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Local Defect Correction for Boundary Integral Equation Methods

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Abstract: The aim in this paper is to develop a new local defect correction approach to gridding for problems with localised regions of high activity in the boundary element method. The technique of local defect correction has been studied for other methods as finite difference methods and finite volume methods. The initial attempts to developing such a technique by the authors for the boundary element method was based on block decomposition and manipulation of the coefficient matrix and right hand side of the system of equations in three dimension. It ignored the inherent global nature of the boundary integral equation, that is, each node of the grid contributes to all the others in the grid through integration. In this paper we present a better approach that takes this into account. We use a new integral approach to defect correction and develop the technique for the boundary element method. The technique offers an iterative way for obtaining the solution on an equivalent composite grid. It uses two grids: a global uniform coarse grid covering the whole boundary and a local fine grid covering the local active boundary. The solution of the local problem on the local fine grid is used to estimate the defect on the fine grid. The effect of the defect onto the right hand side is then considered. We demonstrate the technique's strength using an example and show that it offers a cheaper alternative to either solving on a global uniform grid or directly on a composite grid.

Keywords: integral equations, boundary elements, local error, local defect correction, local activity, gridding.

1 Introduction

Often boundary value problems have small localised regions of high activity where the solution varies very rapidly compared to the rest of the domain. This behaviour

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maybe either due to boundary conditions or due to an irregular boundary. One therefore has to use relatively fine meshes to capture the high activity. Since the activity is localised, one may also choose to solve on a uniform structured grid. That is, instead of a uniform global grid, the solution is approximated using several uniform grids with different grid sizes that cover different parts of the domain. The size of each grid is chosen in agreement with the activity of the solution in that part of the domain. This refinement strategy is called *local uniform grid refinement* [Ferket and Reusken (1996)]. The solution is approximated on a composite grid which is the union of the various uniform local grids. One way of approximating this composite grid solution that is simple and less complex is by *Local Defect Correction* (LDC).

In LDC, at least one grid, the *global coarse grid*, covers the entire domain. Then a uniform *local fine grid* is used in a small part of the domain containing the high activity. In [Ferket and Reusken (1996); Hackbusch (1984)], LDC has been shown to be a useful way of approximating the composite grid solution in which a global coarse grid solution is improved by a local fine grid solution through a process whereby the right hand side of the global coarse grid problem is *corrected* by the *defect* of a local fine grid approximation. This method has been well explored for other numerical methods such as finite differences and finite volumes, see [Ferket and Reusken (1996); Hackbusch (1984); Anthonissen (2001); Minero, Anthonissen, and Mattheij (2006)]. In this paper we explore potential analogues and develop an LDC strategy for boundary integral equation (BIE) methods, in particular, the boundary element method (BEM). The BEM is now a well established technique for potential problems as it leads to a reduction in the dimensionality of the problem, due to the need to discretise only the domain boundary. Moreover it provides accurate solutions due to the use of fundamental solutions and, in electrostatics, has many numerical advantages over other methods as finite differences methods as discussed in Liang and Subramaniam (1997). However, although reduction in dimensionality is guaranteed, the accuracy depends a lot on the mesh used [Guiggiani (1990); Liapis (1994)]. Although a lot has been done for the finite element methods, BEM's main competitor, comparatively little has been done on adaptive mesh refinement in BEM [Carsten and Stephan (1995, 1996)]. This paper therefore seeks to add to a gradually growing literature on mesh refinement in BEM.

The initial attempts on LDC for BEM were in [Kakuba, Mattheij, and Anthonissen (2006)] and [Kakuba and Mattheij (2007)] where an algebraic approach was suggested and studied. The algorithm in [Kakuba, Mattheij, and Anthonissen (2006)] was based on block decomposition and manipulation of the coefficient matrix and right hand side of the BEM equations in three dimension space. That formulation ignores the inherent nature of the Boundary Integral Equation (BIE) that it is glob-

al, that is, each node of the grid contributes to all the others in the grid through integration. In this paper we consider a new and better approach to the LDC algorithm that is based on the global integration. This approach takes into account the global integral properties of BEM in the computation of the defects and in the formulation of the local problem. Since in BEM we discretise the boundary, we are concerned with problems in which the high activity occurs at the boundary.

One of the most important steps in adaptive refinement is error estimation. One way of estimating the error in collocation BEM is by using higher order interpolation at the elements to estimate the exact solutions and then use an appropriate norm of the difference between the BEM solution and the estimated exact solution to measure the error [Kita and Kamiya (2001)]. Since the focus of this paper is to develop the essential steps of the LDC algorithm in BEM, we use simple examples whose exact solutions are known. We also consider high activity due to boundary conditions. Then we use the infinity norm $||\cdot||_\infty$ of the difference between the exact and the BEM solutions as the measure for the error.

The paper is organised as follows: First, we give a brief introduction to the BIE and its discretisation using BEM in Section 2. In Section 3, we develop an LDC strategy for BEM alongside an example. In Section 4 we test the strategy on a typical example and discuss results. In Section 5 we give the advantages of the algorithm and finish in Section 6 with a summary of the concepts and results presented in the paper.

