the x coordinates as $\chi^\varepsilon(x)$ where $x = \varepsilon y + x^1$. Obviously χ^ε is the solution of the following problem

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(a^\varepsilon(x)\frac{\mathrm{d}\chi^\varepsilon(x)}{\mathrm{d}x}\right) = \frac{1}{\varepsilon}\frac{\mathrm{d}a^\varepsilon(x)}{\mathrm{d}x} & \text{in } \Omega_L, \\ \chi^\varepsilon(x) = 0 & \text{on } \partial\Omega_L. \end{cases} \tag{28}$$

Let

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(a^\varepsilon(x)\frac{\mathrm{d}\tilde{\varphi}_i(x)}{\mathrm{d}x}\right) = 0 & \text{in } \Omega_L, i = 1,2, \\ \tilde{\varphi}_i(x) = \bar{\varphi}_i(x) & \text{on } \partial\Omega_L, \end{cases} \tag{29}$$

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(a^\varepsilon(x)\frac{\mathrm{d}\check{u}(x)}{\mathrm{d}x}\right) = f(x) & \text{in } \Omega_L, \\ \check{u}(x) = 0 & \text{on } \partial\Omega_L. \end{cases} \tag{30}$$

Based on the superposition principle, the multiscale solution $\hat{u}^\varepsilon$ in HMM satisfies

$$\hat{u}^\varepsilon(x) = \tilde{\varphi}_1(x)u_1^0 + \tilde{\varphi}_2(x)u_2^0 + \check{u}(x), \tag{31}$$

where u_1^0 and u_2^0 are the values of $u^0(x)$ on the left and right points of Ω_L. Note that $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the multiscale base functions as same as in MsFEM, while $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are the macroscopic linear base functions (Fig. 9).

Let $\chi_i^\varepsilon = \tilde{\varphi}_i - \bar{\varphi}_i, i = 1,2$. According to Eq. 29, χ_1^ε and χ_2^ε solve the following problems respectively

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(a^\varepsilon(x)\frac{\mathrm{d}\chi_1^\varepsilon(x)}{\mathrm{d}x}\right) = -\frac{1}{\varepsilon}\frac{\mathrm{d}a^\varepsilon(x)}{\mathrm{d}x} & \text{in } \Omega_L, \\ \chi_1^\varepsilon(x) = 0 & \text{on } \partial\Omega_L, \end{cases} \tag{32}$$

and

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(a^\varepsilon(x)\frac{\mathrm{d}\chi_2^\varepsilon(x)}{\mathrm{d}x}\right) = \frac{1}{\varepsilon}\frac{\mathrm{d}a^\varepsilon(x)}{\mathrm{d}x} & \text{in } \Omega_L, \\ \chi_2^\varepsilon(x) = 0 & \text{on } \partial\Omega_L. \end{cases} \tag{33}$$

Obviously $\chi^\varepsilon = \chi_2^\varepsilon = -\chi_1^\varepsilon$.

Then we have

$$\begin{aligned} \hat{u}^\varepsilon &= \tilde{\varphi}_1 u_1^0 + \tilde{\varphi}_2 u_2^0 + \breve{u} \\ &= (\bar{\varphi}_1 + \chi_1^\varepsilon)u_1^0 + (\bar{\varphi}_2 + \chi_2^\varepsilon)u_2^0 + \breve{u} \\ &= (\bar{\varphi}_1 u_1^0 + \bar{\varphi}_2 u_2^0) + (\chi_1^\varepsilon u_1^0 + \chi_2^\varepsilon u_2^0) + \breve{u} \\ &= u^0 + \chi_2^\varepsilon(u_2^0 - u_1^0) + \breve{u} \\ &= u^0 + \varepsilon\chi\frac{\mathrm{d}u^0}{\mathrm{d}x} + \breve{u}. \end{aligned} \tag{34}$$

In general, the macroscopic mesh size is larger than the size of local domain Ω_L. Therefore it is assumed that in AHM the macroscopic solution u^0 in Ω_L is approximated by linear interpolation, and its derivative is approximated by the linear combination of the derivative of linear interpolation functions with coefficients u_i^0.

