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Ambarzumyan Type Theorem For a Matrix Valued Quadratic Sturm-Liouville Problem

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Abstract: In this study, Ambarzumyan's theorem for quadratic Sturm-Liouville problem is extended to second order differential systems of dimension $d \geq 2$. It is shown that if the spectrum is the same as the spectrum belonging to the zero potential, then the matrix valued functions both $P(x)$ and $Q(x)$ are zero by imposing a condition on $P(x)$. In scaler case, this problem was solved in [Koyunbakan, Lesnic and Panakhov (2013)].

Keywords: Matrix quadratic Sturm-Liouville equation, spectrum, Ambarzumyan's theorem.

1 Introduction

It is well known that Ambarzumyan's theorem [Ambarzumyan (1929)] is about the boundary value problem

$$-y'' + q(x)y = \lambda y, \ y'(0) = y'(\pi) = 0 \tag{1}$$

with the real potential function $q \in L^2[0, \pi]$. It was proved that if $\lambda_n = n^2, n \geq 0$ is the spectral set of (1), then $q(x) = 0$ on $(0, \pi)$. As an historical viewpoint, this is known as the first result in inverse spectral theory associated with Sturm-Liouville operators. Ambarzumyan's theorem was extended to the second order differential systems of two dimensions in [Chakravarty and Acharyya (1988)], to Sturm-Liouville differential systems of any dimension in [Chern and Shen (1997)], to the Sturm-Liouville equation (which is concerned only with Neumann boundary conditions) with general boundary conditions by imposing an additional condition on the potential function in [Chern, Law and Wang (2001)], and to the multi-dimensional Dirac operator in [C. F. Yang and X. P. Yang (2009)]. In addition, some different results of Ambarzumyan's theorem have been obtained by many authors [Carlson

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and Pvovarchik (2007); Horvath (2001); C. F. Yang and X. P. Yang (2011); Shen (2007); C. F. Yang, Huang and X. P. Yang (2010)].

Ambarzumyan's theorem was extended to the following boundary value problem by imposing to a condition on p

$$-y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 y, x \in [0, \pi] \tag{2}$$

with the homogeneous Neumann boundary conditions

$$y'(0) = y'(\pi) = 0, \tag{3}$$

where λ is a spectral parameter, $p \in W_2^2[0, \pi]$ and $q \in W_2^1[0, \pi]$ by [Koyunbakan, Lesnic and Panakhov (2013)]. This problem is known as diffusion problem or quadratic Sturm-Liouville problem. If $p(x) = 0$, the classical Sturm-Liouville operator is obtained. Some versions of the eigenvalue problem (2), (3) were studied extensively in [Hryniv and Pronska (2012); Gasymov and Guseinov (1981); Guseinov (1985); Nabiev (2007); Koyunbakan and Yilmaz (2008); Panakhov and Sat (2012); Koyunbakan (2011)].

Before giving the main results, we want to mention some physical properties of quadratic equation. The problem of describing the interactions between colliding particles is of fundamental interest in physics. It is interesting in collisions of two spinles particles, and it is supposed that the $s-$wave scattering matrix and the $s-$wave binding energies are exactly known from collision experiments. For a radial static potential $V(E,x)$ and $s-$wave, the Schrödinger equation is written as

$$y'' + [E - V(E,x)]\, y = 0,$$

where

$$V(E,x) = 2\sqrt{E}p(x) + q(x),$$

and we note that with the additional condition $q(x) = -p^2(x)$, the above equation reduces to the Klein-Gordon s -wave equation for a particle of zero mass and energy $\sqrt{E}$ [Jaulent and Jean (1972)].

This paper is organized as follows; Section 2 is devoted to the some known results of matrix quadratic pencil. Section 3 is about some uniqueness theorems and proofs.

2 Matrix Differential Equations

For simplicity, A_{ij} denotes entry of a matrix A at the $i-th$ row and $j-th$ column and I_d is a $d \times d$ identity matrix and 0_d is a $d \times d$ zero matrix.