2 The Boundary Integral Equation and the Boundary Element Method discretisation

The BEM results from a numerical discretisation of a BIE. Consider a domain Ω with boundary $\partial\Omega$ on which we have the following Laplace problem:

$$\begin{cases} \nabla^2 u(r) = 0, & r \in \Omega, \\ u(r) = g(r), & r \in \partial\Omega^1, \\ \dfrac{\partial u}{\partial n}(r) = h(r), & r \in \partial\Omega^2, \end{cases} \tag{1}$$

where $\partial\Omega^1 \cup \partial\Omega^2 = \partial\Omega$ and $\partial\Omega^1 \cap \partial\Omega^2 = \emptyset$ and g and h are given functions, see Fig. 1. If $\partial\Omega^1 \equiv \partial\Omega$ we have a Dirichlet problem and if $\partial\Omega^2 \equiv \partial\Omega$ we have a Neumann problem, otherwise we have a mixed problem. The boundary integral equation gives relations for the potential $u(s)$ at different locations of the domain Ω. These relations have been abundantly derived in literature and are readily available in various books on boundary element methods such as [Paris and Canas

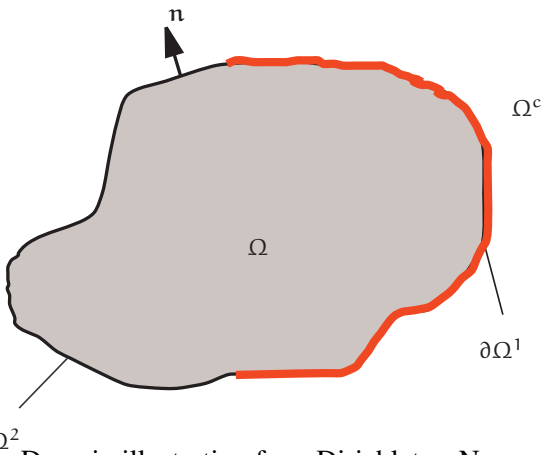


Figure 1: Domain illustration for a Dirichlet or Neumann problem

(1997), Katsikadelis (2002), Pozrikidis (1992)]. The BIE is

$$c(s)u(s) = \int_{\partial\Omega} \left[v(s;r(\chi)) \frac{\partial u}{\partial n}(r(\chi)) - u(r(\chi)) \frac{\partial v}{\partial n}(s;r(\chi)) \right] \mathrm{d}\chi. \tag{2}$$

The coefficient $c(s)$ is given by

$$c(s) := \begin{cases} 1, & s \in \Omega, \\ \dfrac{\alpha(s)}{2\pi}, & s \in \partial\Omega, \\ 0, & s \in \Omega^c, \end{cases} \tag{3}$$

$\alpha(s)$ is the internal angle at s, $\Omega^c = \mathbb{R}^2 \backslash \Omega$ is the complement of Ω and $v(s;r)$ is the fundamental solution of the Laplace equation.

At the boundary, the discretised BIE leads to the linear system of equations

$$\mathrm{Hu} = \mathrm{Gq}, \tag{4}$$

where

$$H_{ij} := c_i \delta_{ij} + \hat{H}_{ij}, \quad (\delta_{ij} \text{ the Kronecker } \delta), \tag{5}$$

$$\hat{H}_{ij} := \int_{\Gamma_j} \frac{\partial v}{\partial n}(r_i;r(\chi))\,\mathrm{d}\chi, \quad \text{and} \quad G_{ij} := \int_{\Gamma_j} v(r_i;r(\chi))\,\mathrm{d}\chi. \tag{6}$$

We also have introduced the vectors

$$\mathrm{u} := (u_1,\ u_2, \ldots, u_N)^T,\ \ \mathrm{q} := (q_1,\ q_2, \ldots, q_N)^T. \tag{7}$$

Use of boundary conditions in (4) leads to the square system

$$\mathrm{Ax} = \mathrm{b}. \tag{8}$$

The solution of the system (8) gives a BEM approximation of the unknowns in x in the grid nodes r_i. We denote by x^L a BEM approximation on a grid of size L. Thus u_j^L (or q_j^L) is a BEM approximation of u_j (or q_j) using a grid of size L. Solving (8) gives the unknown boundary quantities of u and q. Therefore we now have all the boundary quantities. The solution u_i at any point $r_i \in \Omega$ can then be computed using

$$u_i = \sum_{j=1}^{N} G_{ij} q_j - \sum_{j=1}^{N} \hat{H}_{ij} u_j. \tag{9}$$

3 Local Defect Correction

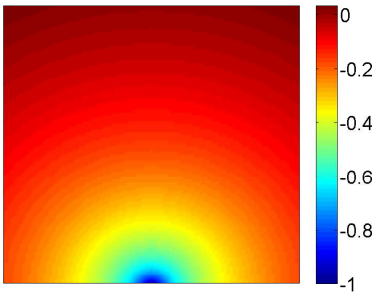
Consider the Neumann problem (10) whose domain is a unit square in two dimension. That is,

$$\begin{cases} \nabla^2 u(r) = 0, & r \in \Omega := [0,\ 1] \times [0,\ 1], \\ \\ q(r) = h(r), & r \in \Gamma, \end{cases} \tag{10}$$