As in Eq. 34, the multiscale solutions in HMM and AHM are equivalent in computation when ignoring the additional part $\breve{u}$. Obviously it holds when there is no source term in Eq. 25. It has been shown that for multiscale problems which do not contain microscopic components in the source term, it is not necessary to consider the source term in the microscopic problems [Hou and Efendiev (2009); Arbogast and Boyd (2006)]. We assume that the source term is a macroscopic function. In this case HMM and AHM provide equivalent multiscale solutions.

Similar result can be extended to higher–dimensional space and we illustrate it in $\mathbb{R}^2$.

Let $\bar{\varphi}_i, i = 1,\ldots,4$ be standard bilinear base functions in a square domain Ω_L (see Fig. 10). We assume that ε is the length of Ω_L. Obviously we have

$$\bar{\varphi}_1 + \bar{\varphi}_4 = -(x_1 - x_1^2)/\varepsilon, \tag{35}$$

$$\bar{\varphi}_2 + \bar{\varphi}_3 = (x_1 - x_1^1)/\varepsilon, \tag{36}$$

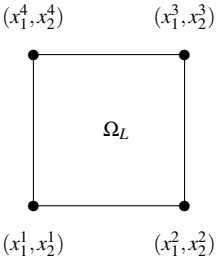


Figure 10: A diagram of Ω_L in $\mathbb{R}^2$

$$\bar{\varphi}_1 + \bar{\varphi}_2 = -(x_2 - x_2^3)/\varepsilon, \tag{37}$$

$$\bar{\varphi}_3 + \bar{\varphi}_4 = (x_2 - x_2^2)/\varepsilon. \tag{38}$$

Similar to Eq. 31, we assume $\hat{u}^\varepsilon = \sum_{i=1}^4 \tilde{\varphi}_i u_i^0 + \check{u}$ in HMM where $\tilde{\varphi}_i$ are multiscale base functions satisfying

$$\begin{cases} -\nabla \cdot (a^\varepsilon(x) \nabla \tilde{\varphi}_i(x)) = 0 & \text{in } \Omega_L, i = 1, \ldots, 4, \\ \tilde{\varphi}_i(x) = \bar{\varphi}_i(x) & \text{on } \partial\Omega_L. \end{cases} \tag{39}$$

And u_i^0 are the values of macroscopic solution at four vertices in Ω_L. $\check{u}$ solves the problem

$$\begin{cases} -\nabla \cdot (a^\varepsilon(x) \nabla \check{u}(x)) = f(x) & \text{in } \Omega_L, \\ \check{u}(x) = 0 & \text{on } \partial\Omega_L. \end{cases} \tag{40}$$

Let

$$\chi^\varepsilon_{x_1,1} = (\tilde{\varphi}_1 + \tilde{\varphi}_4) - (\bar{\varphi}_1 + \bar{\varphi}_4), \tag{41}$$

$$\chi^\varepsilon_{x_1,2} = (\tilde{\varphi}_2 + \tilde{\varphi}_3) - (\bar{\varphi}_2 + \bar{\varphi}_3), \tag{42}$$

$$\chi^\varepsilon_{x_2,1} = (\tilde{\varphi}_1 + \tilde{\varphi}_2) - (\bar{\varphi}_1 + \bar{\varphi}_2), \tag{43}$$

$$\chi^\varepsilon_{x_2,2} = (\tilde{\varphi}_3 + \tilde{\varphi}_4) - (\bar{\varphi}_3 + \bar{\varphi}_4). \tag{44}$$

It is easy to see that

$$\begin{cases} -\nabla \cdot (a^\varepsilon(x) \nabla \chi^\varepsilon_{x_i,1}(x)) = -\frac{1}{\varepsilon} \nabla \cdot (a^\varepsilon(x) e_i) & \text{in } \Omega_L, i = 1, 2, \\ \chi^\varepsilon_{x_i,1}(x) = 0 & \text{on } \partial\Omega_L, \end{cases} \tag{45}$$

$$\begin{cases} -\nabla\cdot(a^\varepsilon(x)\nabla\chi^\varepsilon_{x_i,2}(x)) = \frac{1}{\varepsilon}\nabla\cdot(a^\varepsilon(x)e_i) & \text{in } \Omega_L, i=1,2, \\ \chi^\varepsilon_{x_i,2}(x) = 0 & \text{on } \partial\Omega_L, \end{cases} \tag{46}$$