We are interested in the eigenvalue problem

$$-\phi'' + [2\lambda P(x) + Q(x)]\,\phi = \lambda^2\phi \tag{4}$$

$$A\phi(0) + B\phi'(0) = C\phi(\pi) + D\phi'(\pi) = 0, \tag{5}$$

where $P(x) = diag[p_1(x), p_2(x), ..., p_d(x)]$ and $Q(x)$ are $d\times d$ real symmetric matrix-valued functions, and those $d\times d$ matrices A, B, C and D satisfy the following conditions

$$DC^* \;:\; \text{Self-Adjoint} \tag{6}$$

$$BA^* = 0 \tag{7}$$

$$rank[A, B] = rank[C, D] = d. \tag{8}$$

In this study, we consider the special case of the problem (4), (5) as $A = C = 0_d$ and $B = D = I_d$. Namely, we introduce the following matrix differential equation

$$-Y'' + [2\lambda P(x) + Q(x)]\,Y = \lambda^2 Y, \tag{9}$$

$$Y(0,\lambda) = I_d, Y'(0,\lambda) = 0_d, \tag{10}$$

$$Y'(\pi,\lambda) = 0_d \tag{11}$$

where λ is a spectral parameter, $Y(x) = [y_k(x)], k = \overline{1, d}$ is a column vector, $P \in W_2^2[0,\pi]$ and $Q \in W_2^1[0,\pi]$ are two $d\times d$ real symmetric matrix-valued functions, where $W_2^k[0,\pi]$ $(k = 1, 2)$ denotes a set whose element is a $k-$ th order continuously differentiable function in $L_2[0,\pi]$, $\mu^2 = \lambda$ and $\mu = \sigma + it \in C$. Then, λ is an eigenvalue of (4), (5), if the matrix which is called characteristic function

$$W(\mu) = CY(\pi,\mu) + DY'(\pi,\mu)$$

is singular. In case of $C = 0_d$ and $D = I_d$, the eigenvalues of the problem (9)-(11) are zeros of $W(\mu) = Y'(\pi,\mu) = 0_d$.

In order to describe $W(\mu)$ explicitly, we must know how to express the solution $Y(x,\mu)$. The solution $Y(x,\mu)$ of (9)-(10) can be expressed as [Yang (2012)]

$$Y(x,\mu) = \cos[\lambda x - \alpha(x)] + \int_0^x A(x,t)\cos(\lambda t)dt + \int_0^x B(x,t)\sin(\lambda t)dt \tag{12}$$

where $A(x,t)$ and $B(x,t)$ are symmetric matrix-valued functions whose entries have continuous partial derivatives up to order two respect to x and t. Now, we will give following results that are crucial to obtain our main results. It is pointed out these lemmas were given by [Yang (2012)].

Lemma 2.1. *[Yang (2012)] Let A and B be as in (12). Then, A and B satisfy following conditions*

$$\frac{\partial^2 A(x,t)}{\partial x^2} - 2P(x)\frac{\partial B(x,t)}{\partial t} - Q(x)A(x,t) = \frac{\partial^2 A(x,t)}{\partial t^2} \tag{13}$$

$$\frac{\partial^2 B(x,t)}{\partial x^2} + 2P(x)\frac{\partial A(x,t)}{\partial t} - Q(x)B(x,t) = \frac{\partial^2 B(x,t)}{\partial t^2} \tag{14}$$

$$A(0,0) = 0_d, \;\; B(x,0) = 0_d, \;\; \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0} = 0_d, \tag{15}$$

with $\alpha(x) = \int_0^x P(t)dt$. *Moreover, there holds*

$$2[\cos\alpha(x)A(x,x) + \sin\alpha(x)B(x,x)] = \int_0^x T_1(t)dt \tag{16}$$

and

$$2[\sin\alpha(x)A(x,x) - \cos\alpha(x)B(x,x)] = P(x) - P(0) + \int_0^x T_2(t)dt \tag{17}$$

where

$$T_1(x) = P^2(x) + \cos\alpha(x)Q(x)\cos\alpha(x) + \sin\alpha(x)Q(x)\sin\alpha(x)$$

and

$$T_2(x) = \sin\alpha(x)Q(x)\cos\alpha(x) - \cos\alpha(x)Q(x)\sin\alpha(x).$$

Also, it is well known that [Yang (2012)] the eigenvalues of the problem (9)-(11) are

$$\lambda_n = n + \frac{\alpha_j}{\pi}, n = 0, \pm 1, \pm 2, \pm 3, ..., \; j = \overline{1,d} \text{ and } \alpha_j = \int_0^\pi p_j(x)dx. \tag{18}$$

3 Main Theorems

In this section, some uniqueness theorems are given for the problem (9)-(11). It is shown that an explicit formula of eigenvalues can determine the functions both $Q(x)$ and $P(x)$ be zero by imposing a condition on $P(x)$. Our method is based on [Chern and Shen (1997);Yang (2012)].