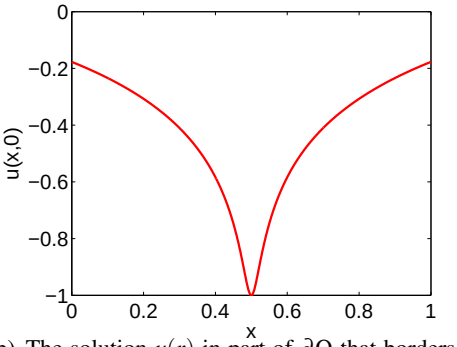
where

$$h(r) = \frac{(r - r_s) \cdot n(r)}{||r - r_s||^2},\ r_s = (0.5, -0.02)^T. \tag{11}$$

The Neumann problem (10) results in a Fredholm integral equation of the second kind [Atkinson (1997); Hackbusch (1995); Kress (1989)]. It is singular and hence has no unique solution [Atkinson (1967)]. To ensure a unique solution in the discretised problem, a value of u is prescribed in one of the nodes [Chen and Zhou (1992); Strese (1984)]. In the implementations in this paper, the value of u in the first element node is prescribed. This will also help us compare the numerical solutions directly with the exact solutions for error measurement. The solution in Ω, shown in Fig. 2, has a small area close to the boundary where it changes rapidly. As a result, the solution $u(r)$ in the boundary has a region of high activity in a small part of the boundary, see Fig. 2. Therefore we can identify a small region



(a) Solution in Ω with small region of high activity.



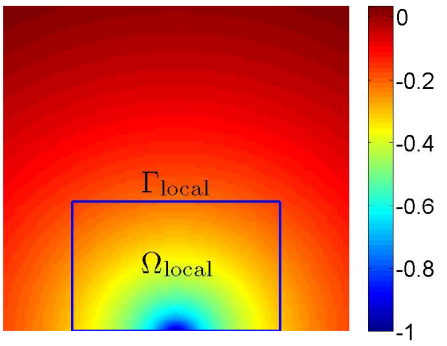
(b) The solution $u(r)$ in part of $\partial\Omega$ that borders the high activity.

Figure 2: Solution in the domain and solution at the boundary $0 \leq x \leq 1, y = 0$, for problem (10).

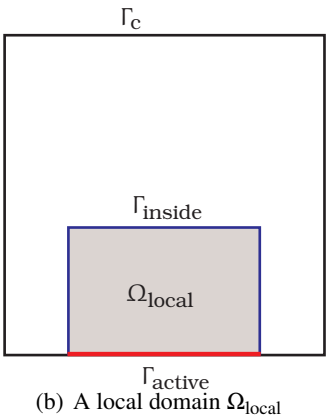
inside Ω which contains the high activity. This region we call the *local domain* and denote it by Ω_{local}, see Fig. 3. Its boundary Γ_{local}, which we will call the *local boundary*, consists of two parts: a part Γ_{active} that is also part of the global boundary and a part Γ_{inside} that is contained in the global domain Ω, Fig. 3(b). We will call the part Γ_{active} the *local active boundary*. For instance, in the problem corresponding to the solutions shown in Fig. 2, the boundary Γ_{active}, may be identified as $\Gamma_{\text{active}} = \{(x,y) : y = 0, x \in [0.2, 0.8]\}$. The part of the global boundary Γ that is outside the active region Γ_{active} will be denoted Γ_{c}, that is, $\Gamma_{\text{c}} := \Gamma \backslash \Gamma_{\text{active}}$.

The interest in BEM is to compute a numerical approximation of $u(r)$ at the boundary as accurately as possible. For such kind of multiscaled variations one is faced with the option of using a global uniform grid with a mesh of relatively small size l in order to capture the high activity. This would result in very large systems which are computationally expensive since BEM matrices are full matrices. Besides, outside the local active boundary Γ_{active}, the variation of the solution is smooth and a relatively coarse grid would suffice. The other option is to use a uniform structured grid designed to capture the different activities. This would be a composite grid with a relatively fine mesh of size l in the local active region and a coarse grid of size L elsewhere.

With LDC we approximate the solution on a composite grid in an iterative way that involves solving a so called *local problem* which is a boundary value problem defined on the local domain. The local problem is solved on a fine mesh whose size is chosen in agreement with the local activity. The solution on the local fine grid



(a) Identify local high activity area Ω_{local}.



(b) A local domain Ω_{local}

Figure 3: An example of a multiscaled solution with localised high activity in 3(a) and, in 3(b), an illustration of a local problem domain. The boundary of Ω_{local} is $\Gamma_{\text{local}} := \Gamma_{\text{active}} \cup \Gamma_{\text{inside}}$.

is combined with the solution on the global coarse grid through *defect correction* to obtain a composite grid solution on Γ. The advantage of this approach is that instead of solving a large composite grid system, two smaller systems; a global coarse grid system and a local fine grid system, are solved independently. For problems with various local activities the local problems can be solved separately in parallel giving a tremendously cheaper way of obtaining a composite grid solution other than solving directly on the composite grid.