where e_i are the canonical bases in the x_i direction. In fact these problems are the auxiliary problems in AHM. And obviously we have $\chi^\varepsilon_{x_i,1} = -\chi^\varepsilon_{x_i,2}, i=1,2$. Furthermore, let (y^1,y^2) be the local coordinates in the corresponding unit cell Y of Ω_L, and $\chi_{y_i}(y^1,y^2) = \chi^\varepsilon_{x_i,2}(x^1,x^2)$. The relationship similar to Eq. 34 could be deduced

$$\begin{aligned} \hat{u}^\varepsilon &= \sum_{i=1}^4 \tilde{\varphi}_i u_i^0 + \breve{u} \\ &= \sum_{i=1}^4 \bar{\varphi}_i u_i^0 + \sum_{i=1}^4 (\tilde{\varphi}_i - \bar{\varphi}_i) u_i^0 + \breve{u} \\ &= u^0 + [(\tilde{\varphi}_2+\tilde{\varphi}_3) - (\bar{\varphi}_2+\bar{\varphi}_3)]\frac{u_2^0+u_3^0}{2} + [(\tilde{\varphi}_1+\tilde{\varphi}_4) - (\bar{\varphi}_1+\bar{\varphi}_4)]\frac{u_1^0+u_4^0}{2} \\ &\quad + [(\tilde{\varphi}_3+\tilde{\varphi}_4) - (\bar{\varphi}_3+\bar{\varphi}_4)]\frac{u_3^0+u_4^0}{2} + [(\tilde{\varphi}_1+\tilde{\varphi}_2) - (\bar{\varphi}_1+\bar{\varphi}_2)]\frac{u_1^0+u_2^0}{2} + \breve{u} \\ &= u^0 + \chi^\varepsilon_{x_1,2}\left(\frac{u_2^0+u_3^0}{2} - \frac{u_1^0+u_4^0}{2}\right) + \chi^\varepsilon_{x_2,2}\left(\frac{u_3^0+u_4^0}{2} - \frac{u_1^0+u_2^0}{2}\right) + \breve{u} \\ &= u^0 + \varepsilon\chi_{y_1}\frac{\partial u^0}{\partial x_1} + \varepsilon\chi_{y_2}\frac{\partial u^0}{\partial x_2} + \breve{u}. \end{aligned} \tag{47}$$

The last term $\breve{u}$ can be eliminated when the source term f is a macroscopic function. In the numerical computation, when the macroscopic solution in Ω_L is represented by bilinear interpolation and its derivatives are represented by constants, the multiscale solutions provided by HMM and AHM are equivalent.

3.3 Computational cost of different methods

Compared with fine FEM, the multiscale methods in the uncoupling framework do comparable work and resolve all the details of the multiscale solution. In contrast, we have the choice whether the multiscale solution is resolved or not by the multiscale methods in the up–down framework. The computational cost is comparable to that of fine FEM when it is necessary to obtain the multiscale solution in the global domain. However the cost will be much less when we merely concern the macroscopic solution. We compare the cost of the four multiscale methods in these two frameworks in this subsection.

As in subsection 3.1, the DOFs of the multiscale methods are compared for convenience. Tab. 1 lists DOFs of the four multiscale methods for a multiscale problem with scale separation assumption. Let H and h be the mesh sizes at the macroscale

Table 1: The cost of different methods

Method	Macroscopic solution	Full multiscale solution
Fine FEM	–	h^{-d}
MsFEM	–	$H^{-d}+(n-1)h^{-d}$
VMS method	–	$H^{-d}+nh^{-d}$
HMM	$H^{-d}+mdH^{-d}(\delta/h)^d$	$H^{-d}+mdH^{-d}(\delta/h)^d+h^{-d}$
AHM	$H^{-d}+mdH^{-d}(\delta/h)^d$	$H^{-d}+dh^{-d}$

Note: d is the spatial dimension, n is the number of nodes in every macroscopic element, and m is the number of quadrature points in every macroscopic element.

and microscale respectively. It is assumed that $h \ll \delta \ll H \ll 1$ where δ is the RVE size in AHM and HMM. In addition, the same number of nodes and the same number of quadrature points in every macroscopic element are assumed.