Consider a second matrix quadratic pencil of Schrödinger problem

$$-\widetilde{Y}'' + \left[2\lambda P(x) + \widetilde{Q}(x)\right]\widetilde{Y} = \lambda^2 \widetilde{Y},\, x \in [0,\, \pi] \tag{19}$$

$$\widetilde{Y}(0,\, \lambda) = I_d,\, \widetilde{Y}'(0,\, \lambda) = 0_d \tag{20}$$

$$\widetilde{Y}'(\pi,\, \lambda) = 0_d, \tag{21}$$

where $\widetilde{Q}$ has the same properties of Q. Solution of this problem can be written as

$$\widetilde{Y}(x,\, \mu) = \cos[\lambda x - \alpha(x)] + \int_0^x \widetilde{A}(x,\, t)\cos(\lambda t)dt + \int_0^x \widetilde{B}(x,\, t)\sin(\lambda t)dt \tag{22}$$

where $\widetilde{A}$ and $\widetilde{B}$ have the same properties of A and B.

The problems (9)-(11) and (19)-(21) will be denoted by $L(P,\, Q)$ and $\widetilde{L}(P,\, \widetilde{Q})$ and spectrums of these problems wil be denoted by $\sigma(P,\, Q)$ and $\widetilde{\sigma}(P,\, \widetilde{Q})$, respectively.

Theorem 3. 1. *Suppose that* $\sigma(P,\, Q) = \widetilde{\sigma}(P,\, \widetilde{Q})$ *and* $\alpha(\pi) = 0,$ *then*
$$\int_0^\pi \left[Q(x) - \widetilde{Q}(x)\right] dx = 0_d$$
almost everywhere on $[0,\, \pi]$.

Proof: Since $\sigma(P,\, Q) = \widetilde{\sigma}(P,\, \widetilde{Q})$, it follows that $\lambda_n \in \sigma(P,\, Q) = \widetilde{\sigma}(P,\, \widetilde{Q})$. Then we can write from (11) that

$$\begin{aligned} Y'(\pi,\, \lambda_n) = &- (\lambda_n - P(\pi))\sin[\lambda_n \pi - \alpha(\pi)] + A(\pi,\, \pi)\cos(\lambda_n \pi) \\ &+ B(\pi,\, \pi)\sin(\lambda_n \pi) + \int_0^\pi A_x(\pi,\, t)\cos(\lambda_n t)\, dt + \int_0^\pi B_x(\pi,\, t)\sin(\lambda_n t)\, dt \end{aligned}$$

and similarly for the problem (19)-(21), we can write

$$\begin{aligned} \widetilde{Y}'(\pi,\, \lambda_n) = &- (\lambda_n - P(\pi))\sin[\lambda_n \pi - \alpha(\pi)] + \widetilde{A}(\pi,\, \pi)\cos(\lambda_n \pi) \\ &+ \widetilde{B}(\pi,\, \pi)\sin(\lambda_n \pi) + \int_0^\pi \widetilde{A}_x(\pi,\, t)\cos(\lambda_n t)\, dt + \int_0^\pi \widetilde{B}_x(\pi,\, t)\sin(\lambda_n t)\, dt. \end{aligned}$$

By substracting $Y'(\pi,\, \lambda_n)$ and $\widetilde{Y}'(\pi,\, \lambda_n)$,

$$\begin{aligned} 0_d = &\left[A(\pi,\, \pi) - \widetilde{A}(\pi,\, \pi)\right]\cos(\lambda_n \pi) + \left[B(\pi,\, \pi) - \widetilde{B}(\pi,\, \pi)\right]\sin(\lambda_n \pi) \\ &+ \int_0^\pi \left[A_x(\pi,\, t) - \widetilde{A}_x(\pi,\, t)\right]\cos(\lambda_n t)\, dt + \int_0^\pi \left[B_x(\pi,\, t) - \widetilde{B}_x(\pi,\, t)\right]\sin(\lambda_n t)\, dt. \end{aligned}$$