Let Γ be the numerical representation of $\partial\Omega$ in BEM. The *global coarse grid* Γ^L is a uniform mesh of N elements each of size L covering the whole of Γ, that is,

$$\Gamma^L := \{\Gamma_1^L, \Gamma_2^L, \ldots, \Gamma_N^L\} \tag{12}$$

where $|\Gamma_j^L| = L$ for all j. The *local fine grid* Γ_{local}^l is a uniform mesh of N_{local} elements each of size l covering Γ_{local}, that is,

$$\Gamma_{\text{local}}^l := \{\Gamma_{\text{local},1}^l, \Gamma_{\text{local},2}^l, \ldots, \Gamma_{\text{local},N_{\text{local}}}^l\} \tag{13}$$

where $|\Gamma_{\text{local},i}^l| = l$ for all $i = 1, 2, \ldots, N_{\text{local}}$. The size of the local fine grid l is chosen in agreement with the activity of the solution in Γ_{active}. Since the solution varies much more rapidly in Γ_{active} than elsewhere, we expect l to be much smaller than L. Part of the grid Γ_{local}^l belongs to Γ_{active} and part belongs to Γ_{inside}. The part that belongs to Γ_{active} is denoted Γ_{active}^l and that that belongs to Γ_{inside} is denoted

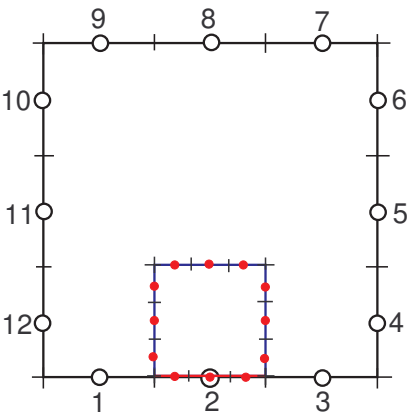


Figure 4: Global coarse and local fine grids. The small dots are the nodes r^l_{local} of the local fine grid Γ^l_{local} and the big circles are the nodes r^L of the global coarse grid Γ^L. Node 2 belongs to $r^L \cap r^l_{\text{active}}$.

Γ^l_{inside}. That is

$$\Gamma^l_{\text{active}} := \{\Gamma^l_{\text{active},1}, \Gamma^l_{\text{active},2}, \ldots, \Gamma^l_{\text{active},N_{\text{active}}}\}, \tag{14a}$$

$$\Gamma^l_{\text{inside}} := \{\Gamma^l_{\text{inside},1}, \Gamma^l_{\text{inside},2}, \ldots, \Gamma^l_{\text{inside},N_{\text{inside}}}\}, \tag{14b}$$

where $\Gamma^l_{\text{active}} \cup \Gamma^l_{\text{inside}} = \Gamma^l_{\text{local}}$ and $N_{\text{active}} + N_{\text{inside}} = N_{\text{local}}$. In constant elements that we discuss here, the collocation nodes are the midpoints of the elements, where the solution is computed. Let us denote the set of nodes of the coarse grid as r^L,

$$r^L := \{r^L_1, r^L_2, \ldots, r^L_N\}. \tag{15}$$

Similarly we denote the set of nodes of the local fine grid as r^l_{local},

$$r^l_{\text{local}} := \{r^l_{\text{local},1}, r^l_{\text{local},2}, \ldots, r_{\text{local},N_{\text{local}}}\}. \tag{16}$$

The sets r^l_{active} and r^l_{inside} are defined similarly. We assume that all the grid nodes of $r^L \cap r^l_{\text{active}}$ belong to r^l_{active}, see Fig. 4. The composite grid nodes $r^{l,L}$ are the union $r^L \cup r^l_{\text{active}}$ of the global coarse grid nodes r^L and the active local fine grid nodes r^l_{active}. The composite grid $\Gamma^{l,L}$ consists of the finest elements that correspond to $r^{l,L}$.

First we discretise the BIE on Γ^L to yield

$$\frac{1}{2}u_i^L + \sum_{j=1}^{N} u_j^L \int_{\Gamma_j^L} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi = \sum_{j=1}^{N} \int_{\Gamma_j^L} q(r(\chi))v(r_i; r(\chi))\,\mathrm{d}\chi, \tag{17}$$

which gives the initial global coarse grid system of equations

$$\mathrm{A}^L \mathrm{u}_0^L = \mathrm{b}^L. \tag{18}$$

Once we have solved (18), the next step is to use the initial global coarse grid solution u_0^L to formulate a local problem on Ω_{local}. This local problem on Ω_{local} satisfies the same operator as in the global problem. The boundary conditions on Γ_{active} are the same as those in the global problem, that is, $q(r) = h(r)$, since $\Gamma_{\text{active}} \subset \Gamma$. On Γ_{inside} we prescribe an artificial boundary condition $\tilde{g}(r)$ defined below. So we have

$$\begin{cases} \nabla^2 u(r) = 0, & r \in \Omega_{\text{local}}, \\ q(r) = h(r), & r \in \Gamma_{\text{active}}, \\ u(r) = \tilde{g}(r), & r \in \Gamma_{\text{inside}}. \end{cases} \tag{19}$$