Since the multiscale base functions in each element $K \in \mathscr{T}_H$ in MsFEM satisfy the formula $\sum_{i=1}^{n} \tilde{\varphi}_{i,K} = 1$, we can solve $n-1$ substructural problems (Eq. 24) and compute the last multiscale base function by the formula. However this is not the case for VMS method. For the convenience of comparing AHM and HMM, the same RVE size δ is assumed in AHM and HMM, and the strategy to obtain effective coefficients by solving auxiliary problems in local domains around each quadrature point is adopted in AHM. As presented in Tab. 1, the cost of AHM is comparable to that of HMM for solving macroscopic solution. However, different manner to resolve multiscale solutions in HMM and AHM leads to different computational cost, and HMM is more economic than AHM. Considering the auxiliary functions could be stored and reused to construct multiscale solution, the cost to solve effective coefficients is included in the cost to resolve multiscale solution in AHM. However this is not the case for HMM.

4 Numerical experiment

To compare the up–down framework and the uncoupling framework further, we consider a numerical example, viz., the thermal conductivity of a two–phase functionally graded material (FGM), which is generated by random morphology description functions with computer introduced in [Vel and Goupee (2010)]. The

structure of the FGM is shown in Fig. 11. The FGM has spatially varying volume fractions in x_1 direction and is random heterogeneous at the microscale. In addition, though the FGM is not uniform at the macroscale, it is stationary random at the microscale and the scale separation assumption is satisfied.

We solve the following multiscale problem without source term

$$\begin{cases} -\nabla\cdot(a^\varepsilon(x)\nabla u^\varepsilon(x)) = 0 & \text{in } \Omega = (0,10)\times(0,1), \\ u^\varepsilon(x) = 0 & \text{on } \partial\Omega_1 = \{x|\ x_1 = 0\}, \\ u^\varepsilon(x) = 10 & \text{on } \partial\Omega_2 = \{x|\ x_1 = 10\}, \\ \nabla u^\varepsilon(x)\cdot n = 0 & \text{on } \partial\Omega\backslash(\partial\Omega_1\bigcup\partial\Omega_2), \end{cases} \tag{48}$$

where n is the outward unit normal vector and

$$a^\varepsilon(x) = \begin{cases} 1, & x\in\Omega_{\text{blue}}, \\ 10, & x\in\Omega_{\text{red}}. \end{cases} \tag{49}$$

To compare the two different frameworks, the multiscale problem (Eq. 48) is solved by AHM and MsFEM. We set the mesh sizes $H = 1$ and $h = 1/128$ in these two methods, and the RVE size in AHM equals to the macroscopic mesh size. Fig. 12 presents the macroscopic mesh for solving macroscopic solution of the multiscale problem.

Fig. 13 shows the multiscale solutions (temperature distribution) computed by fine FEM and two multiscale methods. We could see the difference between the fine FEM solution and other solutions computed by the two multiscale methods. In fact all these methods provide approximations of the true multiscale solution, and fine FEM provides better approximation than AHM and MsFEM. Furthermore, it shows similarities between the multiscale solution of AHM and that of MsFEM, as well as the macroscopic solutions presented in Fig. 14.

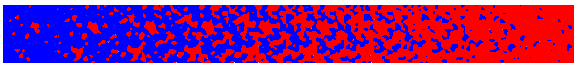
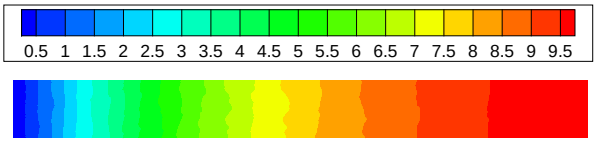


Figure 11: Geometry of a two-phase FGM

The accuracy of these two multiscale methods are further compared by computing the relative errors of temperature solutions and their gradients in the x_1 direction, i.e.,

$$\frac{\|u^{\text{M}}-u^{\text{F}}\|_{L^\infty}}{\|u^{\text{F}}\|_{L^\infty}}, \frac{\|u^{\text{M}}-u^{\text{F}}\|_{L^2}}{\|u^{\text{F}}\|_{L^2}}, \frac{\|u^{\text{M}}-u^{\text{F}}\|_{H^1}}{\|u^{\text{F}}\|_{H^1}}, \tag{50}$$