By using Riemann-Lebesque lemma and for $\lambda_n \to \infty$ in (18), we obtain $A(\pi,\pi) = \widetilde{A}(\pi,\pi)$. On the other hand by Lemma 2.1., we know the following equalities,

$$2\frac{d}{dx}\left[\cos\alpha(x)A(x,x)+\sin\alpha(x)B(x,x)\right] = P^2(x)+Q(x) \tag{23}$$

and similarly

$$2\frac{d}{dx}\left[\cos\alpha(x)\widetilde{A}(x,x)+\sin\alpha(x)\widetilde{B}(x,x)\right] = P^2(x)+\widetilde{Q}(x). \tag{24}$$

After substracting (23), (24) and integrating, we get

$$\int_0^\pi \left[Q(x)-\widetilde{Q}(x)\right]dx = 2\left\{\left[A(\pi,\pi)-\widetilde{A}(\pi,\pi)\right]\cos\alpha(\pi)+\left[B(\pi,\pi)-\widetilde{B}(\pi,\pi)\right]\sin\alpha(\pi)\right\}$$

or

$\int_0^\pi \left[Q(x)-\widetilde{Q}(x)\right]dx = 0_d$. So, this completes the proof.

Theorem 3. 2. *Let* $P(x) = diag[p_1(x), p_2(x), ..., p_d(x)]$ and $Q(x)$ *are two* $d \times d$ *real symmetric matrix-valued functions, and* $\alpha(\pi) = 0$. If $\{0\} \cup \{m_j \;:\; j = 1,2,...\}$ *is a subset of the spectrum of the* $d-$*dimensional problem*

$$-Y'' + [2\lambda P(x)+Q(x)]\,Y = \lambda^2 Y,\; Y'(0) = Y'(\pi) = 0_d \tag{25}$$

where 0 *is the first eigenvalue of (25),* m_j *is a strictly ascending infinite sequence of positive integers, and* 0 *and* m_j *are multiplicity of* n *, then* $P(x) = Q(x) = 0_d$ *almost everywhere on* $(0,\pi)$.

Proof: Suppose for the (25) Neumann problem, then we have infinitely many eigenvalues of the form m_j, m_j are positive integers, $j = 1,2,...$ and each m_j is of multiplicity n. Then, we get

$$Y'(\pi, m_j) = 0_d. \tag{26}$$

On the other hand, by (12), we have

$$\begin{aligned} Y'(x,\lambda_n) = &-(\lambda_n I_d - P(x))\sin[\lambda_n x-\alpha(x)] + A(x,x)\cos(\lambda_n x) + B(x,x)\sin(\lambda_n x) \\ &+ \int_0^x A_x(x,t)\cos(\lambda_n t)\,dt + \int_0^x B_x(x,t)\sin(\lambda_n t)\,dt. \end{aligned} \tag{27}$$

Equations (26) and (27) imply

$$A(\pi,\pi)\cos(m_j\pi) + \int_0^\pi A_x(\pi,t)\cos(\lambda_n t)\,dt + \int_0^\pi B_x(\pi,t)\sin(\lambda_n t)\,dt = 0_d. \tag{28}$$

We have from (28) and Riemann Lebesque lemma that $A(\pi, \pi) = 0_d$. Then, by integration of

$$Q(x) + P^2(x) = 2\frac{d}{dx}\left[\cos\alpha(x)A(x, x) + \sin\alpha(x)B(x, x)\right],$$

we get

$$\cos\alpha(x)A(x, x) + \sin\alpha(x)B(x, x) = \frac{1}{2}\int_0^x \left[Q(t) + P^2(t)\right] dt.$$

or by $\alpha(\pi) = 0$, we have

$$\frac{1}{2}\int_0^\pi \left[Q(t) + P^2(t)\right] dt = A(\pi, \pi) = 0_d$$

and

$$\int_0^\pi Q(x)dx = -\int_0^\pi P^2(x)dx. \tag{29}$$

By using the reality of 0 being the ground state of the eigenvalue problem (25), we may find $d-$ linearly independent constant vectors corresponding to the same eigenvalue 0 by the variational principle and denote them by $\varphi_j, \quad j = 1, 2, ..., d$. Since they should satisfy the equation

$$-\varphi_j'' + [2\lambda_0 P(x) + Q(x)]\,\varphi_j = \lambda_0^2\varphi_j,$$

we obtain

$$Q(x)\varphi_j = 0,\ 0 \le x \le \pi.$$

Thus $Q(x) = 0_d$. If we consider (29) and diagonally property of P, we get $P(x) = 0_d$. This completes the proof.

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