The function $\tilde{g}(r)$ is piecewise constant on Γ_{inside} and takes on values of $u_{\text{inside}}(r_i)$ where r_i is a node of $\Gamma^l_{\text{inside},i}$, that is,

$$\tilde{g}(r) := u_{\text{inside}}(r_i), \quad r \in \Gamma^l_{\text{inside},i} \subset \Gamma_{\text{inside}}. \tag{20}$$

To compute $u_{\text{inside}}(r_i)$ we use the relation

$$u_{\text{inside}}(r_i) := \sum_{j=1}^{N} \int_{\Gamma_j^L} q(r(\chi))v(r_i; r(\chi))\,\mathrm{d}\chi - \sum_{j=1}^{N} u_j^L \int_{\Gamma_j^L} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi, \quad r_i \in \Gamma_{\text{inside}}. \tag{21}$$

Equation (21) means that we use the solution of the initial global coarse grid problem to obtain artificial Dirichlet boundary conditions at Γ_{inside}. Since at Γ_{active} q is known, the local problem is mixed and the BIE for (19) is, for $r,\ r(\chi) \in \Gamma_{\text{local}}$,

$$\frac{1}{2}u(r) + \int_{\Gamma_{\text{active}}} u(r(\chi)) \frac{\partial v}{\partial n}(r; r(\chi))\,\mathrm{d}\chi + \int_{\Gamma_{\text{inside}}} \tilde{g}(r(\chi)) \frac{\partial v}{\partial n}(r; r(\chi))\,\mathrm{d}\chi = \int_{\Gamma_{\text{active}}} q(r(\chi))v(r; r(\chi))\,\mathrm{d}\chi + \int_{\Gamma_{\text{inside}}} q(r(\chi))v(r; r(\chi))\,\mathrm{d}\chi. \tag{22}$$

Discretising (22) on the local fine grid defined in (13) and (14) we have

$$\frac{1}{2}u^l_{\text{local},i}+\sum_j u^l_{\text{active},j}\int_{\Gamma^l_{\text{active},j}}\frac{\partial v}{\partial n}(r_i;r(\chi))\,\mathrm{d}\chi+\sum_j u^l_{\text{inside},j}\int_{\Gamma^l_{\text{inside},j}}\frac{\partial v}{\partial n}(r_i;r(\chi))\,\mathrm{d}\chi=$$

$$\sum_j q^l_{\text{active},j}\int_{\Gamma^l_{\text{active},j}} v(r_i;r(\chi))\,\mathrm{d}\chi+\sum_j q^l_{\text{inside},j}\int_{\Gamma^l_{\text{inside},j}} v(r;r(\chi))\,\mathrm{d}\chi. \quad (23)$$

In (23) we have two vectors on Γ^l_{local}: $\mathrm{u}^l_{\text{local}}$ and $\mathrm{q}^l_{\text{local}}$, where

$$\mathrm{u}^l_{\text{local}}=\begin{bmatrix}\mathrm{u}^l_{\text{active}}\\ \mathrm{u}^l_{\text{inside}}\end{bmatrix},\quad \mathrm{q}^l_{\text{local}}=\begin{bmatrix}\mathrm{q}^l_{\text{active}}\\ \mathrm{q}^l_{\text{inside}}\end{bmatrix}. \quad (24)$$

The vector $\mathrm{u}^l_{\text{inside}}$ is known through (21) and the vector $\mathrm{q}^l_{\text{active}}$ is known because $q(r(\chi))$ is given on Γ_{active}. So if we repeat (23) for all the local nodes we obtain an algebraic system of N_{local} equations. We rearrange the system in matrix form by putting the known quantities on one side to obtain the initial local problem algebraic system of equations

$$\mathrm{A}^l_{\text{local}}\mathrm{x}^l_{0\text{local}}=\mathrm{b}^l_{0\text{local}} \quad (25)$$

where

$$\mathrm{x}^l_{0\text{local}}=\begin{bmatrix}\mathrm{u}^l_{0\text{active}}\\ \mathrm{q}^l_{0\text{inside}}\end{bmatrix}.$$

The solution $\mathrm{u}^l_{0\text{active}}$ is expected to be more accurate than the coarse grid solution u^L_0 in Γ_{active}. The next step of LDC is to use the local fine grid solution to update the global coarse grid problem. In updating, the right hand side of the global coarse grid problem is corrected by the defect of the local fine grid approximation, we will call this step the *defect correction* step. The two approximations are then used to define a composite grid approximation of $u(r)$. The question now is: how do we compute the defect?