Figure 12: The macroscopic mesh

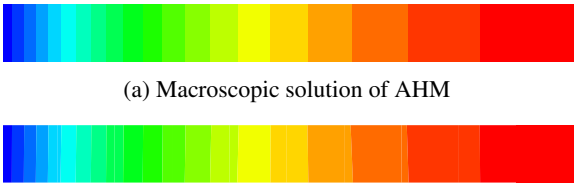


(a) Multiscale solution of fine FEM

(b) Multiscale solution of AHM

(c) Multiscale solution of MsFEM

Figure 13: Multiscale solutions u_{multi} computed by different methods



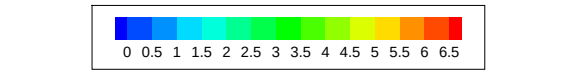
(a) Macroscopic solution of AHM

(b) Macroscopic solution of MsFEM

Figure 14: Macroscopic solutions u_{macro} computed by different methods

Table 2: Comparison of relative errors for different multiscale methods

	AHM		MsFEM	
	u_{macro}	u_{multi}	u_{macro}	u_{multi}
L^∞ error (%)	1.151	1.134	1.176	1.005
L^2 error (%)	0.872	0.864	0.975	0.891
H^1 error (%)	8.938	3.271	8.954	3.271



(a) Multiscale solution of fine FEM

(b) Multiscale solution of AHM

(c) Multiscale solution of MsFEM

Figure 15: Multiscale solutions u_{multi,x_1} computed by different methods

(a) Macroscopic solution of AHM

(b) Macroscopic solution of MsFEM

Figure 16: Macroscopic solutions u_{macro,x_1} computed by different methods

$$\frac{\|u_{x_1}^{\mathrm{M}} - u_{x_1}^{\mathrm{F}}\|_{L^\infty}}{\|u_{x_1}^{\mathrm{F}}\|_{L^\infty}}, \frac{\|u_{x_1}^{\mathrm{M}} - u_{x_1}^{\mathrm{F}}\|_{L^2}}{\|u_{x_1}^{\mathrm{F}}\|_{L^2}}, \frac{\|u_{x_1}^{\mathrm{M}} - u_{x_1}^{\mathrm{F}}\|_{H^1}}{\|u_{x_1}^{\mathrm{F}}\|_{H^1}}. \tag{51}$$

Here u^{M} denotes temperature solution computed by one of the multiscale methods, and u^{F} denotes temperature solution of fine FEM. The gradients of temperature solutions in the x_2 direction are not discussed here because of their unobvious differences. The relative errors of solutions computed by AHM and MsFEM are shown in Tab. 2. As can be seen, the two methods provide both macroscopic solutions and multiscale solutions with comparable accuracy. Moreover, multiscale solutions are better approximations of fine FEM solution than macroscopic solutions for both methods. For this special problem, the differences of macroscopic solutions and multiscale solutions in L^2 norm and L^∞ norm are not large for each multiscale method. But large disparities are shown with H^1 norm. This is caused by the gradients of temperature solutions. Multiscale gradient solutions in the x_1

direction of fine FEM and two multiscale methods are presented in Fig. 15, and macroscopic gradient solutions of these multiscale methods are shown in Fig. 16. Obviously the multiscale gradient solutions given by the multiscale methods are very similar to that of fine FEM, while the macroscopic gradient solutions of the multiscale methods are not. Therefore multiscale solutions are more important in computing accurate gradient solutions. Since the computing resources are limited, only a simple numerical example is considered. However for general multiscale problems, multiscale solutions approximate solutions of the multiscale problems much better than macroscopic solutions because the latter do not contain microscopic information.

5 Conclusions

Differences and similarities of four multiscale methods for elliptic multiscale problems are shown. These methods are divided into two categories. The up–down framework includes AHM and HMM. Methods in this framework consider the macroscopic behavior of the multiscale problem with much less cost than that of fine FEM. Their cost will be comparable to that of fine FEM when all the details of the multiscale solution should be resolved. In contrast, methods in the uncoupling framework resolve the multiscale solution directly and are full fine scale solvers as fine FEM actually. VMS method and MsFEM belong to this framework. In addition, it is found that though the cost to resolve multiscale solution in AHM and HMM is different, these two methods provide similar multiscale solutions in numerical computation. Considering the feature of different multiscale methods, an effective multiscale method should be used to solve the multiscale problem according to the user requirement on the multiscale solution.

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