Consider the coarse grid discretisation (17). If we knew the exact continuous function $u(r)$ and hence the exact solution $u_j := u(r_j)$ in the nodes we would use it in (17) to obtain

$$\frac{1}{2}u_i+\sum_{j=1}^{N}u_j\int_{\Gamma^L_j}\frac{\partial v}{\partial n}(r_i;r(\chi))\,\mathrm{d}\chi=\sum_{j=1}^{N}\int_{\Gamma^L_j}q(r(\chi))v(r_i;r(\chi))\,\mathrm{d}\chi+d^L_i. \quad (26)$$

where d_i^L is the local defect for the i-th equation. We also have the exact BIE as

$$\frac{1}{2}u_i + \sum_{j=1}^{N} \int_{\partial\Omega_j} u(r(\chi)) \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi = \sum_{j=1}^{N} \int_{\partial\Omega_j} q(r(\chi)) v(r_i; r(\chi))\,\mathrm{d}\chi. \tag{27}$$

Subtracting (27) from (26) gives

$$\sum_{j=1}^{N} u_j \int_{\Gamma_j^L} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi - \sum_{j=1}^{N} \int_{\partial\Omega_j} u(r(\chi)) \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi = d_i^L. \tag{28}$$

From (28) we define the local defect per element j as

$$d_{ij}^L := u_j \int_{\Gamma_j^L} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi - \int_{\partial\Omega_j} u(r(\chi)) \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi, \tag{29}$$

so that the total defect at r_i is given by

$$d_i^L := \sum_j d_{ij}^L, \quad i = 1, 2, \ldots, N. \tag{30}$$

Therefore if we know the exact continuous function $u(r)$ we can compute the local defect d_i^L, add it to the right hand side of (17) and solve for the exact solution u_j on each element. However $u(r)$ is not known and therefore we cannot compute the defect using (29). All we can do is estimate d_{ij}^L as accurately as possible using the best solution available, which is

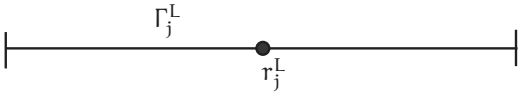
$$u_{\mathrm{best},j}^L = \begin{cases} u_j^L, & \Gamma_j^L \subset \Gamma_c, \\ u_{\mathrm{active},j}^l, & \Gamma_j^L \subset \Gamma_{\mathrm{active}}. \end{cases} \tag{31}$$

So for elements in the high activity region we have the fine grid solution which we can use to estimate the local defect as follows.

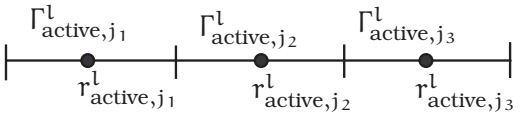
In the case of a square, $\Gamma_j^L \equiv \partial\Omega_j$. Suppose that in the local fine grid $\Gamma_{\mathrm{active}}^l$ a global coarse grid element Γ_j^L is divided into k fine elements $\Gamma_{\mathrm{active},j_k}^l$ such that $\Gamma_j^L = \bigcup_k \Gamma_{\mathrm{active},j_k}^l$, see an illustration in Fig. 5 for $k = 3$. Then the best approximations of the integrals in (29) are

$$u_j \int_{\Gamma_j^L} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi \approx u_{\mathrm{active},j}^l \int_{\Gamma_j^L} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi, \tag{32a}$$

$$\int_{\partial\Omega_j} u(r(\chi)) \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi \approx \sum_k u_{\mathrm{active},j_k}^l \int_{\Gamma_{\mathrm{active},j_k}} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi. \tag{32b}$$



(a) a coarse element Γ_j^L



(b) a refined coarse element Γ_j^L

Figure 5: A coarse element that is refined into three elements in the local fine grid $\Gamma_j^L = \bigcup_{k=1}^{3} \Gamma^l_{\text{active},j_k}$.

Therefore, using the initial fine grid solution, we have the following best approximation of the initial defect per element

$$d^L_{0\,ij} \approx u^l_{0\text{active},j} \int_{\Gamma_j} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi - \sum_k u^l_{0\text{active},j_k} \int_{\Gamma_{\text{active},j_k}} \frac{\partial v}{\partial n}(r_i; r(\chi))\,\mathrm{d}\chi, \tag{33}$$

for $\Gamma_j^L \subset \Gamma_{\text{active}}$ and

$$d^L_{0\,ij} \approx 0$$

for $\Gamma_j^L \subset \Gamma_{\mathrm{c}}$. We can then compute the defect

$$d^L_{0\,i} \approx \sum_j d^L_{0\,ij}, \;\; \Gamma_j^L \subset \Gamma_{\text{active}}, \;\; \text{for all } i = 1,2,\ldots,N. \tag{34}$$

By default, integration in the BIE is global. Each node of the global coarse grid communicates with the active region through integration. So although the activity is local, it affetcs the entire system of equations. The defect $d^L_{0\,i}$ is therefore computed for all nodes of the global coarse grid to generate the local defect vector

$$\mathrm{d}_0^L := (d^L_{0\,1}, d^L_{0\,2}, \ldots, d^L_{0\,N})^T. \tag{35}$$

The next step now is the updating step. The global coarse grid discretisation is updated with the defect of the local fine grid solution. So we have

$$\mathrm{A}^L \mathrm{u}_{\mathrm{I}}^L = \mathrm{b}^L + \mathrm{d}_0^L. \tag{36}$$

Solving (36) gives the updated coarse grid solution u_1^L. At this stage we use the fine grid solution on Γ^l_{active} and the global coarse grid solution to form a composite grid solution $u^{l,L}$ as

$$u^{l,L}_{0,1}(r) = \begin{cases} u^l_{0\,\text{active}}(r), & r \in \Gamma_{\text{active}}, \\ \\ u^L_1(r), & r \in \Gamma_{\text{c}}. \end{cases} \tag{37}$$

The composite grid solution (37) can now be used to compute better boundary conditions on Γ_{inside} and then form and solve the updated fine grid problem

$$\mathrm{A}^l_{\text{local}}\mathrm{x}^l_{1\,\text{local}} = \mathrm{b}^l_{1\,\text{local}}. \tag{38}$$

Then we obtain the updated composite grid solution given by

$$u^{l,L}_{1,1}(r) = \begin{cases} u^l_{1\,\text{active}}(r), & r \in \Gamma_{\text{active}}, \\ \\ u^L_1(r), & r \in \Gamma_{\text{c}}. \end{cases} \tag{39}$$

This step marks the end of one complete cycle of the LDC algorithm. The iteration process is summarised in the following algorithm:

BEM-LDC Algorithm

(i) *Initialisation*

- *Solve the global coarse grid system*

  $\mathrm{A}^L\mathrm{u}^L_0 = \mathrm{b}^L_0$

- *Solve the initial fine grid system*

  $\mathrm{A}^l\mathrm{x}^l_{0\text{local}} = \mathrm{b}^l_0$

- *Compute the initial defect*

  d^L_0

(ii) For $i = 1, 2, \ldots$

- *Solve for the updated coarse grid solution* u^L_i *in*

  $\mathrm{A}^L\mathrm{u}^L_i = \mathrm{b}^L + \mathrm{d}^L_{i-1}$.

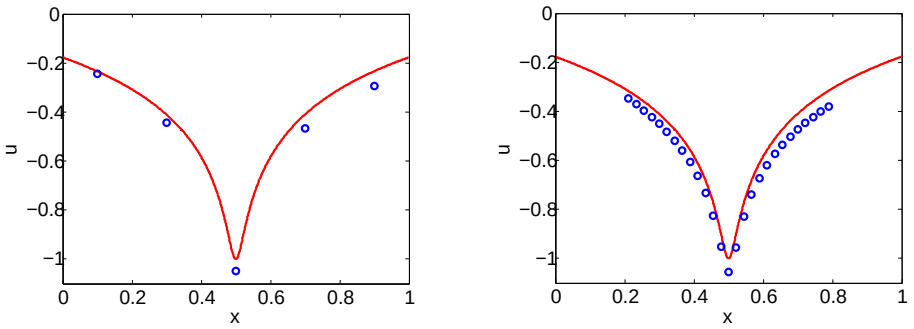
- *Solve for the updated fine grid solution* $\mathrm{x}^l_{i\,\text{active}}$ *in*

  $\mathrm{A}^l_{\text{local}}\mathrm{x}^l_{i\,\text{local}} = \mathrm{b}^l_{i\,\text{local}}$.

- *Form the updated composite grid solution* $u^{l,L}_{i,i}$ *and compute the new defect* d^L_i.

4 Examples and results

The LDC procedure above has been used to solve the problem (10). The results are shown in Fig. 6 and Fig. 7. The figures show the solution on the side $y = 0$ of the unit square. The coarse grid used is of size $L = 0.2$ and the fine grid is of



(a) Initial coarse grid solution u_0^L (b) Initial fine grid solution $u^l_{0\text{active}}$

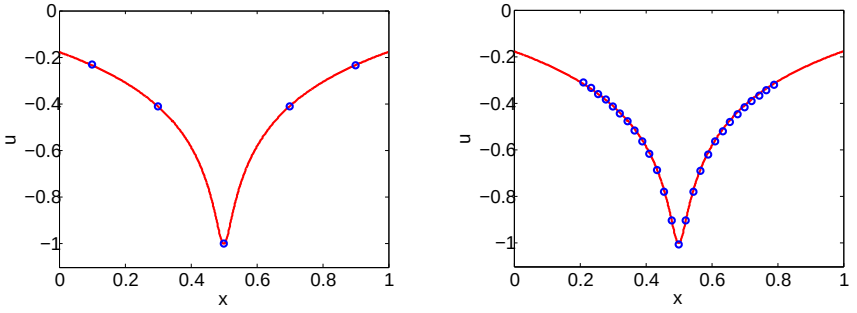
Figure 6: Results of a typical LDC process for a Neumann problem in one iteration. The solid line is the exact solution.

size $l = 0.2/9$. Fig. 6 shows how the initial results compare with the exact solution (the solid line), the initial coarse grid solution in Fig. 6(a) and the initial fine grid solution in Fig. 6(b). Fig. 7 shows the results after the first update, the updated coarse grid solution in Fig. 7(a) is better than the initial one in Fig. 6(a) and the updated local fine grid solution is better than the initial one in Fig. 6(b). Fig. 8 shows how fast the global error decreases. Basically the algorithm has converged already in the first iteration since the error reduction between successive iterations after the first one is much smaller compared to that in the first iteration.

5 Advantages of LDC of the BEM-LDC algorithm

In brief, the LDC iterative process involves the following: solve the global coarse grid problem on Γ, compute u on Γ_{inside}, solve a fine grid problem on Γ_{local} and then update the global coarse grid problem.

Suppose we have p locally active small regions and thus p local problems. Let, for each local problem, M^l be the number of elements on Γ_{local} and M^{in} the number of elements on Γ_{inside}. For instance in the illustration in Fig. 4, $M^l = 3$ and $M^{\text{in}} = 9$. We can increase M^l without necessarily increasing M^{in} since the activity is on



(a) Updated coarse grid solution u_1^L (b) Updated fine grid solution $u^l_{1\,\text{active}}$

Figure 7: Results of a typical LDC process for a Neumann problem in one iteration. The solid line is the exact solution.

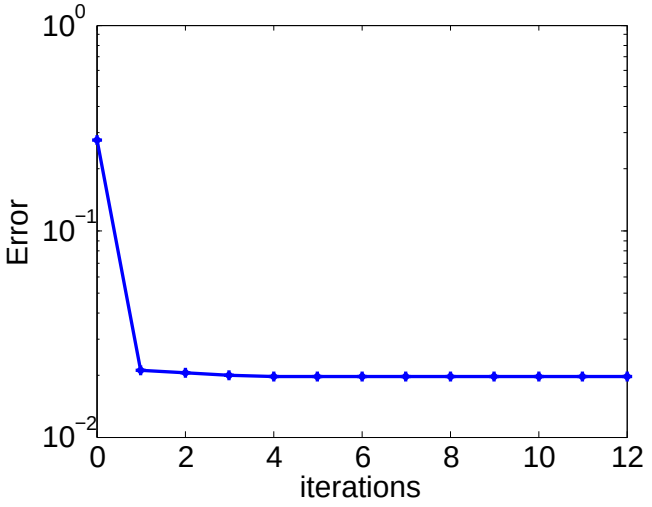


Figure 8: Graph of the global coarse grid error $||u^* - u_i^L||_\infty$, $i = 0,1,2,\ldots,12$ where u^* is the exact solution. A logarithmic scale is used on the error axis.

Γ_{local}. Let M^{in} be so small compared to M^l that the size of the local problem is $M \approx M^l$. Let N be the size of the global problem and N^L_{local} the number of global elements in Γ_{local}. We assume Γ_{local} is such a small part of the global boundary that $N - N^L_{\text{local}} \approx N$. Then the equivalent composite grid on Γ would be of size $pM + N$. The operational count for LU-decomposition is $N^3/3$ for a size N matrix. So the

complexity of the equivalent composite grid problem would be

$$\frac{1}{3}(pM+N)^3 \approx \frac{(p+1)^3}{3}N^3 \text{ if } M \approx N. \tag{40}$$

The BEM-LDC algorithm converges in one step which involves solving two coarse grid problems and p local problems and so has total complexity

$$2 \cdot \frac{1}{3}N^3 + \frac{p}{3}M^3 \approx \frac{N^3}{3}(2+p). \tag{41}$$

So when we compare (40) with (41) we see that the composite problem is $(p+1)^3/(2+p)$ times more expensive than BEM-LDC. Suppose instead we were to refine globally to a grid of size equal to that of the local problems. Then if the refinement ratio is say α, that is, $L/l = \alpha$, the resulting problem would be of complexity $\frac{1}{3}\alpha^3 N^3$. So the resulting problem would be a factor $\alpha^3/(p+2)$ times more complex than using LDC. For instance in the modest case of $\alpha = 2$, this factor is more than one for up to $p = 5$ local problems. Thus BEM-LDC is cheaper than both of its obvious alternatives of either composite gridding or refining uniformly.

Another advantage of LDC over solving on direct composite or fine uniform grids is the memory required. LDC requires less memory than the equivalent composite or uniform grid problems. This is because instead of handling large matrices and vectors of sizes say $(M+N)$ or αN, it handles smaller matrices of sizes N and M at a time.

6 Conclusions

We have presented the technique of Local Defect Correction for BEM, a technique for solving problems with high local activity in the boundary using BEM. The focus of the paper has been to develop the LDC algorith for BEM that takes into account the global integral nature of the BIE, a property that was ignored in Kakuba, Mattheij, and Anthonissen (2006). This technique offers an alternative to solving directly on a composite grid or a uniform fine grid both of which would result in large matrices that are more expensive than using LDC. What is also interesting to note is that one iteration of the algorithm suffices.

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